PRICING RATE OF RETURN GUARANTEES IN A HEATH-JARROW-MORTON FRAMEWORK

KRISTIAN R. MILTERSEN AND SVEIN-ARNE PERSSON

ABSTRACT. Rate of return guarantees are included in many financial products, for example life insurance contracts or guaranteed investment contracts issued by investment banks. The holder of a such contract is guaranteed a fixed periodically rate of return rather than — or in addition to — a fixed absolute amount at expiration.

We consider rate of return guarantees where the underlying rate of return is either (i) the rate of return on a stock investment or (ii) the short term interest rate. Various types of these rate of return guarantees are priced in a general no-arbitrage Heath-Jarrow-Morton framework. We show that there are fundamental differences in the resulting pricing formulas depending on which of the two types of underlying rate of return ((i) or (ii)) the contract is based on.

Finally, we show how the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) occur as special cases in our more general framework based on the Heath, Jarrow, and Morton (1992) model.

1. INTRODUCTION

Interest rate guarantees are included in many financial products. For example many life insurance contracts guarantee the policyholder a fixed annual percentage return. Another example is guaranteed investment contracts sold by investment banks (see e.g., Walker (1992)).

In principle, a guarantee may be connected to any specified rate of return, referred to as the *rate of return process* or *return process*. Real-life examples include rate of returns of stocks and mutual funds, various indexes or interest rates. In this treatment we consider (i) guarantees on return processes connected to assets traded in financial markets and (ii) guarantees on the short term interest rate. Guarantees on stock returns are obvious examples of the first kind of guarantee and we sometimes refer to underlying financial asset simply as a stock.

The very existence of guaranteed return contracts reflects the volatile nature of rates of return. It is reasonable to expect that the interest rates in the economy influence any rate of return process. A proper valuation model should accordingly include a model of the stochastic behavior of the interest rate.

The cashflows of the guarantees are somewhat related to cashflows of European options. We present pricing result for European call options on return processes as well.

We mention two important observations when the return process is based on the short term interest rate. Then the value of the underlying asset is a function of the integral of the short interest rate, hence it is somewhat related to what is known as *Asian* derivatives in the financial literature. Also in this case, the underlying asset, though random, is of finite variation.

KRISTIAN R. MILTERSEN AND SVEIN-ARNE PERSSON

Our model is based on the Heath, Jarrow, and Morton (1992) model of the term structure. This is a rather general model and we show how the term structure models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985) occur as special cases.

This article extends the results of Persson and Aase (1996) in two ways. First, by applying the more general Heath, Jarrow, and Morton (1992) model instead of the Vasicek (1977) model. Second, by also considering guarantees on stock returns.

The paper is organized as follows: In section 2 the set-up is explained. In section 3 pricing results for European call options and guaranteed return processes are obtained. The results are discussed and related to existing literature in Section 4. Section 5 contains some concluding remarks.

2. THE MODEL

The Heath, Jarrow, and Morton (1992) model is based on the definitional relationship between forward rates and market prices of unit discount bonds

$$P(t,T) = e^{-\int_t^T f(t,s)ds}.$$

The major primitive is the family of continuously compounded forward rates $f(t,s)$, $0 \leq t \leq s \leq T$, given by Itô-processes of the form

$$f(t,s) = f(0,s) + \int_0^t \mu_f(v,s)dv + \int_0^t \sigma_f(v,s)d\hat{W}_v.$$

Here $\hat{W}_t$, $0 \leq t \leq T$ is a, possibly multi-dimensional, standard Brownian motion defined on a given filtered probability space. The drift and volatility processes, $\mu_f(t,s)$ and $\sigma_f(t,s)$, $0 \leq t \leq s \leq T$, respectively, are adapted processes satisfying some technical conditions (see Heath, Jarrow, and Morton (1992)).

The short term interest rate (spot-rate) in the economy is given by $r_t = f(t,t)$.

When considering the return process of an asset traded in a financial market, we assume that the underlying market price process of the asset satisfies the stochastic differential equation

$$S(t) = S(0) + \int_0^t \mu_S(v)S(v)dv + \int_0^t \sigma_S(v)S(v)d\hat{W}_v,$$

imposing technical conditions on the adapted processes $\mu_S(t)$ and $\sigma_S(t)$, $0 \leq t \leq T$, so that a strong solution to the above stochastic differential equation exists.

For our purpose it is convenient to define the associated cumulative return process δ_t as

$$\delta_t = \int_0^t (\mu_S(v) - \frac{1}{2}\sigma_S(v)^2)dv + \int_0^t \sigma_S(v)d\hat{W}_v.$$

Then the familiar relationship from deterministic models between market price and return, $S(t) = S(0)e^{\delta_t}$, also holds in this stochastic environment.

Market prices are calculated by the use of the equivalent martingale measure constructed by the Radon-Nikodym derivative

PRICING RATE OF RETURN GUARANTEES IN A HEATH-JARROW-MORTON FRAMEWORK

$$\frac{dQ}{dP} = \exp\left(\int_0^T \lambda d\hat{W}_v - \frac{1}{2}\int_0^T \lambda^2 dv\right),$$

where λ is a vector of the same dimension as $\hat{W}_t$ and λ^2 is to be interpreted as the vector-product of λ by its transpose.

The following examples based on Amin and Jarrow (1992) illustrate how the equivalent martingale is constructed (see the cited source for technical conditions) in the cases relevant for our study.

First assume that $\hat{W}_t$ has dimension (at most) 2, and specify $\sigma_f(t,v) = (\ \sigma_f^1(t,v)\ \ 0\ )$, and $\sigma_S(t) = (\ \sigma_S^1(t)\ \ \sigma_S^2(t)\ )$.

Standard results from arbitrage pricing theory are now employed to determine the time-dependent elements of the vector λ as

$$\lambda_1(t) = \frac{1}{2}\int_0^t \sigma_f(v,t)dv - \frac{\int_0^t \mu_f(v,t)dv}{\int_0^t \sigma_f(v,t)dv}$$

and

$$\lambda_2(t) = \left(\frac{1}{2}\int_0^t \sigma_f(v,t)dv - \frac{\int_0^t \mu_f(v,t)dv}{\int_0^t \sigma_f(v,t)dv}\right)\left(-\frac{\sigma_S^1(t)}{\sigma_S^2(t)}\right) - \frac{\mu_S(t) - r_t}{\sigma_S^2(t)}.$$

The two first special cases are cases with only one source of randomness and the dimension of $\hat{W}_t$ is 1. Without any loss of generality we formally let $\lambda_1(t) = \lambda(t)$ and $\lambda_2(t) = 0$.

In the first case the interest rate is deterministic, hence $\sigma_f^1(t,v) = 0$, moreover, we let $\sigma_S^1(t) = 0$, and $\sigma_S^2(t) = \sigma_S(t)$. We obtain that

$$\lambda(t) = -\frac{\mu_S(t) - r_t}{\sigma_S(t)},$$

which is well known from the Black and Scholes (1973), Merton (1973) model from financial economics. In the second case there is no stock, hence $\sigma_S^1(t) = \sigma_S^2(t) = 0$. We obtain

$$\lambda(t) = \frac{1}{2}\int_0^t \sigma_f(v,t)dv - \frac{\int_0^t \mu_f(v,t)dv}{\int_0^t \sigma_f(v,t)dv},$$

which is well known from the Heath, Jarrow, and Morton (1992) model.

A third potentially interesting special case may be the case where the financial asset is independent of the forward rates, in particular it will also be independent of the short term interest rate. Formally, this condition is obtained by setting $\sigma_S^1(t) = 0$. For notational convenience we also let $\sigma_S^2(t) = \sigma_S(t)$.

In this case

$$\lambda_1(t) = \frac{1}{2}\int_0^t \sigma_f(v,t)dv - \frac{\int_0^t \mu_f(v,t)dv}{\int_0^t \sigma_f(v,t)dv},$$

and

$$\lambda_2(t) = -\frac{\mu_S(t) - r_t}{\sigma_S(t)},$$

i.e., just a combination of the two one-dimensional polar cases.

Under the equivalent martingale measure the processes for the cumulative return and the short interest rate are

$$\delta_t = \int_0^t (r_v - \frac{1}{2}\sigma_S(v)^2)dv + \int_0^t \sigma_S(v)dW_v$$

and

$$r_t = f(0,t) + \int_0^t \sigma_f(v,t) \int_v^t \sigma_f(v,s)ds\,dv + \int_0^t \sigma_f(v,t)dW_v,$$

respectively. Here W_t is a standard Brownian motion under the equivalent martingale measure.

We define a *stock market account* as

$$\alpha(t) = e^{\delta_t}.$$

The similar account involving the short-term interest rate

$$\beta(t) = e^{\int_0^t r_s\,ds} = e^{\int_0^t f(s,s)\,ds}$$

is defined as *the savings account.*

The claims treated here are European call options on the stock market and savings accounts with payoffs at time t $(\alpha(t) - K)^+$ and $(\beta(t) - K)^+$, respectively, where K represent the constant exercise price and the operator $(Z)^+$) returns the non-negative part of Z. The payoffs of the stock market and savings accounts guarantees at time t are $(\alpha(t) \vee e^{gt})$ and $(\beta(t) \vee e^{gt})$, respectively, where g represents the constant guaranteed rate of return, and where the operator $(A \vee B)$ returns the maximum of A and B. Observe the simple relationship between the European call option and the guarantee, $(\alpha(t) \vee e^{gt}) = K + (\alpha(t) - K)^+$, where $K = e^{gt}$. Of course, the same relation holds for the savings account.

3. CLOSED FORM SOLUTIONS IN THE GAUSSIAN CASE

In this section pricing formulas for European call options and guarantees are derived. We assume that forward rates are Gaussian, i.e., $\sigma_f(t,s)$, $0 \leq t \leq s \leq T$, are deterministic functions and that the process $\sigma_S(t), 0 \leq t \leq T$ is deterministic.

3.1. Options on the Stock Market Account—Deterministic Interest. The first case we consider is a European call option on the stock market account payable at time t, where the short term interest rate r_t is deterministic. By this assumption market prices of bonds are given by the formula $P(t,T) = \exp(-\int_t^T r_s\,ds)$. The payoff of this claim is $(\alpha(t) - K)^+$, where the constant K represents the exercise price. The time zero market price for a claim payable at time t is

PRICING RATE OF RETURN GUARANTEES IN A HEATH-JARROW-MORTON FRAMEWORK

$$C_1 = E^Q \left[e^{-\int_0^t r_s \, ds} \left(e^{\delta_t} - K \right)^+ \right]$$

Proposition 3.1. *The time zero market price of a European call option on the stock market account payable at time t is*

$$C_1 = \Phi \left(\frac{1}{\sigma_S^t} (-ln(K) - ln(P(0,t)) + \frac{1}{2}(\sigma_S^t)^2) \right) - KP(0,t)\Phi \left(\frac{1}{\sigma_S^t} (-ln(K) - ln(P(0,t)) - \frac{1}{2}(\sigma_S^t)^2) \right),$$

where $(\sigma_S^t)^2 = \int_0^t [\sigma_S(v)]^2 dv.$

Proof. The payoff resembles a standard European call option where the initial price of the stock is normalized to 1. See Black and Scholes (1973). □

The following corollary follows immediately from the stated relation between the payoffs of European call options and the guarantees.

Corollary 3.2. *The market price at time zero of the claim* $(\alpha(t) \vee e^{gt})$ *is*

$$\pi_1 = \Phi \left(\frac{1}{\sigma_S^t} (-gt - ln(P(0,t)) + \frac{1}{2}(\sigma_S^t)^2) \right) + e^{gt} P(0,t) \Phi \left(\frac{1}{\sigma_S^t} (gt + ln(P(0,t)) + \frac{1}{2}(\sigma_S^t)^2) \right).$$

3.2. Options on the Savings Account. The next case we consider is the case treated by Persson and Aase (1996) involving the payoff$(\beta(t) - K)^+$. Using standard valuation techniques based on the notion of no arbitrage, the time zero market value of a European call option on the savings account is

$$C_2 = E^Q [e^{-\int_0^t r_s \, ds} (\beta(t) - K)^+] = E^Q [(1 - Ke^{-\int_0^t r_s \, ds})^+].$$

Here we remark that

$$\int_0^t r_s ds = ln(P(0,t)) + \frac{1}{2}(\sigma_f^t)^2 + \int_0^t \int_v^t \sigma_f(v,u) du dW_v,$$

where $(\sigma_f^t)^2 = \int_0^t [\int_v^t \sigma_f(v,u) du]^2 dv.$

Proposition 3.3. *The time zero market price of a European call option with expiration at time t on the savings market account is*

$$C_2 = \Phi(\frac{1}{\sigma_f^t}(-ln(K) - ln(P(0,t)) + \frac{1}{2}(\sigma_f^t)^2) - KP(0,t)\Phi(\frac{1}{\sigma_f^t}(-ln(K) - ln(P(0,t)) - \frac{1}{2}(\sigma_f^t)^2).$$

Proof. The result follows by straight forward calculations. □

Corollary 3.4. *The market price at time zero of the claim* $(\beta(t) \vee e^{gt})$ *is*

$$\pi_2 = \Phi(\frac{1}{\sigma_f^t}(-gt - ln(P(0,t)) + \frac{1}{2}(\sigma_f^t)^2) + e^{gt} P(0,t)\Phi(\frac{1}{\sigma_f^t}(gt + ln(P(0,t)) + \frac{1}{2}(\sigma_f^t)^2).$$

3.3. **Options on the Stock Market Account—Stochastic Interest.** The last case we consider is a European call option on the stock market account payable at time t, where the short term interest rate r_t is stochastic.

Proposition 3.5. *The time zero market price of a European call option with expiration at time t on the stock market account with stochastic interest rate is*

$$C_3 = \Phi(\frac{1}{\sigma^t}(-ln(K) - ln(P(0,t)) + \frac{1}{2}(\sigma^t)^2) - KP(0,t)\Phi(\frac{1}{\sigma^t}(-ln(K) - ln(P(0,t)) - \frac{1}{2}(\sigma^t)^2),$$

where $(\sigma^t)^2 = (\sigma_S^t)^2 + 2\int_0^t \sigma_S(v)\int_v^t \sigma_f(v,u)dudv + (\sigma_f^t)^2$.

Proof. See Merton (1973) and Amin and Jarrow (1992).

□

Corollary 3.6. *The market price at time zero of the claim $(\beta(t) \vee e^{gt})$ is*

$$\pi_3 = \Phi(\frac{1}{\sigma^t}(-gt - ln(P(0,t)) + \frac{1}{2}(\sigma^t)^2) + e^{gt}P(0,t)\Phi(\frac{1}{\sigma^t}(gt + ln(P(0,t)) + \frac{1}{2}(\sigma^t)^2).$$

4. RELATION TO EARLIER MODELS

4.1. **The Vasicek (1977) Model.** Under an equivalent martingale measure the SDE of the spot interest rate is given by

$$r_t = r_0 + \int_0^t \kappa(\hat{\theta}_u - r_u)du + \int_0^t \sigma dW_u,$$

where $\hat{\theta}_t = \theta_t - \frac{\sigma\lambda_t}{\kappa}$ is the risk-adjusted mean reversion level. This SDE can be solved as

$$r_t = \hat{\theta}_t + (r_0 - \hat{\theta})e^{-\kappa t} + \int_0^t \sigma e^{-\kappa(t-u)}dW_u.$$

On the other hand the SDE for $f(t,t)$ is given by

$$f(t,t) = f(0,t) + \int_0^t \mu_f(u,t)du + \int_0^t \sigma_f(u,t)dW_u.$$

Moreover, under the equivalent martingale measure the drift of the forward rate, μ_f, is determined as

$$\mu_f(t,s) = \sigma_f(t,s) \cdot \left(\int_t^s \sigma_f(t,v)dv \right).$$

Comparing r_t and $f(t,t)$ gives that σ_f must be specified as

$$\sigma_f(t,s) = \sigma e^{-\kappa(t-s)}.$$

Hence the drift, μ_f, is given by

$$\mu_f(t,s) = \frac{\sigma^2}{\kappa}e^{-\kappa(s-t)}(1 - e^{-\kappa(s-t)}).$$

Matching drift terms in the HJM model and the Vasicek model under the equivalent martingale measure yields

$$f(0,t) + \frac{\sigma^2}{\kappa}(1 - e^{-\kappa t}) - \frac{\sigma^2}{2\kappa^2}(1 - e^{-2\kappa t}) = \theta_t - \frac{\sigma \lambda_t}{\kappa} + (f(0,0) - \theta_t - \frac{\sigma \lambda_t}{\kappa})e^{-\kappa t}.$$

¿From this equation we can find an expression for the risk premium, λ_t, as

$$\lambda_t = \frac{\kappa}{\sigma}\Big(\theta_t - \frac{f(0,t) - f(0,0)e^{-\kappa t}}{1 - e^{-\kappa t}}\Big) - \frac{\sigma}{2\kappa}(1 - e^{-\kappa t}).$$

4.2. **The Cox, Ingersoll, and Ross (1985) Model.** A similar analysis is performed on the Cox-Ingersoll-Ross model in Heath, Jarrow, and Morton (1992, Section 8). Therefore, we will just present how to specify the volatility function of the forward rate process to get the Cox-Ingersoll-Ross model as a special case of the Heath-Jarrow-Morton model.

Under an equivalent martingale measure the SDE of the spot interest rate is given by

$$r_t = r_0 + \int_0^t \kappa(\hat{\theta}_u - r_u)du + \int_0^t \sigma\sqrt{r_u}dW_u,$$

where $\hat{\theta}_t$ is the risk-adjusted mean reversion level. This SDE has a solution but it cannot be written in an explicit form. Cox, Ingersoll, and Ross (1985) show that the zero-coupon bond prices can be calculated as

$$P(t,T) = A(t,T)e^{-B(t,T)r_t},$$

where $B(t,T)$ is given by

$$B(t,T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + \kappa + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

and $\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}$. λ is related to the the the risk premium. $A(t,T)$ is not important for our purpose.

By Itô's lemma the SDE of the zero-coupon bond prices are

$$P(t,T) = P(0,T) - \int_0^t \Big(B(u,T)P(u,T)\kappa(\hat{\theta}_u - r_u) - \frac{1}{2}B(u,T)^2 P(u,T)\sigma^2 r_u$$

$$- e^{-B(u,T)r_u}\frac{\partial}{\partial u}A(u,T) + r_u P(u,T)\frac{\partial}{\partial u}B(u,T)\Big)du$$

$$- \int_0^t B(u,T)P(u,T)\sigma\sqrt{r_u}dW_u.$$

On the other hand the SDE of the zero-coupon bond prices by the Heath-Jarrow-Morton model is given by

$$P(t,T) = P(0,T) + \int_0^t P(u,T)\Big(f(u,u) - \int_u^T \mu_f(u,s)ds + \frac{1}{2}\Big\|\int_u^T \sigma_f(u,s)ds\Big\|^2\Big)du$$

$$- \int_0^t P(u,T)\Big(\int_u^T \sigma_f(u,s)ds\Big) \cdot dW_u.$$

Hence, by matching diffusion terms in these two SDEs yields

$$B(u,T)\sigma\sqrt{f(u,u)} = \int_u^T \sigma_f(u,s)ds.$$

KRISTIAN R. MILTERSEN AND SVEIN-ARNE PERSSON

Differentiating with respect to T gives the expression of how to specify the diffusion term of the Heath-Jarrow-Morton model to get the Cox-Ingersoll-Ross model

$$\sigma_f(t,s) = \sigma\sqrt{f(t,t)}\frac{\partial}{\partial s}B(t,s)$$
$$= \frac{4\sigma\gamma^2 e^{\gamma(T-t)}}{\left((\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma\right)^2}\sqrt{f(t,t)}.$$

Finally, the drift, μ_f, is given by

$$\mu_f(t,s) = \text{To be calculated.}$$

5. CONCLUDING REMARKS

In the paper we have derived a number of pricing formulas for European options on return processes as well as for guarantees on return processes. We believe that our analysis in particular is relevant for life insurance, where many real-life contracts include similar guarantees to the ones treated here.

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DEPT. OF MANAGEMENT, SCHOOL OF BUSINESS AND ECONOMICS, ODENSE UNIVERSITET, CAMPUSVEJ 55, DK–5230 ODENSE M, DENMARK
E-mail address: krm@busieco.ou.dk

INSTITUTE OF FINANCE AND MANAGEMENT SCIENCE, THE NORWEGIAN SCHOOL OF ECONOMICS AND BUSINESS ADMINISTRATION, N–5035 BERGEN-SANDVIKEN, NORWAY
E-mail address: Svein-Arne.Persson@nhh.no