Worst-Case-Optimal Dynamic Reinsurance for Large Claims

by

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A Quotation

Nothing is as practical as a good theory.

(Alexander Rinnooy Kan in his keynote speech on the conference OR 2013 in Rotterdam)

Another Quotation

Two men are preparing to go hiking. While one is lacing up hiking boots, he sees that the other man is forgoing his usual boots in favor of sporty running shoes. "Why the running shoes?" he asks. The second man responds, "I heard there are bears in this area and I want to be prepared." Puzzled, the first man points out, "But even with those shoes, you can't outrun a bear." The second man says, "I don't have to outrun the bear, I just have to outrun you."

(See Hubbard [1], p. 157/158)

Outline

1. Literature Review
2. The Model Setup
3. Worst Case Scenario Optimization
4. Evolution of the Net Reserve Process
5. Computing the Probability of Ruin
6. The Intrinsic Risk–free Rate of Return
7. Comparing Different Business Strategies
8. Outlook

1. Literature Review

1. **Minimising Probability of Ruin**, introduced by Lundberg (1909), Cramér (1930, 1955); for recent results see for instance Hipp and Vogt (2003), Schmidli (2001, 2002, 2004), or Eisenberg and Schmidli (2009): [$\rightarrow$ usually involves PIDE].
2. **Maximise the expected discounted sum of dividend payments until ruin occurs**, see e.g. Gerber (1969), Azcue and Muler (2005), Albrecher and Thonhauser (2008).
3. Another approach is to **maximise expected utility** (e.g. Liu/Ma (2009) or Liang/Guo (2010)). [$\rightarrow$ Conjecture: argmax expected exp-utility is sufficient close to argmin ruin probability, e.g.: Ferguson (1965), Brown (1995), or Fernández et al. (2008)].
4. **Determine worst-case bounds for the performance of optimal investment**, see e.g. Korn and Wilmott (2002), Korn (2005), Korn and M. (2005), M. (2006), Korn and Steffensen (2007), Seifried (2010) [$\rightarrow$ known as Wald's Maximin approach in decision theory, Wald (1945, 1950)].

2. The Model Setup

Assumptions:

- continuously paid premium π.
- claims which occur at random times, modelled by the claims arrival process N^c. Note, that the exact dynamics of N^c plays no role, as long as the process has paths which are RCLL.
- the number of claims arriving in $[0, T]$ are bounded, i.e. we assume
$$N^c(T) \leq N$$
with N being the maximum possible number of claims in $[0, T]$.
- all claims have a non-negative size b_i – bounded above by β, that is $b_i \in [0, \beta]$.
- the insurance company can reinsure the fraction $p(t) \in [0, 1]$ of its business. To do so, it has to pay $\pi\, p(t)\, [1 + \varepsilon]$, where $\varepsilon \geq 0$ is the load or premium. The case of $\varepsilon = 0$ is known as cheap reinsurance compared to $\varepsilon > 0$ which is called non-cheap reinsurance.

Assumptions (continued):

- all insurance policies will be terminated at time T and no claim is possible thereafter.

Thus, the dynamics of the reserve process $R(t)$, the income/outflow from reinsurance $I^p(t)$, and the net reserve process $R^p(t)$ are given by

$$\begin{aligned} dR(t) &= \pi\, dt - \beta\, dN^c \\ dI^p(t) &= -\pi\, p(t)\; [1+\varepsilon]\; dt + \beta\, p(t)\, dN^c \\ \boldsymbol{dR^p(t)} &= dR(t) + dI^p(t) \\ &= \boldsymbol{\pi\, [1 - p(t)\; [1+\varepsilon]]\; dt - \beta\; [1 - p(t)]\; dN^c\; .} \end{aligned} \tag{1}$$

The second assumption ($N^c \leq N$) can be included into our setting above by assuming

$$dR(t) = \pi\, dt$$

after having observed N claims.

3. Worst Case Scenario Optimization

Our aim is to maximize the worst-case expected utility of final reserves, that is

$$\sup_{p\in\mathcal{A}} \inf_{N^c\in\mathcal{B}} \mathbb{E}\left[U\left(R^p\left(T\right)\right)\right] , \tag{2}$$

where

- $\mathcal{A}$ is the set of all predictable processes with respect to the $\sigma-$algebra generated by the reserve process and the jump process which determines how many claims are still possible.
- $\mathcal{B}$ denotes the set of all such possible jump processes.
- the utility function $U(x)$ is strictly increasing and defined on $\mathbb{R}$.

With this, the value function $V^n(t,x)$ of our problem is given by

$$\boldsymbol{V^n(t,x) = \sup_{p\in\mathcal{A}} \inf_{N^c\in\mathcal{B}} \mathbb{E}^{t,x,n}\left[U\left(R^p\left(T\right)\right)\right]} ,$$

where $\mathbb{E}^{t,x,n}$ is the conditional expectation given that $R^p(t) = x$ and given that there are at most n claims left.

Theorem 3.1 (Verification Theorem)

Let the function $U(x)$ be strictly increasing and be defined on $\mathbb{R}$. Given that $n \in \{1, ..., N\}$ possible claims can still occur, the optimal worst-case reinsurance strategy $p^n(t)$ is defined through the following property of the value function

$$\boldsymbol{V^n(t,x)} = \boldsymbol{V^{n-1}\left(t, x - \beta\left(1 - p^n(t)\right)\right)} \tag{3}$$

$$= U\left(x + \pi\left(T - t\right) - \beta \sum_{i=1}^{n} \left(1 - p^i(t)\right)\right). \tag{4}$$

and is given as the unique solution of the (system of) ordinary differential equation

$$\boldsymbol{p_t^n\left(t\right)} = \boldsymbol{\frac{\pi}{\beta}\left[1+\varepsilon\right]\left[p^n\left(t\right) - p^{n-1}\left(t\right)\right]} \tag{5}$$

with boundary conditions

$$p^n(T) = 1 \quad \text{for } n = 1, 2, ..., N \tag{6}$$

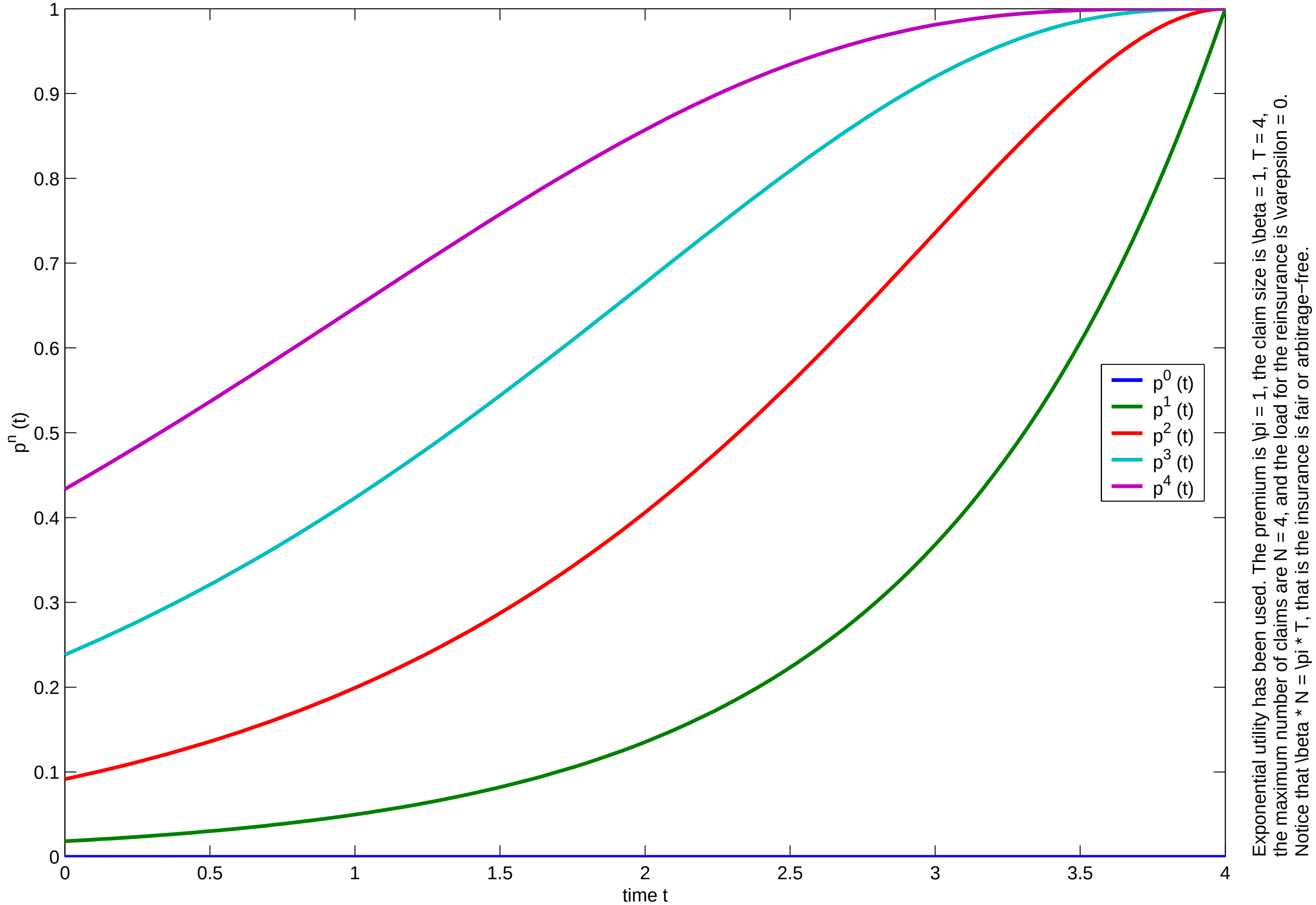
$$p^0(t) \equiv 0 \quad \text{for all } t \in [0, T]. \tag{7}$$

In particular, we have that the value functions are monotonically increasing. Further, with the notation of $\alpha = \frac{\pi}{\beta}\left[1+\varepsilon\right]$, the optimal worst-case reinsurance strategy p^n has the explicit form of

$$\boldsymbol{p^n(t)} = \exp\left(-\alpha\left[T-t\right]\right) \sum_{k=0}^{n-1} \frac{1}{k!} \left(\alpha\left[T-t\right]\right)^k, \quad n = 1, 2, ... \tag{8}$$

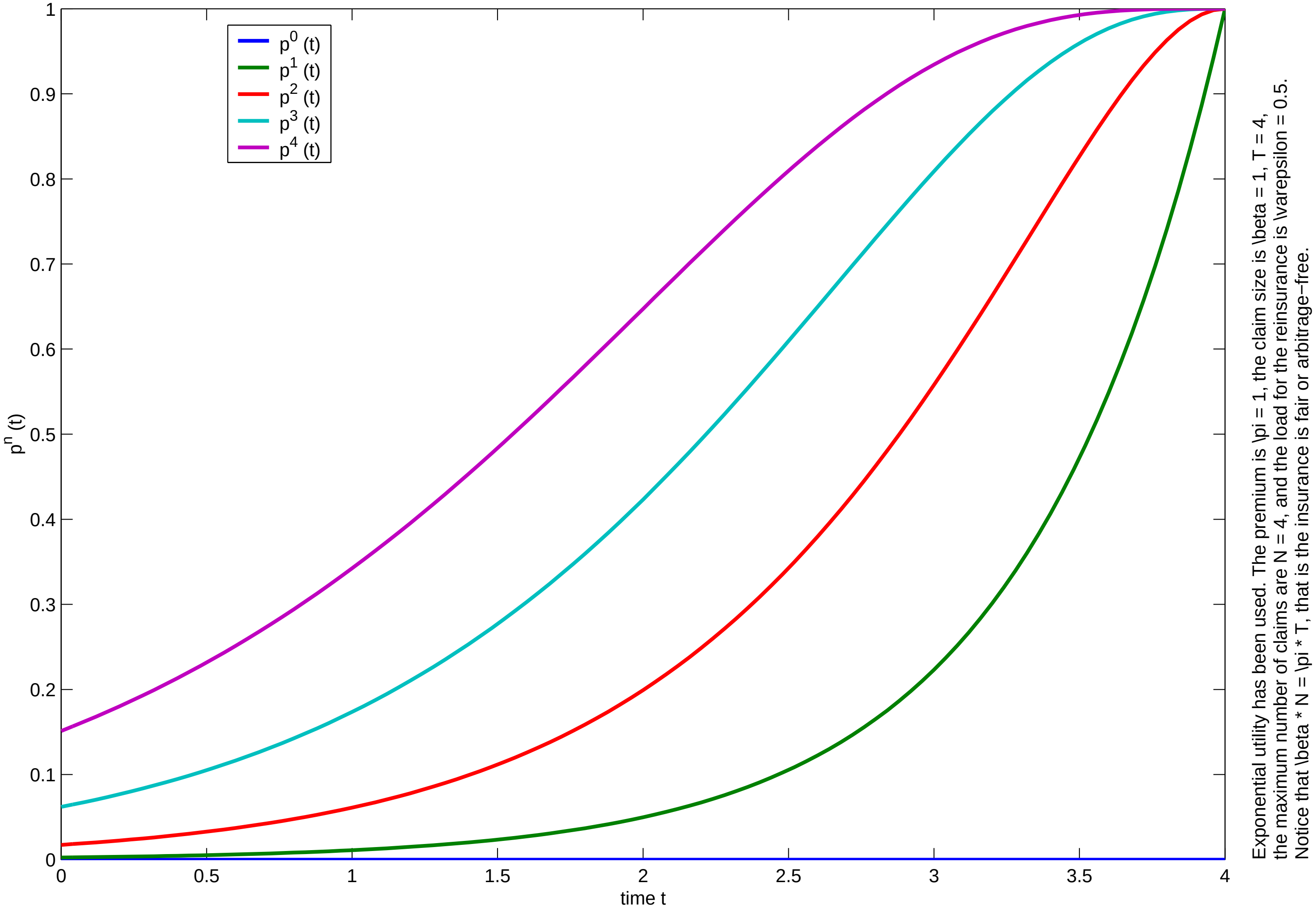
$$p^0(t) = 0. \tag{9}$$

Example $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$



This graphic shows the worst-case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

Example $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$



This graphic shows the worst-case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

Remark 3.2

The strategy p^n can be interpreted as the probability of experiencing less than N claims in an artificial model where the claim numbers are Poisson distributed with intensity α, i.e.

$$p^n(t) = P\left(N^P(T) < N \middle| N^P(t) = N - n\right),$$

where $\left\{N^P(t)\right\}_{t \geq 0}$ is a Poisson process with intensity α.

This is a convenient way to think about the result in the form of a rule-of-thumb:

Calculate an intensity α as the worst-case premium to claim ratio. Then, reinsure the proportion of your portfolio corresponding to the probability of experiencing less than a given number of N claims.

Clearly, N^P is independent of N^c. Indeed, N^P is an artificial construct not belonging to the model setup.

4. Evolution of the Net Reserve Process

Definition 4.3

Let $0 \leq s \leq t \leq T$. Denote by $R(s,t;p,n)$ the net reserve process between time s and t which consists of the incoming premia and the outgoing reinsurance premia given that the insurance company uses the reinsurance strategy $p(u)$ with $u \in [s,t]$, given that at most n claims can come in, and given that no claim occurs in $[s,t]$. If the optimal worst-case reinsurance strategy p^n is considered, the notation is simplified to $R(s,t;p^n) := R(s,t;p^n,n)$.

With this definition, one has that

$$\begin{aligned} R(s,t;p^n) &= \int_s^t \pi \left[1 - p^n(u)\left[1+\varepsilon\right]\right] du = \pi\left[t-s\right] - \pi\left[1+\varepsilon\right] \int_s^t p^n(u)\, du \\ &= \pi\left[t-s\right] - \beta \sum_{k=1}^{n} \left[p^k(t) - p^k(s)\right], \end{aligned}$$

provided the reinsurance strategy p^n is used and no claims occur in $[s,t]$.

There are two special cases which will be considered more closely in the following. The first one is given by setting $s = 0$, that is

$$R(0, t; p^n) \;=\; \pi t - \beta \sum_{k=1}^{n} \left[p^k(t) - p^k(0) \right], \tag{10}$$

which will simply be called the **forward net reserve process without claims** and second,

$$R(t, T; p^n) \;=\; \pi(T - t) - \beta \sum_{k=1}^{n} \left[1 - p^k(t) \right], \tag{11}$$

which will simply be called the **backward net reserve process without claims**. Before continuing let us establish an important property of $R(t, T; p^n)$ which can be verified either by direct computation or by using Theorem 3.1 with $U(x) = x$.

Corollary 4.4 (Backward Reserve Process)
For any $t \in [0, T]$ and $n \in \{1, \dots, N\}$ the following holds

$$R(t, T; p^n) \leq R\left(t, T; p^{n-1}\right) - b_n \left(1 - p^n(t)\right), \tag{12}$$

where $b_n \leq \beta$ denotes the actually observed claim size for the $N - n + 1$–st claim. Equality holds in (12) if and only if $b_n = \beta$.

Proposition 4.5 (Forward Reserve Process)
Denote by τ^i the arrival time of the $N-i+1$–st claim and its size by b_i, then $0 \leq \tau^N < \ldots < \tau^i < \ldots < \tau^{n+1} \leq t^* \leq T$ and $b_i \leq \beta$ with $i = n+1, \ldots, N$. Assuming that the insurance follows the worst–case reinsurance strategy – that is up to the first claim p^N, between the first and second claim p^{N-1}, etc., the reserve process at time t^* computes to

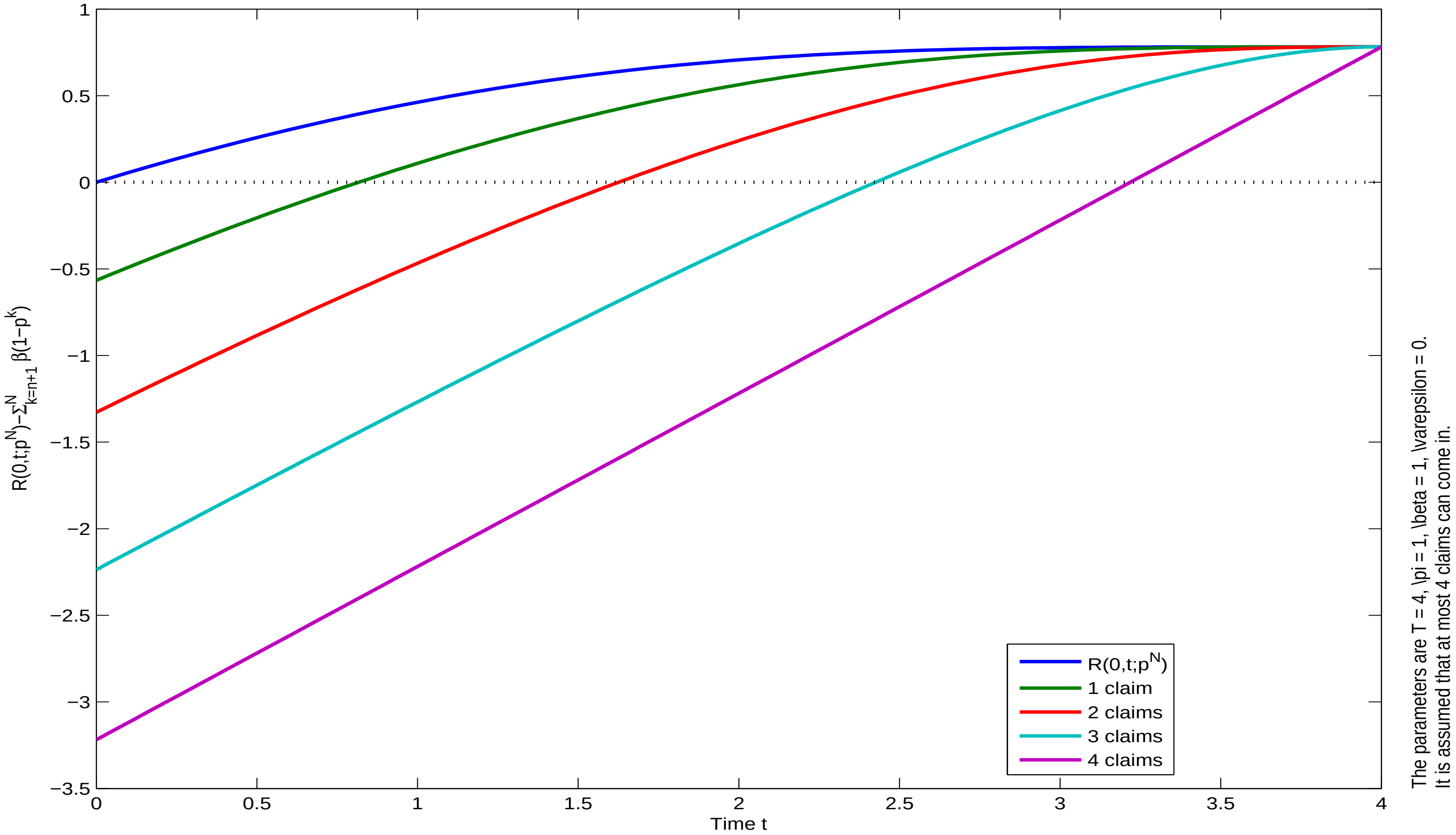
$$\begin{aligned} & R\left(0, \tau^N; p^N\right) - b_N \left[1 - p^N\left(\tau^N\right)\right] + \sum_{i=n+1}^{N-1} \left\{R\left(\tau^{i+1}, \tau^i; p^i\right) - b_i \left[1 - p^i\left(\tau^i\right)\right]\right\} + \\ & + R\left(\tau^{n+1}, t^*; p^n\right) \\ = \;& R\left(0, t^*; p^N\right) + \sum_{k=n+1}^{N} (\beta - b_k) \left[1 - p^k\left(\tau^k\right)\right] - \beta \sum_{i=n+1}^{N} \left[1 - p^i\left(t^*\right)\right] . \end{aligned} \tag{13}$$

In particular, the lower bound or the **worst-case bound** *for the reserve process at time T is given by*

$$\boldsymbol{R\left(0, T; p^N\right) = \pi T - \beta \sum_{k=1}^{N} \left[1 - p^k\left(0\right)\right]} , \tag{14}$$

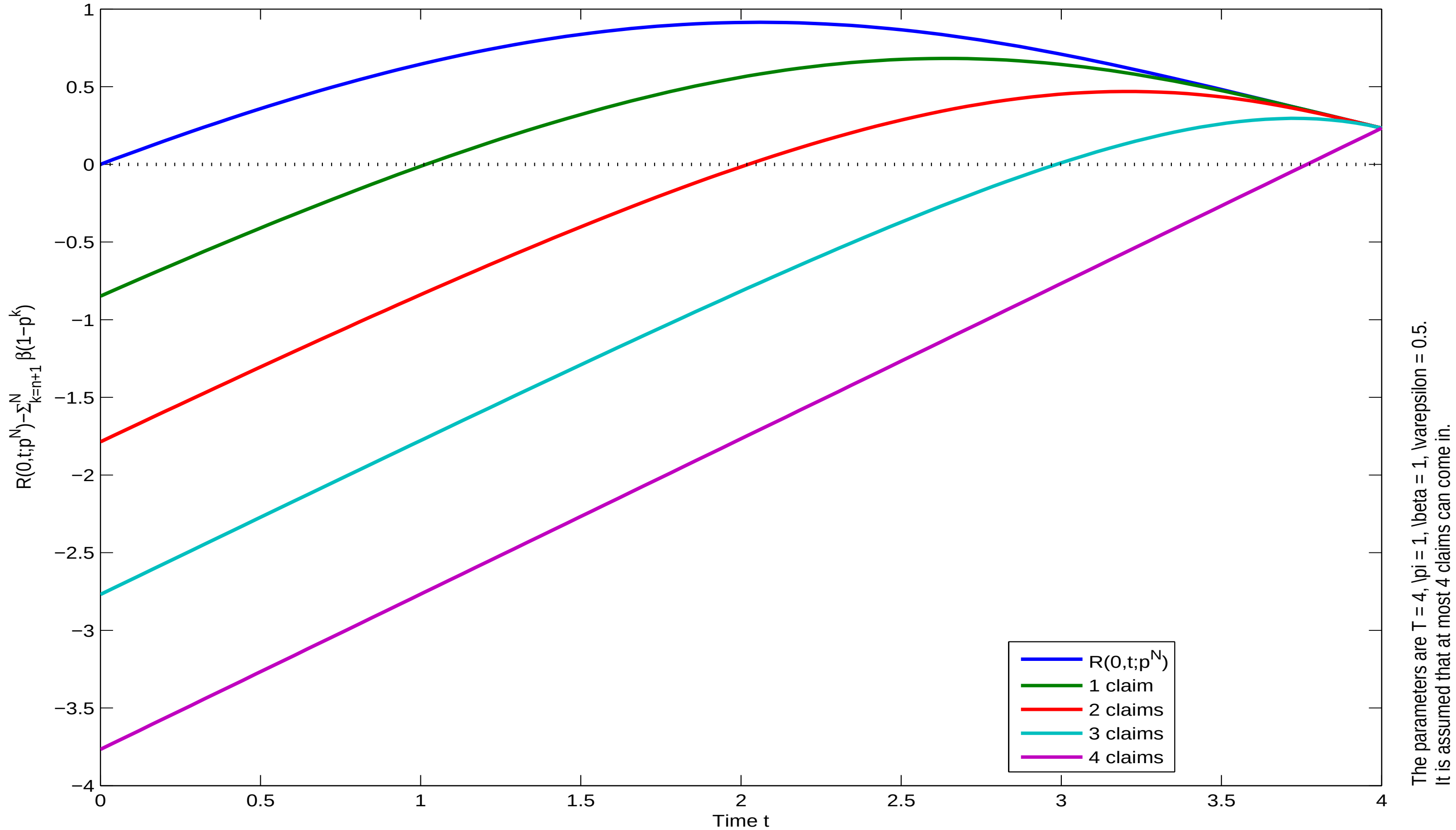
assuming that the insurance follows the worst-case reinsurance strategy (p^n), $n = 0, 1, \ldots, N$. This bound will be reached if either no claim is made or if all claims that are made are of the worst-case size β.

Possible Scenarios for the Net Reserve Process with $\varepsilon = 0$



This graphic shows the possible evolution of the net reserve process $R(0,t;p^N) - \beta \sum_{k=n+1}^{N} \left[1 - p^k\left(t_0^n\right)\right]$ using the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

Possible Scenarios for the Net Reserve Process with $\varepsilon = 0.5$



This graphic shows the possible evolution of the net reserve process $R(0,t;p^N) - \beta \sum_{k=n+1}^{N} \left[1 - p^k\left(t_0^n\right)\right]$ using the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

5. Computing the Probability of Ruin

Lemma 5.6 (Average Upper Bound for the Reinsurance Strategy) *Given that the initial reserve is $R(0) = y \geq 0$ and that the* **weak solvency condition** $R(0, t; p, N, y) := y + R(0, t; p, N) \geq 0$ *holds for all $t \in [0, T]$, then the reinsurance strategy p satisfies*

$$\frac{1}{t}\int_0^t p(s)\, ds \;\leq\; \frac{1}{1+\varepsilon} + \frac{y}{\pi\,(1+\varepsilon)\,t} \qquad \text{for all } t \in [0, T]\,,$$

which is a strict bound for $\varepsilon > 0$ and $y = 0$. Moreover, the worst-case reinsurance strategy satisfies this weak solvency condition if and only if

$$\beta N \leq y + \pi T + \beta \sum_{k=1}^{N} \left[1 - p^k(0)\right] . \tag{15}$$

In order to identify the scenarios where the reserve level is negative, the zeros of Equation (13), adjusted by the initial capital y, are calculated for $n = 0, 1, \ldots, N-1$. The zeros are denoted by $t_0^n(y)$ or simply t_0^n as they clearly depend on the initial capital y. This means that $N-n$ claims have been made so far (at times $0 \leq \tau^N \leq \tau^{N-1} \leq \ldots \leq \tau^{n+1} \leq t_0^n \leq T$) and n claims might still be made. This can be written as

$$\pi t_0^n + \beta \sum_{k=1}^{n} \left[1 - p^k\left(t_0^n\right)\right] \;=\; \beta \sum_{k=1}^{N} \left[1 - p^k(0)\right] - \sum_{k=n+1}^{N} (\beta - b_k) \left[1 - p^k\left(\tau^k\right)\right] - y\,,$$

which can be solved explicitly just for $n = 0$:

$$t_0^0 = \frac{\beta}{\pi} \sum_{i=1}^{N} \left[1 - p^i(0)\right] - \frac{y}{\pi} - \frac{1}{\pi} \sum_{k=1}^{N} (\beta - b_k) \left[1 - p^k\left(\tau^k\right)\right]. \qquad (16)$$

Given the initial reserve y and the reinsurance strategy p, denote by $\psi_p(y)$ the probability of ruin. With this, the probability of survival is given by $\delta_p(y) = 1 - \psi_p(y)$.

Ruin occurs for instance, if one claim is made between time 0 and t_0^{N-1}, or two claims are made between time 0 and t_0^{N-2}, and so on. In general, ruin occurs if $N-n$ claims are made between time 0 and t_0^n for $n = 0, 1, \ldots, N-1$. Hence,

$$\psi_{(p^n)}(y) = \frac{\mathbb{P}\left(R\left(0, t; p^N, y\right) < 0\right) + \sum_{n=0}^{N-1} \mathbb{P}\left(N^c\left(t_0^n(y)\right) = N - n\right)}{\mathbb{P}\left(N^c(T) \leq N\right)}. \tag{17}$$

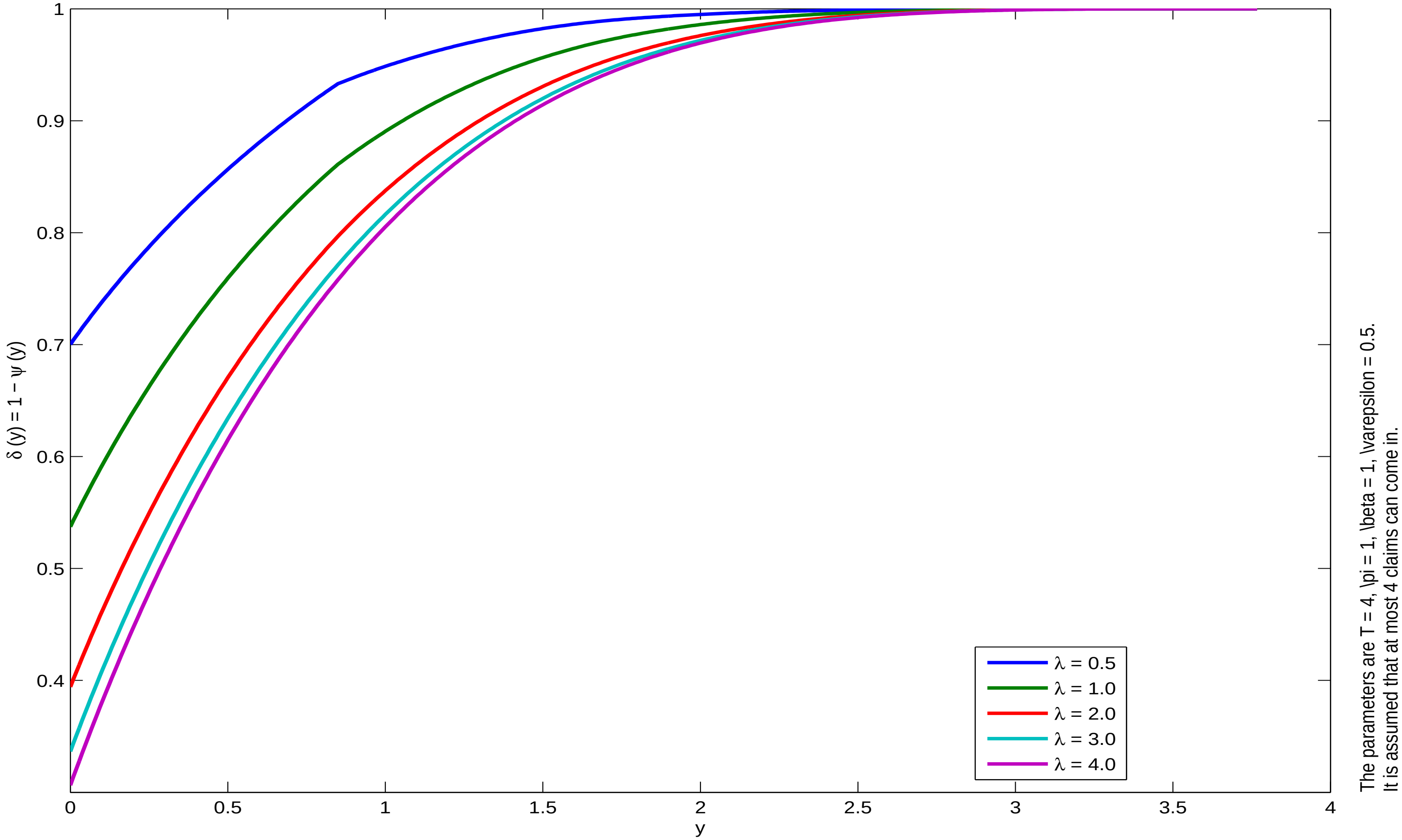
The first term on the right side is due to the possibility that ruin may occur even with no claims being made. The denominator is due to the assumption that at most N claims can be made. Note that $\mathbb{P}\left(R\left(0, t; p^N, y\right) < 0\right) = 0$ if $t_0^N < 0$ or if Lemma 5.6 holds.

Example 5.7

Assuming that Condition (15) holds (that is Lemma 1 holds) and that N^c is Poisson distributed with parameter λ, the **worst-case bound of the probability of ruin** (that means setting $b_i = \beta$ for all i) calculates to

$$\psi_{(p^n)}(y) = \frac{\sum_{n=0}^{N-1} \frac{\left(\lambda t_0^n(y)\right)^{N-n}}{(N-n)!}}{\sum_{k=0}^{N} \frac{(\lambda T)^k}{(k)!}} .$$

Probability of Survival with $\varepsilon = 0.5$



This graphic shows the worst-case bound for the probability of survival for various initial reserves y and various λ if the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ is used for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

6. The Intrinsic Risk–free Rate of Return

The insurance cannot go bankrupt if the initial reserve satisfies

$$y \geq \beta \sum_{k=1}^{N} \left[1 - p^k(0)\right] =: y^* .$$

Note, that $y^* \in [0, \beta N]$. Hence, the **worst-case bound for the return** or the **intrinsic risk–free rate of return** is given by

$$\frac{R\left(0, T; p^N\right)}{y^*} = \frac{\pi T}{\beta \sum_{k=1}^{n} \left[1 - p^k(0)\right]} - 1 = \frac{\pi T}{y^*} - 1 .$$

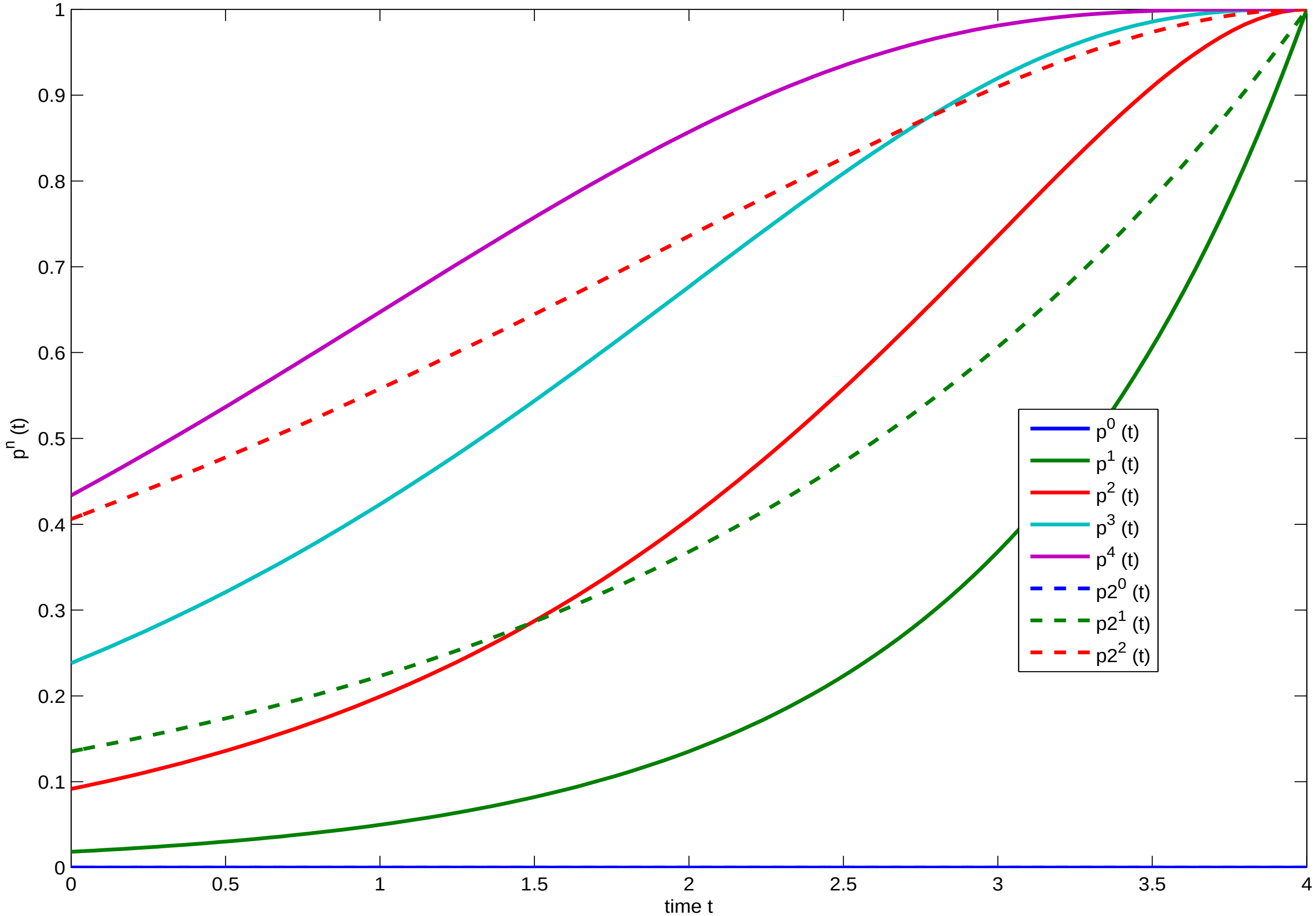
Profitability of the Worst Case Reinsurance strategy with $\pi T = \beta N$

ε	0	0.1	0.5	0	0.1	0.5
β	1	1	1	2	2	2
N	4	4	4	2	2	2
$R(0, T; p^N)$	0.7815	0.6232	0.2330	1.0827	0.9307	0.4979
y^*	3.2185	3.3768	3.7670	2.9173	3.0693	3.5021
risk–free return	0.2428	0.1845	0.0619	0.3711	0.3032	0.1422

This table is calculated using $\pi = 1$ and $T = 4$.

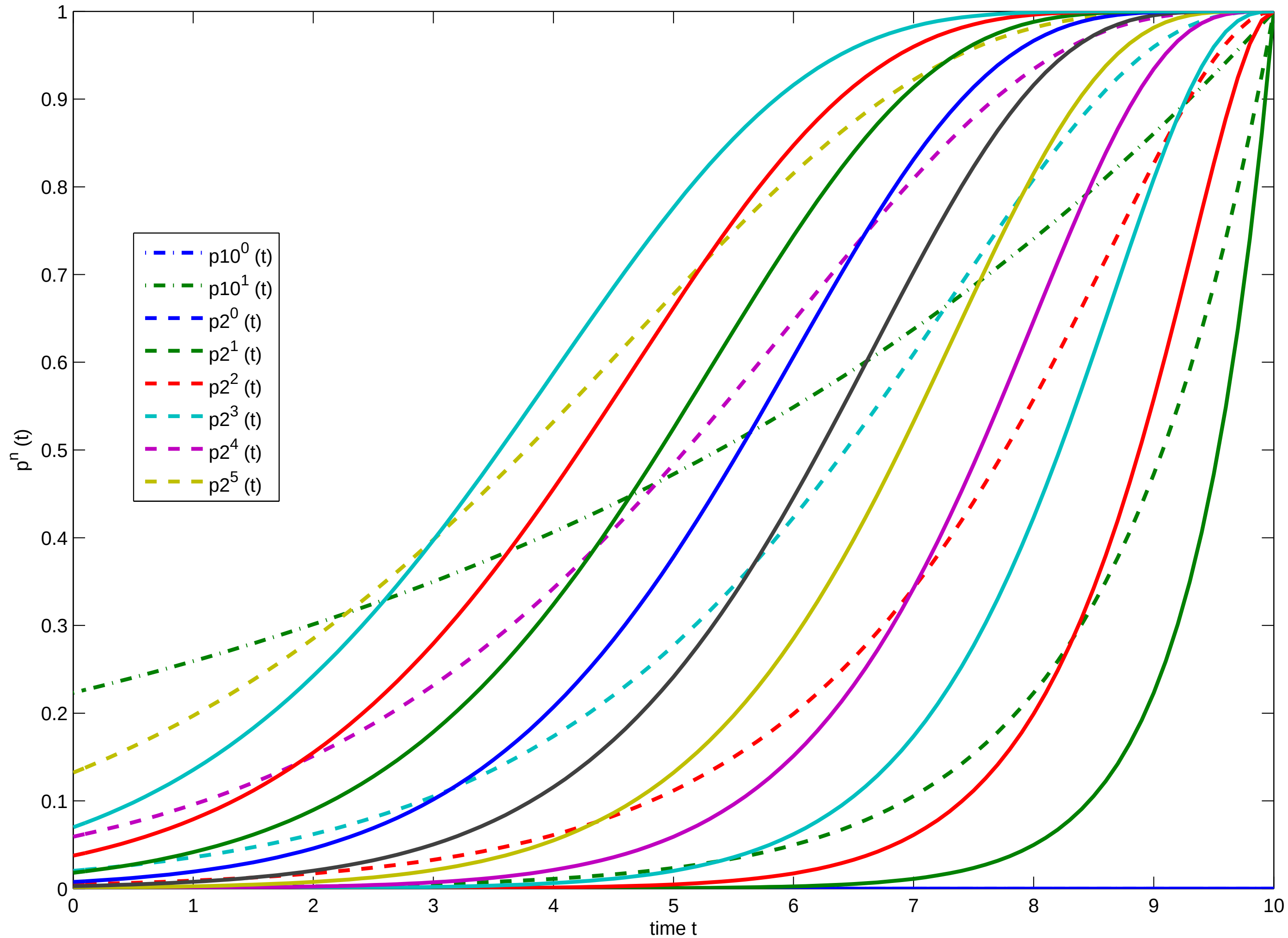
7. Comparing Different Business Strategies

Example $\pi = 1$, $T = 4$, and $\varepsilon = 0$



This graphic shows the worst-case optimal reinsurance strategy for $\pi = 1$, $T = 4$, and $\varepsilon = 0$ for $\beta = 1$, $N = 4$ (solid lines) and $\beta = 2$, $N = 2$ (dashed lines).

Example $\pi = 1$, $T = 10$, and $\varepsilon = 0.5$



This graphic shows the worst-case optimal reinsurance strategy for $\pi = 1$, $T = 10$, and $\varepsilon = 0.5$ for $\beta = 1$, $N = 10$ (solid lines), $\beta = 2$, $N = 5$ (dashed lines), and $\beta = 10$, $N = 1$ (dashed-dotted line).

Let be K, $L \in \mathbb{N}$ with K, $L \geq 1$. Compare the following business strategies:

- the case of K contracts (or possible claims) with potential worst-case claim size $\beta = b$ with
- the case of $K+L$ contracts with potential worst-case claim size $\beta = \frac{Kb}{K+L}$.

Setting $\pi T = bK = \frac{Kb}{K+L}(K+L)$, it is clear that both business strategies have the same volume. If Markowitz's principle is **not** true, the following inequality should hold

$$R\left(0, T; p^{K}\right) - R\left(0, T; p^{K+L}\right) \;\geq\; 0\,.$$

Define

$$\boldsymbol{f_{K,L}(x) := \sum_{l=0}^{K-1} (K-l)\,\frac{x^l}{l!} - e^{-\frac{L}{K}x}\frac{K}{K+L}\sum_{l=0}^{K+L-1}(K+L-l)\,\frac{\left[\frac{K+L}{K}x\right]^l}{l!}}\,.$$

Proposition 7.8

(i) *The business strategy of having K contracts gives a higher worst-case bound than the strategy of having $K+L$ contracts with the same turnover volume, if αT is sufficiently large.*

(ii) *In the special case $K=1$, the business strategy of having only $K=1$ contract is always superior to having $1+L$ contracts (with $L \geq 1$) given that both business strategies generate the same turnover volume.*

Thus, diversification can have a negative effect, that is the Principle of Markowitz – don't put all your eggs in one basket – does not always hold.

8. Conclusion and Outlook

We demonstrated the attractive properties of this model, specifically

- explicitly computable, worst-case optimal reinsurance strategies,
- robustness against choice of utility function,
- robustness against modeling of claim sizes and claim numbers,
- and giving fresh insights on the aspects of diversification.

It is interesting, in future studies to include further aspects of the non-life insurance company's decision making, including

- investment risk modeling and control,
- small claims modeling and control, e.g. by a Gaussian process for the small claims surplus, thereby adding noise to the system and the results, and
- alternative ways of formalizing the worst-case, e.g. comparing with worst–case bounds on the (claim–number independent) intensity.

References

1. Douglas Hubbard, *The Failure of Risk Management: Why It's Broken and How to Fix It*, John Wiley & Sons, 2009.
2. Ralf Korn, Olaf Menkens, and Mogens Steffensen, Worst-Case-Optimal Dynamic Reinsurance for Large Claims. *European Actuarial Journal* 2(1), 21–48, 2012.

Thank you

very much

for your attention!