Minimization of the Total Required Capital by Reinsurance

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Abstract

Reinsurance reduces the required capital of the primary insurer but increases that of the reinsurer. Capital is costly. All capital costs, including that of the insurer and the reinsurer, are ultimately borne by primary policyholders. Reducing the total capital of insurers and reinsurers brings down the total capital cost and the total primary policy premium. A reinsurance arrangement is considered optimal, if it minimizes the total required capital. This optimal reinsurance is an attracting equilibrium under price competition. Evidence suggests an inverse relationship between the total required capital and the correlation between the losses held by different insurers and reinsurers. Examples are constructed to examine this and other properties of the optimal reinsurance.

Keywords

Required capital, capital cost, optimal reinsurance, subadditive risk measure, correlation between losses

1 Introduction

A new kind of optimal reinsurance is introduced in this paper. Reinsurance serves many purposes, one of which is to reduce the required capital by reducing the volatility of losses. From shareholders' point of view, capital is costly due to income taxes and agency costs. Shareholders pay income taxes two times on their capital investment, first at the corporate level and then at the personal level when they sell the stock. They would not owe the first tax if they invested directly in the securities market. Agency costs exist because of the separation of ownership and control. They include monitoring and bonding expenditures and other losses in profits due to misalignment of managers' decisions and shareholders' welfare. Taxes and agency costs, altogether called capital costs, are generally directly related to the amount of capital. See Jensen and Meckling (1976), Perold (2005), Chandra and Sherris (2007), Zhang (2008) for discussions on this. Thus carrying less amount of capital is desirable.

Reinsurance transfers losses from a ceding company to a reinsurer. Such losses are often highly volatile. So this transfer of losses increases the capital requirment of a reinsurer while reduces that of a ceding company. Consequently, capital costs of the reinsurer increase and those of the ceding company decrease. The total capital cost, the sum of that of both companies, may go either way. I will examine an ideal insurance market where all insurance policies are fairly priced. Primary policy premiums include exact amounts to cover primary insurers' capital costs; reinsurance premiums include exact amounts to cover reinsurers' capital costs. But reinsurance premiums are funded through premiums of primary policies. Therefore, the total capital costs of primary insurers and reinsurers are ultimately borne by primary policyholders. If a treaty reduces a ceding company's capital costs more than it increases the reinsurer's, the total capital cost is reduced, which benefits primary policyholders. A treaty, or a set of treaties, is optimal, if it minimizes the total capital cost. Such optimal reinsurance arrangements are the subject of this paper.

Under simplifying assumptions, minimization of the total capital cost is equivalent to minimization of the total amount of capital carried by all companies. This latter problem may be directly solved by simulating loss scenarios. Suppose (1) a capital requirement is defined using a statistical risk measure (TVaR will be the preferred one in this paper), (2) the joint distribution of all losses involved (losses of primary insurers and reinsurers) is known, and (3) a set of permissible reinsurance treaties (types of treaties and lines of business to be covered) is given. Then for each of the treaties the total required capital can be calculated. And by comparing the total capital across treaties, the optimal treaty is easily found.[1]

The risk measure TVaR belongs to a desirable class called the coherent risk measures, defined in Artzner et al. (1999). For such a risk measure there is an absolute lower bound for the total capital—regardless of reinsurance arrangements, the total capital must be greater than this lower bound. It can be shown that if the losses of the insurers have a certain correlation called comonotonicity, then the

[1] In a real-world situation, usually only a finite number of treaty options are practically available, e.g., a few quota share treaties with certain ceding percentages, or a few excess-of-loss treaties with certain retentions and limits.

total capital attains the lower bound. This observation leads to a discussion of the relationship between optimal reinsurance and correlated losses. Evidence suggests that an optimal reinsurance is one that makes the losses of insurers and reinsurers as correlated as possible. (Such correlation needs only occur at the tail.)

Numerous authors have written about optimal reinsurance, and have given various criteria of optimality. Our criterion is noticeably different. Usually an optimal reinsurance is defined from the ceding company's point of view. The ceding insurer seeks a treaty to maximize the risk-adjusted return (Lampaert and Walhin 2005, Fu and Khury 2010), to minimize the variance of the net loss (Kaluszka 2001, Lampaert and Walhin 2005), or to minimize the tail risk of the net loss (Gajek and Zagrodny 2004, Cai and Tan 2007), under the constraint of a given premium principle that links the ceded premium to the ceded loss. This line of reserach is valuable. However, it does not pay enough attention to the interest of the reinsurer. Although the premium principles include risk margins that reflect the volatility of the ceded loss, they generally ignore the fact that the reinsurer needs to put up more capital thus incurs greater capital costs. Our approach treats the ceding insurer and the reinsurer on an equal footing. It addresses capital costs directly. A reinsurance arrangement that minimizes the total capital is the best deal for the combined welfare of primary insurers, reinsurers and policyholders. It is an attracting equilibrium under market forces: competition among insurers and reinsurers tends to produce reinsurance contracts with less total capital.

The main part of the paper is organized as follows. Section 2 shows that minimization of the total primary insurance premium leads to minimization of the total capital. In Section 3, a lower bound of the total required capital is introduced, and an inverse relationship between the total capital and the correlation between losses is observed. Examples are given in Section 4 to illustrate interesting properties of the optimal reinsurance and to further examine the relationship between the total capital and the correlation.

2 Why Minimize the Total Required Capital

2.1 Lower total capital means lower premium for policyholders

Policyholders purchase insurance to protect themselves against potential losses. At the same time, they also provide funds to cover all operating costs of the insurance company, including underwriting and claim expenses, income taxes, agency costs, and reinsuance costs. The reinsurance costs, in turn, cover the reinsurer's expenses, taxes and agency costs, and its reinsurance costs, if there are retrocessions. Ultimately, it is the primary insurance policyholders that bear the operating costs of primary insurers and reinsurers. For the insurance/reinsurance market as a whole, reinsurance treaties rearrange these costs among all insurers and reinsurers. Some reinsurance arrangements result in lower total costs than others. A reinsurance arrangement is optimal if the total cost is minimized, in which case the primary policyholders pay the lowest aggregate premium.

This paper focuses on minimizing the total capital cost, consisting of income taxes and agency costs. To cleanly study the capital cost, we assume that the aggregate underwriting and claim expenses remain constant under all reinsurance arrangements. Therefore these expenses can be excluded from consideration. The gross insurance premium of a policy can be decomposed into the following components

$$p = \mathrm{PV(Loss)} + \mathrm{PV(Tax)} + \mathrm{PV(Agency\ Cost)} + \mathrm{Reinsurance\ Premium} \qquad (2.1)$$

The p in (2.1) represents the fair premium, which is the exact amount to fund all insurer's costs related to the policy. Equation (2.1) is a form of the net present value principle. Some slightly different formulas for the fair premium have appeared in the literature (Myers and Cohn 1987, Taylor 1994, Vaughn 1998). Each term on the right-hand side of (2.1) provides the exact amount to cover that specific type of cost. The present values are risk-adjusted. The loss in the first term is the net loss. It is assumed here that the present value of insured loss satisfies the following two basic requirements of the fair value accounting: (1) The value PV(Loss) is independent of the carrier of the insurance policy.[2] (2) The function $\mathrm{PV}(\cdot)$ is additive. The two conditions together eliminate the possibility of arbitrage. In particular, they imply that PV(Gross Loss) = PV(Net Loss) + PV(Ceded Loss).

I now examine the relationship between the gross fair premium and the total amount of capital held by insurers and reinsurers. Consider a one-year model containing only one loss to be shared between a primary insurer and a reinsurer. Let p be the gross premium charged by the primary insurer at the beginning of the year, and L the random gross loss paid at the end of the year. The primary insurer collects the premium p then cedes an amount p_c to the reinsurer, retaining $p_n = p - p_c$. Denote by L_c the ceded loss, and L_n the net loss. Then $L_n + L_c = L$.

[2] The risk-adjusted PV can be viewed as the risk-free discounted expected cash flow plus a risk margin, where the risk margin reflects the market, or systematic risk of the cash flow. It is sometimes argued that the fair value of losses should be affected by its carrier's default risk. In this paper, we only consider insurance firms that hold the required level of capital, whose risk of default is negligible.

The income tax is the sum of two charges, one on the income generated by premium, which equals the underwriting profit plus the investment income on premium, and the other on the investment income generated by capital. Let e_{Pr} and e_{Re} be the capital amounts carried by the primary insurer and the reinsurer, respectively. Then the present value of tax for the primary insurer is of the form $t_{Pr}(p_n - \mathrm{PV}(L_n)) + u_{Pr}e_{Pr}$, and that for the reinsurer is $t_{Re}(p_c - \mathrm{PV}(L_c)) + u_{Re}e_{Re}$. The t's are the average tax rates for the corresponding incomes, and the u's are the average tax rates on the corresponding capital investments times a constant $r_f/(1+r_f)$, where r_f is the risk-free rate. (A derivation of this constant can be found in Cummins 1990). The agency cost in equation (2.1) consists of various costs related to the insurer's holding capital, including monitoring and bonding expenditures and other losses in profits due to misalignment of managers' decisions and shareholders' welfare (Perold 2005, Chandra and Sherris 2007, Zhang 2008). Agency costs generally increase with the amount of capital.[3] Assume there are constants s_{Pr} and s_{Re} such that the present value of agency cost is $s_{Pr}e_{Pr}$ for the primary company, and $s_{Re}e_{Re}$ for the reinsurer.

Following (2.1), for the primary insurer we have

$$p = \mathrm{PV}(L_n) + t_{Pr}(p_n - \mathrm{PV}(L_n)) + u_{Pr}e_{Pr} + s_{Pr}e_{Pr} + p_c \tag{2.2}$$

and for the reinsurer (if there is no retrocession)

$$p_c = \mathrm{PV}(L_c) + t_{Re}(p_c - \mathrm{PV}(L_c)) + u_{Re}e_{Re} + s_{Re}e_{Re} \tag{2.3}$$

A formula for the fair gross premium p can be obtained by substituting (2.3) into (2.2). It is composed of the following four terms.

1. The present value of loss, $\mathrm{PV}(L_n) + \mathrm{PV}(L_c) = \mathrm{PV}(L)$, which does not vary with reinsurance.

2. The tax on the incomes generated by premium, $t_{Pr}(p_n - \mathrm{PV}(L_n)) + t_{Re}(p_c - \mathrm{PV}(L_c))$. On the condition that the tax rates are equal, $t_{Pr} = t_{Re} = t$, this term is $t(p - \mathrm{PV}(L))$, which decreases as p decreases.

3. The tax on the incomes generated by capital, $u_{Pr}e_{Pr} + u_{Re}e_{Re}$. If the applicable tax rates are the same, then $u_{Pr} = u_{Re} = u$, and the term equals $u(e_{Pr} + e_{Re})$, which decreases if a reinsurance contract lowers the sum of capitals $e_{Pr} + e_{Re}$.

[3] An important type of capital cost is the cost of financial distress, which increases as capital becomes more insufficient. But all firms considered in this paper satisfy a given capital requirement. So the cost of financial distress is ignored.

4. The agency cost $s_{Pr}e_{Pr}+s_{Re}e_{Re}$. If the cost factors are equal, $s_{Pr} = s_{Re} = s$, then the term equals $s(e_{Pr}+e_{Re})$, again a direct function of the total capital $e_{Pr} + e_{Re}$.

To sum up, as the reinsurance varies, the loss component $\mathrm{PV}(L)$ remains constant, while the fair premium p varies because taxes and agency costs vary. (These costs are all called capital costs.) p is lower if the present values of taxes and agency costs are lower. Under the above assumptions, this is equivalent to a less amount of total capital $e_{Pr} + e_{Re}$. The optimal reinsurance is then defined as the one that minimizes $e_{Pr} + e_{Re}$. An optimal reinsurance creates the least gross premium, so is best to the policyholder.

This definition can be generalized to an insurance market with many primary insurers and reinsurers, and many primary policyholders. Assume each primary insurer covers a given set of policyholders. There are a great number of ways in which each insurer buys reinsurance and each reinsurer enters retrocession agreements. A set of reinsurance/retrocession arrangements is called optimal, if it minimizes the total capital cost of the insurers and reinsurers. On condition that all companies have identical tax rates and agency cost factors, this criterion is equivalent to minimizing the total amount of capital.[4]

2.2 Price competition leads to optimal reinsurance

Minimization of the total capital cost is a new optimality criterion. Criteria in the existing literature are very different, see Kaluszka (2001), Gajek and Zagrodny (2004), Lampaert and Walhin (2005), Cai and Tan (2007), Fu and Khury (2010) for a sample of recent papers. In these papers a reinsurance is considered optimal if it minimizes the risk of the net loss under a given constraint on the reinsurance cost (or the ceded premium). This line of research is valuable for reinsurance purchase decisions, but is incomplete. A major concern of reinsurance has been missing. The reinsurer needs additional capital to accommodate the increased risk from the assumed loss, which increases its capital cost. This cost is transfered to the ceding company through reinsurance premium. To the ceding company, if this extra cost is not offset by the reduction of its own capital cost, the deal is not acceptable. Our method treats the ceding insurer and the reinsurer equally. The optimal treaty is fair to both firms, and is the most beneficial to the primary policyholder. Obviously, an optimal reinsurance so defined cannot be calculated by

[4] If tax rates or agency cost factors are not all equal, or the costs are not all linear to the capital, then the optimal reinsurance is one that minimizes an increasing function of the capitals.

either company, since one company cannot model the other company's aggregate loss distribution. Fortunately, as explained in the following examples, it is not necessary to explicitly calculate the optimal treaty terms. As long as each company correctly prices its own policies, the optimal treaty is automatically attained through price competition.

Let us begin with a simple scenario. Assume a primary insurer has written a line of business and would like to purchase reinsurance to reduce risk. Denote by f_{Pr} the amount of capital cost saved by a reinsurance. The reinsurer incurs extra capital costs associated with the assumed loss. It charges the primary insurer an additional premium, denoted by f_{Re}, to cover these costs.[5] So the primary insurer would pay an amount of premium f_{Re} to save an amount of cost f_{Pr}. The reinsurance only makes sense if $f_{Re} \leq f_{Pr}$, which means the sum of the capital costs of both companies must decrease.

Assume further that there are two competing reinsurers; a treaty placed with reinsurer 1 would cost the primary insurer a premium $f_{Re,1}$ to save a capital cost $f_{Pr,1}$, and one placed with reinsurer 2 would cost $f_{Re,2}$ to save a capital cost $f_{Pr,2}$. The immediate (present value) benefits from the treaties are $f_{Pr,1} - f_{Re,1}$ and $f_{Pr,2} - f_{Re,2}$, respectively. The insurer would choose the reinsurer with the greater benefit, which is the one producing the lower total capital cost.

Now look at an example where primary insurers choose reinsurance to compete with each other for business. Suppose that a line of business is on the market, and two insurers are bidding. Suppose each insurer has a set of available reinsurance options. As proved in Section 2.1, the gross fair premium includes a capital cost component that equals the present value of the total capital cost of the insurer and the reinsurer. So, to win the bid an insurer looks for a reinsurance with as low a total capital cost as possible. Eventually the business will go to the insurer that is able to secure a reinsurance with so low a total capital cost that the other cannot match. Obviously, an insurer's ability of getting a more competitive reinsurance deal depends on its existing business and capital structure.

The above analysis shows that market competition always favors a reinsurance structure that produces less total capital cost. Consequently, a reinsurance structure with the least total capital cost is an attracting equilibrium.

[5] Rigorously, f_{Pr} and f_{Re} represent risk-adjusted present values of the corresponding capital cost cash flows.

3 A Lower Bound of Total Required Capital

3.1 Capital requirements by coherent risk measures

Suppose a uniform capital requirement is imposed on all insurers by regulation. We will only deal with the loss risk, that is, the risk that L will become very large. The required capital can be defined by a risk measure on the loss distribution. A class of risk measures that are often considered desirable are the coherent risk measures. According to Artzner et al. (1999), a risk measure ρ is called coherent, if it satisfies the following conditions

- Monotonicity: for any two losses L_1 and L_2, if $L_1 \leq L_2$ then $\rho(L_1) \leq \rho(L_2)$.
- Positive homogeneity: for any loss L and a constant $a > 0$, $\rho(aL) = a\rho(L)$.
- Translation invariance: for any loss L and a constant b, $\rho(L + b) = \rho(L) + b$.
- Subadditivity: for any two losses L_1 and L_2, $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$.

All these properties have simple intuitive meanings. Most important to our study is subadditivity. Subadditivity implies diversification: when two risks are pooled together, the required capital of the pool is less than the sum of the required capitals of each risk.

A typical property/casualty loss is a continuous random variable, that is, its cumulative distribution function $F_L(x)$ is continuous. The p-quantile of L is defined by

$$Q_p(L) = \min\{x | F_L(x) \geq p\}, \qquad p \in (0, 1) \tag{3.1}$$

and the Tail Value-at-Risk (TVaR) at level p is

$$\text{TVaR}_p(L) = E[L | L \geq Q_p(L)], \qquad p \in (0, 1) \tag{3.2}$$

The TVaR is the most well-known coherent risk measure for continuous risks. (The quantile, also called the value-at-risk, does not always respect subadditivity.) The TVaR will be used in our illustrative examples.

Suppose a coherent risk measure ρ is selected by the regulator. Then $\rho(L)$ is the amount of asset that a company is required to hold. In a one-year model, the premium provides part of the asset at the beginning of the year, the required capital thus equals the required asset minus the premium. Following Section 2, we want to examine reinsurance structures that minimize the sum of the required capitals of the insurer and the reinsurer. This is equivalent to the problem of minimizing the sum of their required assets[6], i.e., minimizing the sum of their risk measures. Note that the required asset should be calculated from the loss distribution at the end of the year and discounted back to the beginning of the year. I will ignore the discounting for simplicity.

3.2 Lower bound and comonotonicity

Reconsider the simplified model with a single loss L, one primary insurer and one reinsurer. The primary insurer issues a policy to cover the entire loss L, and cedes part of it to the reinsurer. Thus L is split between the two insurers, $L = L_{Pr} + L_{Re}$. For a given coherent risk measure ρ, by the rule of subadditivity, $\rho(L) \leq \rho(L_{Pr}) + \rho(L_{Re})$. This inequility provides an absolute lower bound for the sum of capitals: however L is split between the two insurers, the sum of their required assets is no less than $\rho(L)$. To minimize the total required capital is to get the sum $\rho(L) \leq \rho(L_{Pr}) + \rho(L_{Re})$ as close to $\rho(L)$ as possible.

The lower bound can be attained by many reinsurance arrangements. One trivial case is that $L_{Pr} = L$ and $L_{Re} = 0$, or $L_{Pr} = 0$ and $L_{Re} = L$, that is, only one insurer holds all of L. This fact is no surprise, for if there is only one insurer and all losses are insured with it, the effect of diversification is maximized, and the least amount of capital is required. An extention of this fact is that an insurance market with few insurers requires less total amount of capital than a market with many insurers. But few insurers means less competition, and insurers have less incentive to price policies fairly.

The lower bound is also reached by the quota share reinsurance. If a is the quota share ceding fraction ($0 < a < 1$), then $L_{Pr} = (1-a)L$ and $L_{Re} = aL$. The equality $\rho(L) = \rho(L_{Pr}) + \rho(L_{Re})$ follows from the rule of positive homogeneity of ρ. More generally, if two losses L_1 and L_2 are perfectly linearly correlated, that is, their linear (Pearson) correlation coefficient equals 1, then $\rho(L_1 + L_2) = \rho(L_1) + \rho(L_2)$. Therefore, if a reinsurance treaty splits L into two linearly correlated parts, then the sum of their required capitals is minimized. The condition of perfect linear correlation can rarely be fulfilled. Fortunately, it can be much relaxed in the following two steps. First, although some kind of perfect correlation has to exist between two losses L_1 and L_2 for their risk measures to add up, the

[6] This can be explained using equations (2.2) and (2.3). The asset for the insurer is $p_n + e_{Pr}$, and that for the reinsurer is $p_c + e_{Re}$. It is proved in Section 2.1 that the total (gross) fair premium $p_n + p_c$ decreases as the total capital $e_{Pr} + e_{Re}$ decreases. So, if a reinsurance treaty minimizes the total required asset, it must simultaneously minimizes the total required capital and the total fair premium.

correlation does not have to be linear—any monotonic and increasing relationship suffices. Second, a perfect correlation only needs to exist at the tail, for large values of L_1 and L_2. Mathematically, both these issues have been well treated in the literature, as explained below.

A set in the n-dimentional space R^n is called comonotonic, if, for any two points in the set, $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, $x_i < y_i$ for some i implies $x_j \leq y_j$ for all j. A vector of n random variables $X_1, \ldots, X_n$ are called comonotonic, if its support in R^n (that is, the set of all possible values of the random vector) is a comonotonic set. A good overview of comonotonicity and its application in risk theory is Dhaene et al. (2006). The support of a random vector can be visualized by drawing a scatter plot (in R^n). A scatter plot of a random vector is merely a small, randomly selected subset of its support. For a comonotonic random vector, the support is essentially a one-dimentional curve (or a subset of a curve) in R^n. When a point moves along such a set, all its coordinates move simultaneously up or down (some coordinates may remain constant). If n random variables are perfectly linearly correlated, the support lies in a straight line. Thus the perfect linear correlation is a special case of comonotonicity. Comonotonicity is a much more general relationship. For example, if X is any positive random variable, then X and X^2 are comonotonic but not linearly correlated. The support of (X, X^2) is contained in the parabola $y = x^2$. The Spearman rank correlation coefficient is a more meaningful measure than the linear correlation coefficient to characterize such a nonlinear relationship. The rank correlation coefficient of two comonotonic random variables equals 1 (see Wang 1998), while their linear correlation coefficient is typically less than 1.

The TVaR is a coherent risk measure and is also additive for comonotonic risks: if two losses L_1 and L_2 are comonotonic, then $\text{TVaR}_p(L_1 + L_2) = \text{TVaR}_p(L_1) + \text{TVaR}_p(L_2)$ for any p (Dhaene et al. 2006).[7] In our one-insurer-one-reinsurer model, assume the required asset is determined by a risk measure ρ that is coherent and additive for comonotonic risks. If L is split in such a way that L_{Pr} and L_{Re} are comonotonic, then $\rho(L_{Pr}) + \rho(L_{Re})$ reaches its lower bound $\rho(L)$. We have seen that the quota share reinsurance splits the loss this way. Another example is the stop-loss reinsurance, which is defined by

$$L_{Pr} = \min(L, k), \qquad L_{Re} = \max(L - k, 0) \tag{3.3}$$

where $k > 0$ is the attachment point. It is easy to check that the three variables

[7] There are other risk measures that are coherent and additive for comonotonic risks, e.g., the concave distortion risk measures. The VaR is additive for comonotonic risks but is not coherent, see Dhaene et al. (2006).

L, L_{Pr} and L_{Re} are comonotonic, and $\rho(L) = \rho(L_{Pr}) + \rho(L_{Re})$.

Risk measures like $Q_p(L)$ and $\mathrm{TVaR}_p(L)$ are determined by large values of L. When considering how to split L into L_{Pr} and L_{Re} to minimize the total capital, one should focus on large losses. The condition of comonotonicity requires the entire support of the random vector to be a comonotonic set. This condition is too strong. Cheung (2009) introduces the concept of upper comonotonicity, only requiring the upper tail of the support to be comonotonic. If L_{Pr} and L_{Re} are upper comonotonic, then $\rho(L) = \rho(L_{Pr}) + \rho(L_{Re})$, where ρ is either Q_p or TVaR_p and p is sufficiently close to 1. In general, the amount of total capital corresponding to a reinsurance structure is determined by large losses only.

3.3 Optimal reinsurance in a general setting

I now apply the concepts developed so far to formulate a general problem about optimal reinsurance. In the real world, a primary insurer does not have the option or the intension to buy reinsurance on its entire book of business. It only seeks coverages for lines or accounts that have the potential of generating very undesirable results. On the other hand, a reinsurer assumes losses from many insurers and reinsurers. For the primary insurer, the correlation between the retained loss and the ceded loss determines how much capital can be shed. For the reinsurer, the correlation between the newly assumed loss and the existing loss determines how much additional capital is needed. A treaty is beneficial from the capital cost point of view if the deduction of capital from the primary insurer exceeds the addition of capital to the reinsurer.[8]

Assume the primary insurer initially carries losses $X + Z$, where X will be entirely retained and Z may be partially ceded. The reinsurer holds a loss Y before assuming a part of Z. A reinsurance treaty splits Z into a net and a ceded part $Z = Z_n + Z_c$. Without reinsurance, the total required asset of the insurer and the reinsurer is $\rho(X+Z)+\rho(Y)$. After a reinsurance, the total required asset is $\rho(X + Z_n) + \rho(Y + Z_c)$. A treaty is optimal if the latter sum is minimized.

If ρ is a coherent risk measure, an absolute lower bound for $\rho(X + Z_n) + \rho(Y + Z_c)$ is $\rho(X + Y + Z)$. In general, the distributions of the losses and the correlations between them are complex. Even the optimal ceding arrangement

[8] Note that the"correlation" here is used in a broad sense. A precise measure of the correlation is yet to be found. It will be clear from later discussions that the familiar linear correlation coefficient and rank correlation coefficient are not proper measures in the study of capital. One reason is that these correlation coefficients encompass all ranges of losses, but capital is only associated with large losses.

may not take down the value of the sum $\rho(X + Z_n) + \rho(Y + Z_c)$ to anywhere near this lower bound. Moreover, in the reinsurance market only a few types of treaties are commonly placed, including the quota share, excess-of-loss, catastrophe and stop-loss treaties. Minimizing the sum $\rho(X + Z_n) + \rho(Y + Z_c)$ for a given set of available treaties is mathematically a conditional optimization problem.

From the preceeding section, we see that if a ceding arrangement makes $X+Z_n$ and $Y+Z_c$ comonotonic (upper comonotonicity suffices), then the sum of required capitals attains its minimum value $\rho(X+Y+Z)$. In other words, the minimum sum of capitals corresponds to the maximum correlation between the losses (their rank correlation equals 1). This suggests that the value of $\rho(X + Z_n) + \rho(Y + Z_c)$ may be inversely related to the correlation between $X + Z_n$ and $Y + Z_c$. A reinsurance that makes the total capital small must make the correlation large. This intuition is important in understanding the optimal reinsurance. In the appendix at the end of the paper I will provide a graphic reasoning to further support this linkage between the total capital and the correlation.

4 Optimal Reinsurance by Examples

In the following examples I will first describe the loss model, solve for the optimal reinsurance, and then discuss the result. In particular, I will examine whether minimization of the total capital is closely related to maximization of the correlation between the losses.

4.1 A multivariate normal example

Let X, Y and Z be three jointly normally distributed variables. X and Z are losses written by the primary insurer, X will be retained and Z may be partially ceded; Y is the existing loss of the reinsurer. Suppose only quota share treaties may be placed on Z. Although this is not a realistic situation (actual losses do not take negative values as the normal distribution does), discussion of this tractable example can provide us valuable insights.

Let X, Y and Z have the following parameters: means μ_x, μ_y and μ_z, standard deviations σ_x, σ_y and σ_z, and pairwise correlation coefficients γ_{xz}, γ_{yz} and γ_{xy}. If a quota share treaty is placed and a is the ceding fraction, then the primary company's net loss is $L_{Pr} = X + (1-a)Z$, and the reinsurer's total loss is $L_{Re} = Y + aZ$. These two losses are also normal random variables. Their means and standard deviations are as follows.

$$\begin{aligned}
\mu_{Pr} &= E(L_{Pr}) = \mu_x + (1-a)\mu_z \\
\sigma^2_{Pr} &= \mathrm{Var}(L_{Pr}) = \sigma^2_x + (1-a)^2\sigma^2_z + 2(1-a)\gamma_{xz}\sigma_x\sigma_z \\
\mu_{Re} &= E(L_{Re}) = \mu_y + a\mu_z \\
\sigma^2_{Re} &= \mathrm{Var}(L_{Re}) = \sigma^2_y + a^2\sigma^2_z + 2a\gamma_{yz}\sigma_y\sigma_z
\end{aligned}$$

For a given confidence level p, the risk measures Q_p and TVaR_p of a normal random variable can be easily obtained. In fact, they can be written as $Q_p = \mu + h_p\sigma$ and $\mathrm{TVaR}_p = \mu + k_p\sigma$, where h_p and k_p are constants independent of μ and σ. For example, $Q_{0.99} = \mu + 2.33\sigma$ and $\mathrm{TVaR}_{0.99} = \mu + 2.67\sigma$. Therefore, if the risk measure ρ is of the quantile or the TVaR type, minimizing the sum $\rho(L_{Pr})+\rho(L_{Re})$ is equivalent to minimizing the sum $\sigma_{Pr} + \sigma_{Re}$. The latter problem will be solved below.

The variances of the insurer and the reinsurer can be written in a simpler form

$$\begin{aligned}
\sigma^2_{Pr} &= \sigma^2_z((a - A_{Pr})^2 + B^2_{Pr}) \\
\sigma^2_{Re} &= \sigma^2_z((a + A_{Re})^2 + B^2_{Re})
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
A_{Pr} &= 1 + \gamma_{xz}\sigma_x/\sigma_z, \qquad B^2_{Pr} = (1-\gamma^2_{xz})\sigma^2_x/\sigma^2_z \\
A_{Re} &= \gamma_{yz}\sigma_y/\sigma_z, \qquad B^2_{Re} = (1-\gamma^2_{yz})\sigma^2_y/\sigma^2_z
\end{aligned} \tag{4.2}$$

The sum of standard deviations is thus

$$\sigma_{Pr} + \sigma_{Re} = \sigma_z\left(((a - A_{Pr})^2 + B^2_{Pr})^{1/2} + ((a + A_{Re})^2 + B^2_{Re})^{1/2}\right).$$

To minimize this sum is to minimize the following function $f(a)$

$$f(a) = ((a - A_{Pr})^2 + B^2_{Pr})^{1/2} + ((a + A_{Re})^2 + B^2_{Re})^{1/2}.$$

where the ceding fraction a is between 0 and 1. The derivative of $f(a)$ is

$$f'(a) = \frac{a - A_{Pr}}{((a - A_{Pr})^2 + B^2_{Pr})^{1/2}} + \frac{a + A_{Re}}{((a + A_{Re})^2 + B^2_{Re})^{1/2}}.$$

Setting the right-hand side of the equation equal to zero, moving one of the terms to the other side and squaring the terms, we have

$$\frac{(A_{Pr} - a)^2}{(a - A_{Pr})^2 + B^2_{Pr}} = \frac{(a + A_{Re})^2}{(a + A_{Re})^2 + B^2_{Re}}.$$

Simplifying this gives

$$(A_{Pr} - a)^2 B^2_{Re} = (a + A_{Re})^2 B^2_{Pr}.$$

We make the assumption that $\gamma_{xz} \geq 0$ and $\gamma_{yz} \geq 0$, meaning that the losses X, Y and Z are not negatively correlated, a condition likely to be true in the real world. Mathematically, this implies $A_{Pr} \geq 1$ and $A_{Re} \geq 0$. If we assume $-A_{Re} \leq a \leq A_{Pr}$, then $A_{Pr} - a \geq 0$ and $a + A_{Re} \geq 0$. Taking the square root in the above equation we get the solution

$$a^* = \frac{A_{Pr}B_{Re} - A_{Re}B_{Pr}}{B_{Pr} + B_{Re}} \tag{4.3}$$

This is the unique zero of $f'(a)$ between $-A_{Re}$ and A_{Pr}, and is the unique minimum point of $f(a)$. The function $f(a)$ strictly decreases from $-A_{Re}$ to a^* and strictly increases from a^* to A_{Pr}. Note that the optimal ceding fraction does not depend on how X and Y are correlated. This statement is obviously true for any distributions of X, Y and Z.

Now we examine a few special cases. First, suppose Z is uncorrelated with both X and Y, that is, $\gamma_{xz} = \gamma_{yz} = 0$. From the equations (4.2), $A_{Pr} = 1$, $B_{Pr} = \sigma_x/\sigma_z$, $A_{Re} = 0$ and $B_{Re} = \sigma_y/\sigma_z$. Using (4.3) we obtain the optimal ceding fraction $a^* = \sigma_y/(\sigma_x + \sigma_y)$. So, in this case, to minimize $\sigma_{Pr} + \sigma_{Re}$, Z should be shared between the primary insurer and the reinsurer in proportion to the standard deviations of their "fixed" losses, σ_x and σ_y.

Another interesting case is when Z is highly correlated to X, but almost uncorrelated to Y. Then $\gamma_{xz} \approx 1$ and $\gamma_{yz} \approx 0$. These imply that $A_{Pr} \approx 1 + \sigma_y/\sigma_z$, $B_{Pr} \approx 0$, $A_{Re} \approx 0$ and $B_{Re} \approx \sigma_y/\sigma_z$. By (4.3), $a^* \approx 1 + \sigma_x/\sigma_z$. This a^* is greater than 1. Thus, to minimize $\sigma_{Pr} + \sigma_{Re}$, Z should be 100% ceded. On the other hand, since Z and X are highly correlated, the more Z is ceded to the reinsurer, the greater is the (linear) correlation between $X + (1-a)Z$ and $Y + aZ$. This correlation is maximized at $a = 100\%$. In this example, the reinsurance is optimized at the same ceded ratio where the correlation between the losses is maximized.

A parallel result is that, if Z is highly correlated to Y, but almost uncorrelated to X, then the optimal ceded ratio is 0%. At this ceded ratio the correlation between the losses is again maximized.

Now we plug in some numerical values. Assume $\sigma_x = 300$, $\sigma_y = 500$ and $\sigma_z = 100$; $\gamma_{xz} = 0.4$, $\gamma_{yz} = 0.4$ and $\gamma_{xy} = 0.2$. Using (4.2) we compute $A_{Pr} = 2.20$, $B_{Pr} = 2.75$, $A_{Re} = 2.00$ and $B_{Re} = 4.58$. Substituting these into equation (4.3) we obtain the optimal ceding fraction $a^* = 62.5\%$. However, this a^* does not provide the maximum correlation between $X + (1-a)Z$ and $Y + aZ$. Using simulation we get that the maximum linear correlation coefficient is 0.290, and is reached at the ceded ratio of 30.5%. Therefore, the minimum total capital does not always correspond to the maximum correlation. As mentioned before, this result is not really a surprise because the capital is determined by large losses, while the linear or rank correlation coefficient does not distinguish between large and small losses (or even negative losses, in this example).

4.2 A numerical example

If the joint distribution of losses X, Y and Z is known, and a set of available reinsurance treaties is given, the optimal treaty can be found by simulation. To have an easy control on correlations between the losses, I will assumed the losses are jointly lognormal. I will look at two common types of treaties, the quota share and the stop-loss.

Let the variables X, Y and Z be jointly lognormal, in the sense that $\ln(X)$, $\ln(Y)$ and $\ln(Z)$ are jointly normal. The mean μ^0 and the standard deviation σ^0 of these normal variables are as follows

	$\ln(X)$	$\ln(Y)$	$\ln(Z)$
μ^0	19.5	20.0	17.0
σ^0	0.16	0.25	1.10

The mean, the standard deviation and quantiles of X, Y and Z can be computed from above with simple formulas. I will denote a parameter for a normal random variable with a superscipt 0, and the same parameter for the corresponding lognormal variable without the superscript. For example, μ_x^0 is the mean of $\ln(X)$ and μ_x the mean of X. These formulas are well known: $\mu_x = \exp(\mu_x^0 + (\sigma_x^0)^2/2)$, and $\sigma_x = \exp(\mu_x^0 + (\sigma_x^0)^2/2)(\exp((\sigma_x^0)^2) - 1)^{1/2}$. The p-quantile of X can be written as $Q_p(X) = \exp(\mu_x^0 + h_p\sigma_x^0)$, where h_p is the p-quantile of the standard normal distribution. More complex measures of the lognormals, like $\mathrm{TVaR}_p(X)$ or the standard deviation of $X + Y + Z$, are more easily estimated using simulation. Some useful statistics for X, Y and Z are shown in the following table (loss amounts are in millions).

	X	Y	Z
μ	298	501	44
σ	48	127	68
CV	0.16	0.25	1.53
$Q_{0.99}$	427	868	312
$\mathrm{TVaR}_{0.99}$	451	941	477

I will choose $\rho = \text{TVaR}_{0.99}$ as the risk measure. In addition to the known μ and σ, if the linear correlation coefficients γ_{xz}, γ_{yz} and γ_{xy} are also given, then the distribution of the triplet (X, Y, Z) is completely determined. Following our naming convention, γ^0_{xz} is the linear correlation coefficient between $\ln(X)$ and $\ln(Z)$. γ^0_{xz} determines γ_{xz}, and vise versa. A greater γ^0_{xz} corresponds to a greater γ_{xz}. The strongest correlation between X and Z is attained when $\ln(X)$ is a linear function of $\ln(Z)$ with a positive slope. In this case $\gamma^0_{xz} = 1$, but γ_{xz} is generally less than 1.[9]

A straightforward sampling method is used to find the optimal ceding term. For μ and σ in the above table and known γ_{xz}, γ_{yz} and γ_{xy}, a large random sample of (X, Y, Z) is drawn (using Excel with the @RISK add-in or with a macro performing the Cholesky decomposition). Applying a given reinsurance treaty on the sample data we get samples of losses of the primary insurer and the reinsurer, from which the TVaR of the losses can be estimated. Table 1 displays results for quota share treaties. Five scenarios of different γ^0_{xz}, γ^0_{yz} and γ^0_{xy} are analyzed. For each scenario, a sample of 20,000 points of the triplet (X, Y, Z) is drawn, 101 quota share fractions, a, ranging from 0% to 100% with 1% increments, are applied, the measures $\rho(X + (1-a)Z)$ and $\rho(Y + aZ)$ are estimated, and the least sum of them is found by comparison, which gives the optimal quota share term. (Loss amounts in Table 1 are in millions.)

Table 1: Optimal Quota Share Fractions

	(1)	(2)	(3)	(4)	(5)
γ^0_{xz}	0.9	0.9	0	0.1	0
γ^0_{yz}	0	0.1	0.99	0.9	0
γ^0_{xy}	0	0	0	0	0
$\rho(X + Y + Z)$	1,540	1,570	1,764	1,721	1,422
a^* (optimal ceding)	100%	100%	0%	36%	75%
$\rho(X + (1 - a^*)Z) + \rho(Y + a^*z)$	1,596	1,624	1,771	1,756	1,529

In the table $\rho(X + Y + Z)$ is the absolute lower bound of the total required asset, for any type of reinsurance. In scenario (3) the optimal total asset $\rho(X + (1 - a^*)Z) + \rho(Y + a^*Z)$ is close to $\rho(X + Y + Z)$. But in general the gap

[9] The exact formula is $\gamma_{xz} = [\exp(\sigma^0_x \sigma^0_z \gamma^0_{xz}) - 1]/[(\exp((\sigma^0_x)^2) - 1)(\exp((\sigma^0_z)^2) - 1)]^{1/2}$. If $\gamma^0_{xz} = 1$, γ_{xz} is generally less than 1, but the Spearman rank correlation coefficient between X and Z equals 1.

between the two is sizable. In scenarios (1) and (2), Z is strongly correlated to X but weakly correlated Y. Ceding out the entire Z ($a = 100\%$) would maximize the correlation between $X + (1 - a)Z$ and $Y + aZ$.[10] This supports the claim that the optimal treaty is the one that creates the strongest correlation between the insurer's and the reinsurer's losses. Similar relationship holds in scenario (3), where Z is strongly correlated to Y but weakly correlated to X. The optimal term is to cede nothing, which again corresponds to the strongest correlation between the two losses. However, in scenario (5), the optimal ceding ratio is 75%, while, as can be shown, the maximum correlation is reached at $a = 55\%$. The two ratios are different.

I now consider the same five correlation scenarios and perform a similar analysis for stop-loss treaties. In each scenario, let the primary insurer's retention, k, vary from 20 million to 250 million, with 5 million increments. The ceded loss is $Z_c = \max(Z - k, 0)$, and the retained $Z_n = Z - Z_c = \min(Z, k)$. Comparing the total asset $\rho(X + Z_n) + \rho(Y + Z_c)$ for all these k we get the optimal retention k^*. The results are summarized in Table 2 (loss amounts are in millions).

Table 2: Optimal Stop-Loss Retentions

	(1)	(2)	(3)	(4)	(5)
γ^0_{xz}	0.9	0.9	0	0.1	0
γ^0_{yz}	0	0.1	0.99	0.9	0
γ^0_{xy}	0	0	0	0	0
$\rho(X + Y + Z)$	1,540	1,570	1,764	1,721	1,422
k^* (optimal retention)	20	20	250	250	85
$\rho(X + Z^*_n) + \rho(Y + Z^*_c)$	1,598	1,625	1,804	1,770	1,557

$Z^*_n = \min(Z, k^*)$, $Z^*_c = \max(Z - k^*, 0)$

In the first two scenarios, Z is highly correlated to X; in the next two scenarios, it is highly correlated to Y. Thus, intuitively, in the first two scenarios, the correlation (at the right tail) between $X + Z_n$ and $Y + Z_c$ increases as more of Z is ceded. In fact, the sample linear correlation is indeed the highest at $k = 20$. This again supports the claim that the optimal treaty maximizes the correlation.

[10] It can be proved mathematically that, if γ_{xz} is very close to 1, then the greater is the ceded ratio a, the greater is the linear correlation between $X + (1 - a)Z$ and $Y + aZ$. The intuition behind this result is that, if Z behaves very similarly to X, then adding Z to any variable Y makes it behave more similarly to X (and to Z), that is, $Y + Z$ is more correlated to X than Y is.

This statement holds too in the next two scenarios, where the optimal treaty is to cede the least of Z. However, in scenario (5), the maximum linear correlation is attained at the retention $k = 115$, which is different from the optimal retention $k^* = 85$.

Finally, let us look at scenario (5) and compare the two types of treaties. The optimal total required asset for the stop-loss treaties is 1,557, and for the quota share treaties it is 1,529. So the quota share is more effective in cutting the total capital.[11] This appears contradicting the general belief that a stop-loss treaty reduces volatility more effectively than a quota share treaty. The fact is, however, although the stop-loss treaty cuts more capital from the primary insurer, it adds even more to the reinsurer, which results in an increase in the total required capital.

5 Conclusions

I have proposed to call a reinsurance arrangement optimal if it minimizes the total capital of the primary insurer and the reinsurer. This optimal reinsurance produces the cheapest price for primary insurance policies, so is an attracting equilibrium under market competition. An interesting relationship is observed between the total capital and the tail correlation between the losses of the insurer and the reinsurer. A multivariate normal model and a numerical example are analyzed to get some insight into the nature of an optimal treaty.

This paper fills a gap in the existing literature on optimal reinsurance, in which the capital cost of the reinsurer has not been adequately addressed. Our approach establishes a close link between reinsurance and pricing of insurance and reinsurance policies. In a competitive market, reinsurance not only provides the ceding insurer a tool of risk transfer, but also satisfies the reinsurer with a fair amount of profit, and benefits primary policyholders by reducing the cost of insurance.

Tail correlation between losses has been widely discussed in relation to risk measurement and management. In this paper it is linked to the size of the total capital. This should continue to be an important area of research.

[11] The quota share structure is better in other four scenarios as well. But those results are of no surprise. As the stop-loss retention is limited to between 20 and 250, ceding the whole of Z and ceding none of Z are excluded, yet the optimal quota share terms in these scenarios fall into these extremes.

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Appendices

A More on the Linkage Between the Total Capital and Correlation

We have shown that the TVaR is a subadditive risk measure: if $\rho = \text{TVaR}_p$ then $\rho(X) + \rho(Y) \geq \rho(X+Y)$, and the equality holds if X and Y (representing the losses of a primary insurer and a reinsurer) are comonotonic. Following this fact we propose that a linkage exists between the total asset $\rho(X) + \rho(Y)$ and the correlation between X and Y, that is, the greater the tail correlation, the closer is $\rho(X) + \rho(Y)$ to $\rho(X+Y)$. In this appendix, I will use the scatter plot to further explain why there should be such a link.

Figures 1 through 3 provide scatter plots of a pair of losses X and Y correponding to three different correlation scenarios. (The correlations are actually only different at the right tail.) Each loss is in the range $[0, 100)$. In Figure 1, X and Y are comonotonic at the tail. In Figure 2 they are not comonotonic, but are still highly correlated at the tail: as X moves up from about 80, Y generally moves up as well, although it sometimes moves in the opposite direction (down) slightly. In Figure 3, X and Y have little correlation at the tail.

Let the risk measure be $\rho = \text{TVaR}_{0.9}$. There are one hundred points in each figure. The point labeled A has the eleventh largest x-coordinate, and the one labeled B has the eleventh largest y-coordinate. The quantile $Q_{0.99}(X)$ is the x coordinate of A, and $Q_{0.99}(Y)$ the y-coordinate of B. $\rho(X)$ is the average of the x-coordinates of the points to the right of A, and $\rho(Y)$ the average of the

y-coordinates of the points higher than B. $\rho(X+Y)$ is the average of the largest ten $x+y$ of all points.

In Figure 1, A and B are actually the same point $(78, 76)$ (coordinates are rounded), and the points to the right of A are the same as those higher than A, which are also the ten points with the largest $x+y$. Thus, $Q_{0.99}(X) + Q_{0.99}(Y) = Q_{0.99}(X+Y) = 78+76 = 154$, and $\rho(X)+\rho(Y) = \rho(X+Y)$ $(= 178)$. This explains that if X and Y are perfectly correlated at the tail, then $\rho(X)+\rho(Y) = \rho(X+Y)$.

In Figure 2, the upper-right tail is a rather "thin" set. Thus the two points A and B are close to each other. Further, the following three sets of points are similar (contain mostly the same points): those to the right of A, those higher than B, and the ten points with the largest $x+y$. This implies that $\rho(X)+\rho(Y)$ is close to $\rho(X+Y)$. (Here $\rho(X) = 95.3$, $\rho(Y) = 88.8$ and $\rho(X+Y) = 183.9$.) This example shows that if X and Y are highly correlated at the tail, then $\rho(X)+\rho(Y)$ is (greater than but) close to $\rho(X+Y)$.

The upper-right tail in Figure 3 is not a thin set, and the two points A and B are generally far apart. Also, it is likely that the three sets—the one to the right of A, the one higher than B, and the one with the largest $x+y$ —contain very different points. So $\rho(X)+\rho(Y)$ can be much larger than $\rho(X+Y)$. (Here $\rho(X) = 95.3$, $\rho(Y) = 82.4$ and $\rho(X+Y) = 174.1$.) This is what normally happens when X and Y are not correlated at the tail.

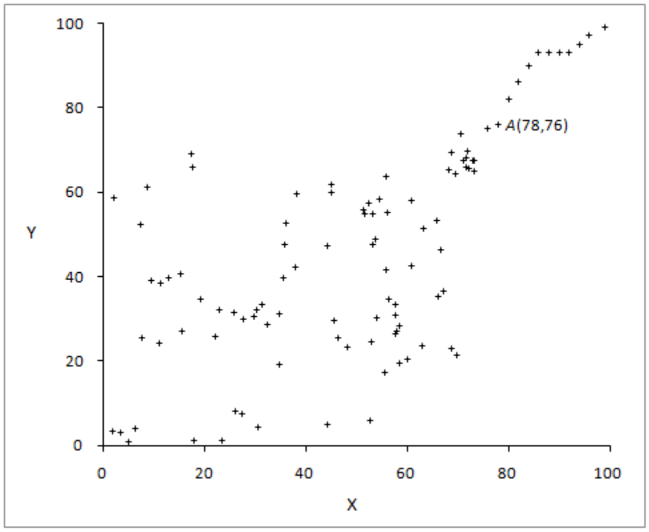

Figure 1: X and Y are comonotonic at the tail. $\rho(X) + \rho(Y) = \rho(X + Y)$

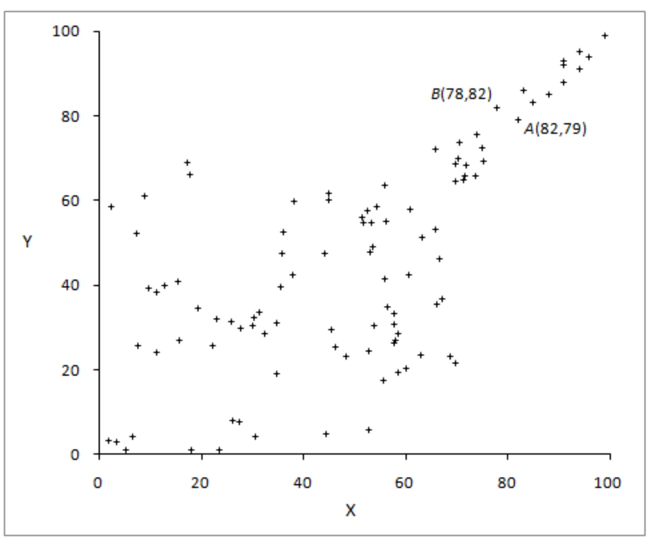

Figure 2: X and Y are highly correlated at the tail. $\rho(X) + \rho(Y)$ is close to (but greater than) $\rho(X + Y)$

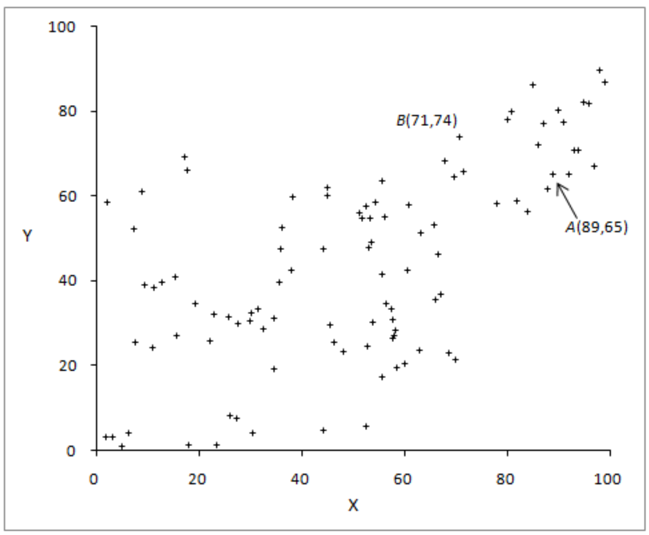

Figure 3: X and Y are not correlated at the tail. $\rho(X) + \rho(Y)$ is generally much greater than $\rho(X + Y)$