

# **Pricing Futures and Futures Options with Basis Risk**

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## **Abstract**

In this article, assuming the stochastic behavior of basis as a modified Brownian bridge process, we obtain closed-form solutions of futures and futures options generalizing the Black (1976). The arrangement permits the formulas of futures and futures options to be functions of spot price, volatility of spot return, initial basis, basis volatility, as well as the correlation coefficient between basis and spot return. In the meantime, it also ensures the basis to be zero at maturity of futures contract.

From the numerical test, the futures call option price is positively related with the correlation coefficient between basis and spot return but is a negatively in the initial basis value. Meanwhile, the sign of correlation coefficient determines the relationship between the basis volatility and the futures call price. Finally, we empirically tested our model with S&P 500 futures call options daily data. Comparing with the Black (1976) model, our model outperforms the Black's model due to eliminate systematic moneyness and time-to-maturity biases and has better prediction power. In total sample data, the mean errors in terms of index and percentage are 0.973 and 1.0% for our model, -4.468 and -27.1% for Black's model.

**Keyword :** Futures、 Futures Options、 Basis risk、 Brownian bridge

## **I. Introduction**

In recent years, there has been a steady growth in the number of financial assets, which one might properly call derivative assets, that are available for trading on the organized exchanges. Among these, the most notable are contracts of futures and options written on futures contracts. In 1999, total trading volume of futures and futures options on equity and index was 48 million contracts in Chicago Mercantile Exchange. In 2004, contract volume reached 313 million contracts a year. This represents an average annual growth rate of 41% for the past 5 years. This spectacular growth prompts many researchers to take a close look at pricing between the futures contracts and the underlying index.

Traditionally, the pricing of stock index futures has been based upon the Cornell and French (1983) which is known as cost-of-carry model. Assuming that markets are perfect, Cornell and French (1983) derive the futures prices for a stock or portfolio of stocks with constant dividend payout and interest rate. They also extend their model by introducing the forward rate, seasonal dividends, and a simple tax structure. After the prominent study of Cornell and French (1983), Ramaswamy and Sundaresan (1985) provide closed-form solutions for options on futures contracts with stochastic interest rate model of Cox, Ingersoll, and Ross (1985), they also argue that mispricing of options on S&P 500 futures can be better explained by stochastic interest rate models. Hemler and Longstaff (1991) develop a general equilibrium model of stock index futures prices with stochastic interest rates and market volatility. Their model allows the stock index futures price to depend on the variance of return on the market, instead of just the prices of traded assets.

For commodity futures, Gibson and Schwartz (1990) develop a two-factor model where the first factor is the spot price of the commodity, and the second factor is the instantaneous convenience yield. Schwartz (1997) extends this model by introducing

a third stochastic factor, the stochastic interest rate. Hilliard and Reis (1998) investigate the pricing of commodity futures and futures options under the stochastic convenience yield, stochastic interest rate, and jumps in the spot price. Nevertheless, all of their models leave the market price of convenience yield risk as a parameter in their pricing formulas. Meanwhile, standard no-arbitrage arguments leave no room for explicit modeling of mean reversion via the drift of the spot commodity price.

Therefore, by assuming normality of continuously compound forward interest rates and mean-reversion convenience yields and log-normality of the spot price of the underlying commodity, Miltersen and Schwartz (1998) obtain closed-form solutions for the pricing of options on futures prices as well as forward prices. Yan (2002) proposes a general commodity valuation models for futures and futures options to allow for stochastic volatility and simultaneous jumps in the spot price and spot volatility. They find that the derived closed-form solution of futures price is not a function of either spot volatility or jumps, however numerical examples show that in pricing options on futures.

In sum, the above articles use the tax structure, convenience yield (or dividend), the term structure of interest rates, market volatility, as well as jumps in the spot price and spot volatility to indirectly model the difference between log futures price and log spot price (the basis)\*. It is worth to notice that we may not assure how many state variables should be included in the basis function.

In this article, assuming the stochastic behavior of basis as a modified Brownian bridge process, we obtain closed-form solutions of futures and futures options generalizing the Black (1976). **A Brownian bridge is a stochastic process that is like a Brownian motion except that with probability one it reaches a specified point at a specified time. Under no arbitrage assumption, the spot price and the futures price**

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\* The definition is used by Yan (2002).

will converge at the expiration of the futures contract. It means that the basis value is zero. The modified Brownian bridge process ensures the basis to be zero at maturity of futures contract. The setup allows for the prices of futures and futures options to be functions of spot price, volatility of spot return, initial basis, basis volatility, as well as the correlation coefficient between basis and spot return.

From the numerical analysis, the futures call option price is positively related with the correlation coefficient between basis and spot return but is negatively in the initial basis value. Meanwhile, the sign of correlation coefficient determines the relationship between the basis volatility and the futures call price. Finally, we empirically tested our model with S&P 500 futures call options daily data. Comparing with the Black (1976) model, the empirical test shows clear evidence supporting the occurrence of basis risk. The futures option model with basis risk outperforms the Black's model by producing smaller bias and better goodness of fit. It not only eliminates systematic moneyness and time-to-maturity biases produced by Black model, but also has better prediction power. In total sample data, the mean errors in terms of index and percentage are 0.973 and 1.0% for our model, -4.468 and -27.1% for Black's model.

An outline of this study is as follows: Section II presents the model specification for futures and futures options. Section III shows the numerical analysis for the basis risk call option model. Section IV empirically tests the basis risk model. Section V extends the model to discuss some Greek letters, hedge strategy, and the futures put options with basis risk, while Section VI summarizes the paper.

## **II. The valuation framework**

In order to price futures options with the basis risk, the futures formula should be remodeled. We assume that the futures price is affected by both of underlying asset

and basis risk which is a modified Brownian bridge process. Therefore, deriving the price of basis risk is the first stage and we remodel the futures and futures call options later.

### **The process of underlying asset**

Under spot martingale measure  $Q$ , we assume the underlying security follows a geometric Brownian motion with continuous dividend-yield  $\delta$ , and constant instantaneous drift  $r$  and volatility  $\sigma_S$ . The process is:

$$dS(t) = (r - \delta)S(t)dt + \sigma_S S(t)dW_S^Q(t). \quad (1)$$

$W_S^Q$  stands for an one-dimensional standard Brownian motion defined on a filtered probability space  $(\Omega, F, P^Q)$ .

### **The futures price with basis risk**

#### **1. Defining the basis**

Using the definition proposed by Yan (2002), the basis,  $Z(t, U)$ , is defined as from the log futures price with delivery date  $U$  subtract the log spot price. That is:

$$Z(t, U) = \ln F(t, U) - \ln S(t) \quad (2)$$

Both  $F(t, U)$  and  $S(t)$  are considered in finance to be lognormal. Therefore, the  $Z(t, U)$  is a normal distribution.  $Z(0, U)$  is the basis at time zero with maturity at time  $U$  and an adapted random variable. In the absence of arbitrage opportunity, the futures prices will agree with the spot price at maturity. That is  $Z(U, U)$  should be zero.

#### **2. The basis follows modified Brownian bridge process**

We assume that the basis in the article follows modified Brownian bridge process. A Brownian bridge is a stochastic process that is like a Brownian motion except that with probability one it reaches a specified point at a specified time. This process

ensures that the basis is known with certainty at the time the futures is initiated and at maturity. That is, the basis is known with certainty at the time the futures is initiated and approaches to zero at maturity under no arbitrage assumption.

Let the basis  $Z(t, U)$  be a modified Brownian bridge in the time interval  $[0, U]$  with  $Z(0, U) = \ln F(0, U) - \ln S(0)$ , and  $Z(U, U) = 0$ . The process is provided in Theorem I.

*Theorem I: Under spot martingale measure  $Q$ , the basis process is as follows:*

$$dZ(t, U) = \frac{-Z(t, U)}{U-t} dt + \sigma_Z(t, U) \left[ \rho dW_s^Q(t) + \sqrt{1-\rho^2} dW_z^Q(t) \right] \quad (3)$$

where  $\sigma_Z(t)$  is the basis volatility,  $\rho$  is the correlation coefficient between the underlying security and the basis, and  $dW_z^Q$  is one-dimensional Brownian motion defined on the filtered probability space  $(\Omega, \mathcal{F}, P^Q)$ . We furthermore assume  $E(dW_s^Q dW_z^Q) = 0$ . Therefore, we have  $E(dS dZ) = \rho dt$ . The modified Brownian Bridge process considers the basis volatility and allows for interactions between the underlying asset return and the basis.

The solution of (7) is as follows:

$$Z(t, U) = \frac{(U-t)Z(0, U)}{U} + (U-t) \int_0^t \frac{\sigma_Z(v, U)}{U-v} \left[ \rho dW_s(v) + \sqrt{1-\rho^2} dW_z(v) \right], \quad 0 < t < U \quad (4)$$

The proof of Theorem 1 is shown in Appendix A. The basis is modeled directly without knowing how many state variables should be included. Moreover,  $Z(t, U)$  will approach to zero almost surely<sup>†</sup> when  $t = U$ .

### 3. Pricing futures with basis risk

Now we derive the closed-form solution for futures with basis risk. In view of

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<sup>†</sup> The feature is verified by Klebaner (1988), page 124.

(2), the value of futures is  $S(t)$  multiplies by exponential of  $Z(t,U)$  and displayed in Theorem II.

*Theorem II: The solution for futures price is as follows:*

$$F(t,U) = S(0) \exp \left[ (r - \delta - \frac{1}{2} \sigma_s^2) t + Z(t,U) + \sigma_s W_s^Q(t) \right], \quad \forall t \in [0,U] \quad (5)$$

Where  $F(t,U)$  is the futures price at time  $t$  with expiration date  $U$  on the contract.

Using the technique of moment generating function, the mean and variance of futures value with basis risk are provided as follows:

$$E_Q[F(t,U)] \equiv F(0,U) \exp \left[ (r - \delta) t + \mu_{BASIS}(0,t) \right] \quad (6)$$

$$V_Q[F(t,U)] \equiv \left[ F(0,U) \right]^2 \exp \left[ 2(r - \delta) t + 2\mu_{BASIS}(0,t) \right] \left( \exp \left[ \sigma_{SZ}^2(0,t,U) \right] - 1 \right) \quad (7)$$

where

$$\mu_{BASIS}(0,t) = \frac{-t \times Z(0,U)}{U} + \frac{1}{2} \int_0^t \left[ \frac{2\rho(U-v)\sigma_z(v,U)\sigma_s}{U-v} + \frac{(U-v)^2 \sigma_z^2(v,U)}{(U-v)^2} \right] dv \quad (8)$$

$$\sigma_{SZ}(0,t,U)^2 = \int_0^t \left[ \sigma_s^2 + \frac{2\rho(U-v)\sigma_s\sigma_z(v,U)}{U-v} + \frac{(U-v)^2 \sigma_z^2(v,U)}{(U-v)^2} \right] dv$$

(9)

The proof of Theorem II is verified in Appendix B.

The model has some important characteristics of observed futures process. First, the expectation of futures price is a function of spot price, volatility of spot return, initial basis, basis volatility, as well as the correlation coefficient of basis and spot return. However, the futures formula derived by Yan (2002) is not a function of spot volatility. Second, the basis is modeled directly without knowing how many state



variables should be included. Third, the basis is not only stochastic but also a zero value at the maturity of futures contract. It means that spot price and the futures price converge at the expiration of futures contract.

### **Martingale property for futures option with basis risk**

In this section, we verify that a discounted call option with basis risk is a martingale under spot martingale measure  $Q$ .

Assuming that the call price with basis risk is a function of  $F$  and  $t$ ,  $C^{BASIS} = C(F, t)$ , one can create a riskless hedge using the standard technique as outlined in Merton (1973). We construct a hedged portfolio by short selling a basis risk call option and buying  $n$  units of futures. The value of this portfolio  $V(t)$  is

$$G(t) = -C(t) + nF(t, U) \quad (10)$$

Where  $F(t, U) = S(t)e^{Z(t, U)}$ . By Ito's lemma we obtain

$$dG(t) = \left\{ (n - C_F)F(t, U) \left[ (r - \delta) - \frac{Z(t, U)}{U - t} + \rho \sigma_S \sigma_Z(t, U) + \frac{1}{2} \sigma_Z^2(t, U) \right] - \left[ C_t - \frac{1}{2} C_{FF} F^2(t, U) \sigma_{SZ}(0, t, U)^2 \right] \right\} dt + (n - C_F) \left[ (\sigma_S + \rho \sigma_Z(t, U)) dw_S^Q(t) + \sigma_Z(t, U) \sqrt{1 - \rho^2} dw_Z^Q(t) \right] \quad (11)$$

Where  $\sigma_{SZ}^2(t, U)$  is defined in equation (9). Equation (11) is verified in Appendix C.

When  $n = C_F$ , the hedging portfolio is riskless. Under no arbitrage assumption, the hedging portfolio can earn the riskless interest rate. Therefore, we have the following partial differential equation:

$$C_F rF + C_t + \frac{1}{2} C_{FF} F^2 \sigma_{SZ}^2(0, t, U) = rC$$

(12)

Equation (12) is the same PDE as presented in Black and Scholes (1973). By

Feynman-Kac formula, we obtain the call price on futures at time zero with maturity  $T$  under the spot martingale measure  $Q$ :

$$C_0^{Basis} = e^{-rT} E_Q \left[ (F(T,U) - k)^+ \right] \quad (13)$$

### The price of futures option with basis risk

The call price with basis risk is verified that it has martingale property in above section. Using (13) and  $F(T,U)$  in (5), we have futures call price with basis risk at time zero and the model is presented in Theorem III.

*Theorem III: Futures European call option valuation with basis risk is as follows:*

$$C_0^{Basis} = F(0,U) \exp \left[ -\delta T + \mu_{BASIS}(0,T) \right] N(d_1^{BASIS}) - ke^{-rT} N(d_2^{BASIS}), \quad 0 < T < U \quad (14)$$

Where

$$d_1^{BASIS} = \frac{\ln \frac{F(0,U)}{K} + (r - \delta)T + \mu_{BASIS}(0,T) + \frac{1}{2} \sigma_{SZ}^2(0,T,U)}{\sqrt{\sigma_{SZ}(0,T,U)^2}}, \quad (15)$$

$$d_2^{BASIS} = d_1^{BASIS} - \sqrt{\sigma_{SZ}(0,T,U)^2}, \quad (16)$$

and  $N(\cdot)$  is the cumulative probability of standard normal distribution.

The proof of Theorem III is shown in Appendix D.

The basis risk model in (14) is somewhat different with the model derived by the

Black (1976)<sup>‡</sup>. The basis risk model has the term  $\exp[\mu_{BASIS}(0, T)]$  rather than  $\exp(-rT)$  in options formula. This term captures the basis risk in futures contract.

### III. Extension

Taking the basis risk into consideration, we re-proof the Greek letters of delta and gamma, and futures European put options model.

#### Computation of Greek letters

Firstly, we present Delta and gamma for a futures European call option with basis risk. The solution is proposed in next theorem.

*Theorem IV: Delta and Gamma of the basis risk call option are as follows:*

$$\text{Delta} \equiv \frac{\partial C}{\partial F} = \exp[-\delta T + \mu_{BASIS}(0, T)] \times N(d_1^{BASIS}) \quad (17)$$

$$\text{Gamma} \equiv \frac{\partial^2 C}{\partial F^2} = \frac{\exp[-\delta T + \mu_{BASIS}(0, T)]}{F(0, U) \sqrt{2\pi\sigma_{SZ}^2(0, T, U)}} e^{\frac{-1}{2} \times (d_1^{BASIS})^2} \quad (18)$$

Where  $N(\cdot)$  and  $d_1^{BASIS}$  are defined in Theorem III.

The proofs are showed in Appendix E. From equation (11) and (17), there is an implication for hedging traders. When  $n = C_F$ , the hedging portfolio created in (10) is riskless. That is, the traders of the basis risk option should buy  $C_F$  units of futures as defined in (17) to construct a hedging portfolio to hedge the risk induced by one unit of basis risk option. Comparing Delta in (17) and Black(1976), we know that the hedger should buy more shares of futures when we take into account the basis risk.

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<sup>‡</sup> Black (1976) derived the futures call option as follows:

$$C_0 = \exp(-rT) \left[ F(0, U) \exp(-\delta T) N(d_1) - kN(d_2) \right]$$

### Futures European put option with basis risk

We display the futures European put option formula by put-call parity in this section. Firstly, we derive the put-call parity under basis risk environment. Then, using the relationship between call and put and call options derived in theorem III, we have the futures European put option formula. The closed-form solutions of put-call parity and put options are presented in Theorem V.

*Theorem V: The put-call parity of basis risk options is as follows:*

$$P_t + F(t, U) \exp[-\delta T + \mu_{BASIS}(t, T)] = ke^{-r(T-t)} + C_t \quad (19)$$

*The closed-form solution of futures European put option is as follows:*

$$P_0^{BASIS} = ke^{-rT} N(-d_2^{BASIS}) - F(0, U) \exp[-\delta T + \mu_{BASIS}(0, T)] N(-d_1^{BASIS}), \quad 0 < T < U \quad (20)$$

Where  $\mu_{BASIS}(t, T)$ ,  $d_1^{BASIS}$ , and  $d_2^{BASIS}$  are defined in Theorem III.

The proof of Theorem V is shown in Appendix F.

## IV. Numerical analysis

To investigate the properties of futures option with basis risk, we show graphically numerical results in Figure 1 and 2. The option prices are computed using  $F(0, U) = 100$ ,  $K = 95$ ,  $r = 0.03$ ,  $\delta = 0.02$ ,  $T = 0.3$ ,  $U = 0.5$ ,  $\sigma_s = 0.25$  and substituting these in equation (14). To explore the effects of initial basis value  $Z(0, U)$  and correlation between basis and stock return  $\rho$  on futures call option price, we let  $\sigma_z = 0.09$ ,  $\rho$  range from -1 to +1 and  $Z(0, U)$  range from -0.2 to +0.2.

The result is shown in Figure 1. The solution of call price in (14) is a plane that increase with respect to  $\rho$ , but decreases with respect to  $Z(0, U)$ . In words, the call price is an increasing function of correlation coefficient but decreasing function of

initial basis value. When the initial basis value is less than zero, the futures price is less than spot price. Holding other parameters unchanged (including futures price), the initial basis values change from  $< 0$  to  $> 0$ , the spot prices decline. Therefore, the call prices also decline. Figure 2 shows the impact of basis volatility  $\sigma_z$  and correlation between basis and stock return on option price. Let  $Z(0,U)=0.1$ ,  $\rho$  range from -1 to +1 and  $\sigma_z$  range from 0 to +0.2. In view of Figure 2, the solution of option price in (14) is a surface that increase with respect to  $\rho$ . However, the sign of correlation coefficient determines the influence of the volatility of basis on the value of call price. When  $\rho > 0$ , the value of call price with basis risk is increasing with the increase of the volatility of basis. If  $\rho < 0$ , the value of call price with basis risk is decreasing with the increase of the volatility of basis.

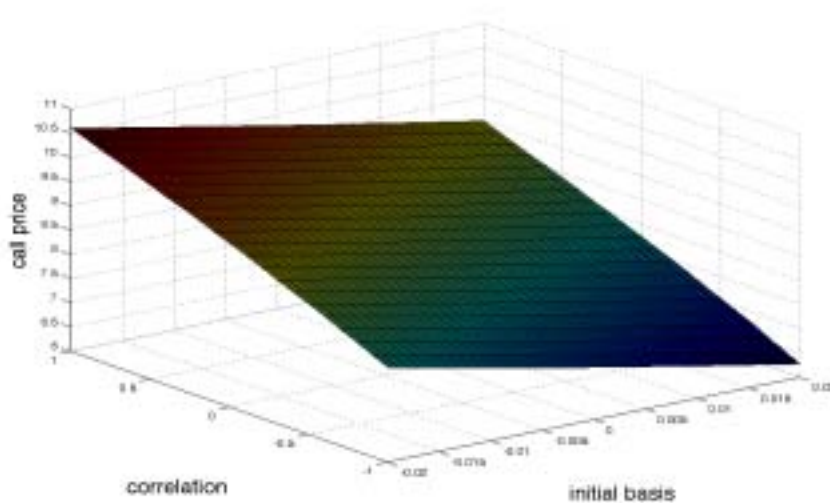


Figure 1

The effect of initial basis value and correlation between basis and stock return on futures call option price

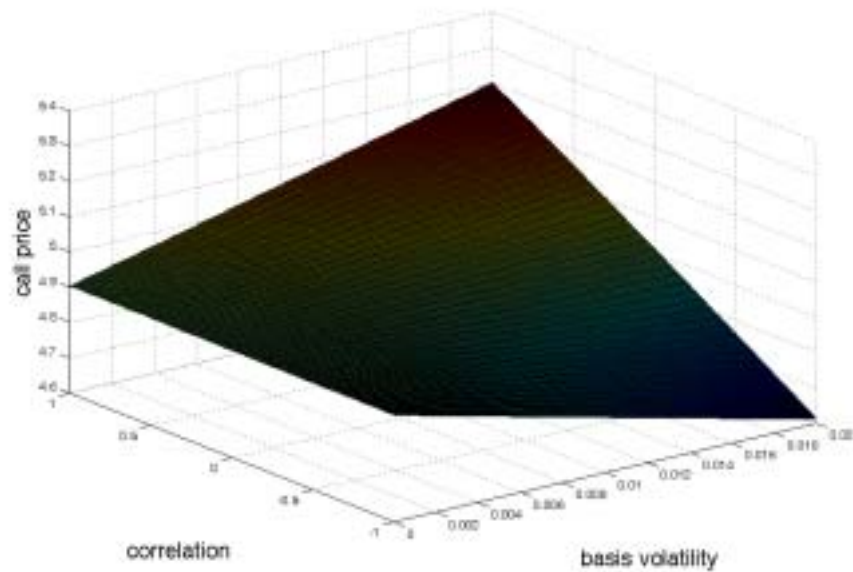


Figure 2

The impact of basis volatility and correlation between basis and stock return on futures call option price

## V. Empirical Test

We empirically test Futures European call option model with basis risk modeled in section III. As a comparison, the Black model is used to be the competing model in the empirical test.

### Empirical Data

The empirical test of the model is performed with daily trading prices of S&P 500 futures options at the Chicago Mercantile Exchange (CME) from January 3, 2005 to June 17, 2005. We chose the data based on two considerations as Lim and Guo (2000). Firstly, the options have great liquidity in American. Secondly, S&P 500 index, S&P 500 futures, and S&P 500 futures options are used widely in existing literature. Both the underlying futures and the options on futures expired on June 17, 2005.

Option prices are matched with the nearest corresponding futures price preceding the option transactions. Both the futures and options prices are quoted in index points. The futures prices in the sample period range from 1139.8 to 1229.8. The strike price of the options ranged from 800 to 1400. The widely ranged strike price may cause large forecasting error, but that actually responses the efficiency of models in all situations. There were 4138 call option prices in the sample.

One common filtering rule was applied to the raw sample before the data were used in the empirical test. That is, call prices that are less than 0.5 are not used to mitigate the impact of price discreteness (the tick size for S&P 500 futures option is 0.05). Most option-pricing models assume continuous price movements, while in real world the prices move in ticks. Nandi (1996) and Lim and Guo (2000) use this rule with their S&P 500 futures option data. The filtered sample consists of 3999 call prices. A more-detailed profile of the raw data is listed in Table 1.

The dividend yield of S&P 500 composite at 2004 is 1.53% to 1.96% proposed by Uhlfelder (2004), Tripp (2005), and Bary (2005). We use 1.7% as the estimate of S&P 500's dividend yield in 2005. The risk-free rate is calculated from quotes of U.S.

Table 1. Descriptive Statistics

	$T \geq 90$		$90 > T \geq 50$		$T < 50$		All Maturity	
Moneyneess $F/K$	Basiscall	BLcall	Basiscall	BLcall	Basiscall	BLcall	Basiscall	BLcall
<0.97								
Number	413		413		251		1239	
Mean	7.931	5.523	7.931	5.523	4.892	4.044	7.702	5.479
SD	4.549	3.521	4.549	3.521	2.863	2.433	4.899	3.712
097-100								
Number	297		297		321		835	
Mean	22.261	16.958	22.261	16.958	11.673	9.977	19.239	14.953
SD	5.727	4.923	5.727	4.923	8.745	7.696	9.206	7.173
100-103								

Number	254		254		285		721	
Mean	40.894	32.782	40.894	32.782	26.988	24.146	36.154	29.787
SD	7.237	6.381	7.237	6.381	10.669	9.726	11.350	8.930
1.03-1.06								
Number	152		152		155		387	
Mean	64.300	54.151	64.300	54.151	53.345	50.029	60.879	52.941
SD	8.212	7.607	8.212	7.607	8.943	8.623	10.321	8.148
$\geq 1.06$								
Number	299		299		346		806	
Mean	170.658	157.952	170.657	157.952	183.332	179.069	170.334	160.806
SD	81.597	81.677	81.597	81.677	95.367	95.342	85.176	85.918
All Moneyness								
Number	1415		1415		1358		3999	
Mean	57.262	50.218	57.262	50.218	62.372	59.718	53.300	47.859
SD	71.913	68.939	71.913	68.939	86.954	86.024	72.289	70.268

The sample data are daily trading prices of S&P 500 futures options at the Chicago Mercantile Exchange (CME) from January 3, 2005 to June 17, 2005. There are divided into 3 time-to-maturity and 5 moneyness groups so we have 24 groups altogether.

Treasury Bill prices. The U.S. 13-weeks Treasury Bills used in this study were auctioned by the Bureau of the Public Debt from Jan 1, 2005 to Jun 27, 2005. The average of discounted quotes is used to calculate the yield.

### Parameter Estimation

To implement option-pricing models, some unobservable parameters need to be estimated using the observed-traded-option prices in the sample period. For simplicity, we assume the parameters are constant. However, as monthly estimations greatly increased the fit, we follow the spirit of many existing empirical articles to use this method<sup>§</sup>. For basis risk model, four parameters of the process  $\sigma_S, \sigma_Z, \rho$ , and  $Z(0, U)$  need to be estimated from the realized data. But the  $\sigma_S$  is the only parameter needed

<sup>§</sup> See Bakshi et al. (1997), Chang et al. (1998), and Lim & Guo (2000).



to be estimated in Black-Scholes model.

### Test of Model Performance

The model's performance is examined using empirical test. We divided the time-to-maturity into three subgroups, and the moneyness into five. Including both the subtotals and grand totals, there are 24 groups. We present the Mean error, Mean of Absolute Error (MAE), and Root Mean Square Error (RMSE) statistics, in terms of both index and percentage to assess the efficiency of the two competing options models. The index point error is defined as the difference between model price and actual price, and the percentage error is the index point error divided by the model price. That is as follows:

$$e_{index} = C_{mo} - C_{actual} \quad (21)$$

$$e_{per} = \frac{e_{index}}{C_{mo}} \times 100\% \quad (22)$$

Where  $C_{mo}$  is the model call option price,  $C_{actual}$  is the actual call option price,  $e_{index}$  is the index point error, and  $e_{per}$  is the percentage error.

Table 2. Empirical Test Result

Moneyness $F/K$	$T \geq 90$		$90 > T \geq 50$		$T < 50$		All Maturity	
	Basiscall	BLcall	Basiscall	BLcall	Basiscall	BLcall	Basiscall	BLcall
<i>Panel A: Index-Point Error</i>								
<0.97								
Mean	-0.912	-3.519	1.624	-0.790	2.969	1.930	0.718	-1.505
MAE	2.388	3.727	2.477	2.015	3.126	2.331	2.566	2.872
RMSE	3.009	4.808	3.391	2.794	4.123	2.994	3.387	3.890

097-1.00									
Mean	-1.749	-8.473	2.081	-3.222	1.892	0.195	1.013	-3.273	
MAE	3.900	8.473	3.715	4.941	3.013	2.450	3.493	4.901	
RMSE	4.192	9.471	4.869	5.429	4.192	3.081	4.444	6.119	
1.00-1.03									
Mean	-2.935	-12.387	1.520	-6.591	0.857	-1.985	0.133	-6.233	
MAE	4.647	12.387	3.310	7.365	2.468	2.863	3.315	6.853	
RMSE	5.435	13.325	4.303	7.838	3.530	3.470	4.348	8.439	
1.03-1.06									
Mean	-1.952	-14.066	1.794	-8.356	1.403	-1.913	0.785	-7.152	
MAE	3.847	14.066	3.264	8.610	2.584	2.809	3.132	7.598	
RMSE	4.346	14.683	4.154	9.310	3.443	3.616	3.940	9.354	
$\geq 1.06$									
Mean	0.392	-14.554	3.926	-8.781	1.472	-2.788	2.167	-7.362	
MAE	3.027	14.554	3.997	8.781	1.697	2.816	2.816	8.867	
RMSE	3.519	14.924	4.381	9.006	2.210	3.340	3.419	8.909	
All Moneyiness									
Mean	-1.265	-7.944	2.203	-4.840	1.712	-0.942	0.972	-4.468	
MAE	3.183	8.042	3.290	5.724	2.535	2.649	3.009	5.464	
RMSE	3.843	9.916	4.189	6.776	3.537	3.281	3.887	7.172	

*Panel B: Percentaget Error*

<0.97									
Mean	-0.170	-0.805	0.169	-0.253	0.323	0.142	0.043	-0.429	
MAE	0.374	0.870	0.324	0.439	0.467	0.485	0.376	0.648	
RMSE	0.578	1.226	0.407	0.638	0.626	0.690	0.538	0.964	

**Table II. Empirical Test Result (continued)**

Moneyiness $F/K$	$T \geq 90$		$90 > T \geq 50$		$T < 50$		All Maturity	
	Basiscall	BLcall	Basiscall	BLcall	Basiscall	BLcall	Basiscall	BLcall
097-1.00								
Mean	-0.064	-0.435	0.070	-0.238	-0.098	-0.395	-0.029	-0.350
MAE	0.150	0.435	0.162	0.317	0.412	0.586	0.255	0.451
RMSE	0.161	0.484	0.201	0.360	0.725	1.220	0.472	0.824
1.00-1.03								
Mean	-0.067	-0.364	0.030	-0.216	0.011	-0.085	-0.002	-0.212
MAE	0.107	0.364	0.082	0.237	0.093	0.119	0.093	0.231
RMSE	0.124	0.392	0.105	0.258	0.129	0.144	0.120	0.274
1.03-1.06								

Mean	-0.029	-0.254	0.026	-0.160	0.023	-0.043	0.012	-0.136
MAE	0.038	0.170	0.033	0.108	0.024	0.030	0.051	0.144
RMSE	0.052	0.218	0.053	0.146	0.051	0.064	0.064	0.176
$\geq 1.06$								
Mean	-0.004	-0.141	0.025	-0.073	0.009	-0.020	0.012	-0.064
MAE	0.024	0.141	0.026	0.073	0.011	0.021	0.018	0.062
RMSE	0.029	0.157	0.029	0.084	0.017	0.028	0.025	0.089
All Moneyness								
Mean	-0.104	-0.546	0.077	-0.195	0.044	-0.100	0.010	-0.271
MAE	0.225	0.577	0.154	0.270	0.212	0.270	0.195	0.364
RMSE	0.405	0.882	0.244	0.404	0.448	0.669	0.373	0.669

The empirical test result is bisected into two panels. Panel A shows the index points error in each subgroup as well as subtotals and grand totals. Panel B presents the percentage error based on model price. Model prices are calculated using the estimated parameters. Those are figured out from the sample data in the same month.

The empirical test is to assess the performance of the model prices compared to actual prices using the parameter estimated from the realized data in the same month. The time-to-maturity and the moneyness biases are examined to analyze the magnitude of misspecification.

Table 2 presents the empirical results. The basis risk model shows better good-of-fit with no evidence of time-to-maturity or moneyness bias. In opposition, the Black's model demonstrates evidence of time-to-maturity and moneyness biases.

*Bias.* The overall empirical performances show no significant bias for the basis risk model, but the Black's model has significant bias. The mean errors in index-point terms and percentage terms in the sample for the basis risk model are small, but not for Black's model. For grand totals group, those are 0.973 and 1.0% for the basis risk model, -4.468 and -27.1% for Black's model.

From the angle of subtotals and subgroups, the basis risk model had no significant time-to-maturity and moneyness bias. However, we noticed that the Black's model presented systematic bias in both terms of time-to-maturity and

moneyness. In every time-to-maturity group, the Black model underprices (shown by a negative value) the option prices. The farther the maturity is, the severer the options underprices. In every moneyness groups, the Black's model also generally underpriced the options. In index-point terms, the degree of underpricing is proportional to the moneyness. In percentage term, the degree of underpricing, however, is opposite proportional to moneyness.

A few groups are excluded from the consistent result. In the first subtotal group in Table II, where  $F/K < 0.97$ , the Black model outperforms the basis risk model in  $90 > T \geq 50$  and  $T < 50$  subgroups. But both models have large mean errors in  $F/K < 0.97$  subtotal group. In particular, the Black's model has mean error of -80.5%, where  $T \geq 90$ , and the basis risk model has 32.3%, where  $T < 50$ . This may be due to deep-out-of-money causing illiquidity bias.

We also found that the relative content between the MAE and its mean error in each subgroup of index-point terms was larger for the basis risk model than the Black's model. For example, in the seventeen subgroup in Table II (Panel A), where  $T \geq 90$  and  $F/K \geq 1.06$ , the Black's model generates an MAE of 14.554 and a Mean of -14.554, while the basis risk model has 3.207 and 0.392, respectively. If we take MAE/Mean as Lim and Guo (2000), we obtain 1.0 for Black's model and 8.2 for the basis risk model. This can be interpreted as the evidence of the Black model's bias. When most of the errors are of the same sign in each subgroups, the mean error in index-point terms will be close to its MAE. The basis risk model's prices are more distributed around the actual prices, so the mean error is much smaller than MAE in index-point terms.

*Goodness of Fit.* The MAE of the Black model in grand total group of index-point terms is 5.464, while it is 3.009 for the basis risk model. In percentage terms, the MAE is 36.4% for the Black model and 19.5% for the basis risk model. The

results in the RMSE statistics are consistent with this. The basis risk model dominates the Black's in most of the groups.

According to the empirical test, we make a short conclusion here. The smaller bias and the better goodness of fit in the empirical test for the basis risk model over the Black's shows that there is evidence of basis risk in the S&P 500 futures options. The basis risk model is a better specification than the model without basis risk.

## **VI. Conclusion**

This study formulates an alternative futures European option model. This model differs from the Black (1976) by assuming that the future prices are influenced by the processes of underlying asset and basis risk. We further assume the basis risk dynamics follow modified Brownian bridges which are like Brownian motions except that with probability one these reach a specified point at a specified time. The futures and futures options model with basis risk has some features. First, the basis is modeled directly without knowing the function form and how many state variables should be included. Second, the basis is not only stochastic but also zero at the maturity of futures contract. Unlike Cornell and French (1985), the basis is constant or deterministic. Lastly, the futures price is a function of spot price, volatility of spot return, initial basis, basis volatility, as well as the correlation coefficient of basis and spot return. The futures model derived by Yan (2002) is not a function of spot volatility. Theoretically, the option model with basis risk is superior to assuming that the future prices follow standard cost-of-carry model. Because it has many characteristics stated above, uses more parameters and therefore allows for more degree of freedom.

The numerical test shows that the futures call option price is positively related with the correlation coefficient between basis and spot return but is a negatively in the

initial basis value. Meanwhile, the sign of correlation coefficient determines the relationship between the basis volatility and the futures call price.

This theoretical superiority has been empirically tested by S&P 500 futures options daily data. Comparing with the Black's model, the empirical test shows clear evidence supporting the occurrence of basis risk in futures options on stock index. Our model outperforms the Black's model by producing smaller bias and better goodness of fit. It not only eliminates systematic moneyness and time-to-maturity biases produced by Black model, but also has better prediction power. In overall sample data, the mean errors in terms of index and percentage are 0.973 and 1.0% for our model, -4.468 and -27.1% for Black's model.

Lastly, we derive the delta, gamma, and futures European put option, and discuss the hedge strategy while taking the basis risk into consideration.

## Appendix A

The basis  $Z(t, U)$  in (3) can be rewritten as follow:

$$dZ(t, U) = \frac{-Z(t, U)}{U-t} dt + \sigma_Z(t, U) dW^*(t) \quad (\text{A-1})$$

Where

$$dW^*(t) = \rho dW_S(t) + \sqrt{1-\rho^2} dW_Z(t) ,$$

We use the following technique to derived the solution of  $Z(t, U)$ . The technique is presented by Klebaner (1998) at page 121. Consider general linear stochastic differential equation (SDE) in one dimension

$$dZ(t, U) = \left[ \alpha(t) + \beta(t)Z(t, U) \right] dt + \left[ \gamma(t) + \delta(t)Z(t, U) \right] dW^*(t) \quad (\text{A-2})$$

Where functions  $\alpha, \beta, \gamma, \delta$  are given adapted processes, and are continuous functions

of  $t$ . Then  $Z(t,U)$  is found to be

$$Z(t,U) = X(t) \left[ Z(0,U) + \int_0^t \frac{\alpha(u) - \delta(u)\gamma(u)}{X(u)} du + \int_0^t \frac{\gamma(u)}{X(u)} dW^*(u) \right] \quad (\text{A-3})$$

Where  $X(t)$  is stochastic exponential SDE's and as follow:

$$dX(t) = \beta(t)X(t)dt + \delta(t)X(t)dW^*(t)$$

From (A-1), and (A-2), we have  $\alpha(t) = 0$ ,  $\beta(t) = \frac{-1}{U-t}$ ,  $\gamma(t) = \sigma_z(t,U)$ ,  $\delta(t) = 0$ .

From previous condition, we use (A-3) to obtain Eq. (4).

## Appendix B

From (2), we have

$$F(t,U) = S(t) \exp[Z(t,U)] \quad (\text{B-1})$$

Using (4) and the solution of (1), the Eq. (5) is derived.

By (B-1),  $F(0,U)$  can be obtained, and substituted into (5). The transformed function for (5) turns out to be:

$$F(0,U) = F(0,U) \exp \left[ (r - \delta - \frac{1}{2} \sigma_s^2) t - \frac{tZ(0,U)}{U} + \int_0^t \sigma(v) \bullet dW^Q(v) \right] \quad (\text{B-2})$$

Where

$$\sigma(v) = \begin{bmatrix} \sigma_s + \frac{(U-t)\rho\sigma_z(v,U)}{U-u} \\ \frac{(U-t)\sqrt{1-\rho^2}\sigma_z(v,U)}{U-u} \end{bmatrix}, \text{ and } dW^Q(v) = \begin{bmatrix} dW_S^Q(v) \\ dW_Z^Q(v) \end{bmatrix}$$

According to (B-2) and moment generating function of lognormality, we obtain Eq. (6), and (7).

## Appendix C

Under spot martingale measure  $Q$ , the stock price follows equation (1). Applying Itos' lemma to (5), we get:

$$\begin{aligned} \frac{dF(t,U)}{F(t,U)} = & \left[ (r-\delta) - \frac{Z(t,U)}{U-t} + \rho\sigma_s\sigma_z(t,U) + \frac{1}{2}\sigma_z(t,U)^2 \right] dt + (\sigma_s + \rho\sigma_z(t,U))dW_s^Q(t) \\ & + \sigma_z(t,U)\sqrt{1-\rho^2}dW_z^Q(t) \quad (\text{C-1}) \end{aligned}$$

By assuming the call option with basis risk is a function of  $F$  and  $t$ . Applied Itos' lemma, we obtain

$$\begin{aligned} dC = & \left\{ C_{FF}(t,U) \left[ (r-\delta) - \frac{Z(t,U)}{U-t} + \rho\sigma_s\sigma_z(t,U) + \frac{1}{2}\sigma_z(t,U)^2 \right] + \frac{1}{2}C_{FF} \left[ F^2(t,U) (\sigma_s^2 + 2\rho\sigma_s\sigma_z(t,U) + \sigma_z(t,U)^2) \right] + C_i \right\} dt \\ & (\sigma_s + \rho\sigma_z(t,U))dW_s^Q(t) + \sigma_z(t,U)\sqrt{1-\rho^2}dW_z^Q(t) \quad (\text{C-2}) \end{aligned}$$

From (10) we know

$$dG(t) = -dC(t) + ndF(t,U) \quad (\text{C-3})$$

Substitute (C-1) and (C-2) into (C-3), Equation (11) is computed.

## Appendix D

### Proof of Theorem III.

Substitute (5) into (13), the futures European call options at time zero are as follows:

$$\begin{aligned} C_0 = & e^{-rT} E_Q \left( \left[ F(T,U) - k \right]^+ \right) \\ = & e^{-rT} E_Q \left[ F(T,U) 1_D \right] - k e^{-rT} E_Q(1_D) \quad (\text{D-1}) \end{aligned}$$

We divide the RHS of (D-1) into two terms.

For the first term,  $I_1$ , on the RHS of (D-1):

$$\begin{aligned} I_1 = & e^{-rT} E_Q \left( F(0,U) \exp \left[ (r-\delta - \frac{1}{2}\sigma_s^2)T - \frac{TZ(0,U)}{U} + \int_0^T \sigma(v) \bullet dW^Q(v) \right] 1_D \right) \\ = & F(0,U) \exp \left[ (\delta - \frac{1}{2}\sigma_s^2)T - \frac{T \times Z(0,U)}{U} + \frac{1}{2} \int_0^T |\sigma(v)|^2 du \right] \times \\ & E_Q \left\{ \exp \left[ \int_0^T \sigma(v) \bullet dW^Q(v) - \frac{1}{2} \int_0^T |\sigma(v)|^2 dv \right] 1_D \right\} \quad (\text{D-2}) \end{aligned}$$

Where  $|\cdot|$  denotes the Euclidean norm in  $Q^2$ .

Assume that there exists a unique spot martingale measure  $R$  on  $(\Omega, F)$ , which



is given by the Radon-Nikodym derivative

$$\xi_T = \frac{dR}{dQ} = \exp \left[ \int_0^T \sigma(v) dW^Q(v) - \frac{1}{2} \int_0^T |\sigma(v)|^2 dv \right] \quad (D-3)$$

where  $\sigma \in R^2$  is the vector of market prices of risks corresponding to the sources of randomness in the economy.

By Girsanov's theorem, the process  $W_t^R$ , defined by

$$dW^R(t) = dW^Q(t) - \sigma(t)dt, \quad (D-4)$$

is a standard Brownian motion under probability measure  $R$ . Hence,

$$I_1 = F(0,U) \exp \left[ -\delta T + \mu_{BASIS}(0,T) \right] \times E_R [F(T,U) \geq K] \quad (D-5)$$

Under  $R$ -measure ,

$$\ln F(T,U) = \ln F(0,U) + (r - \delta + \frac{1}{2} \sigma_{SZ}^2(t,U))T + \mu_{BASIS}(0,T) + \int_0^T \sigma(v) dW^R(v) \quad (D-6)$$

Where  $\mu_{BASIS}(0,t)$ , and  $\sigma_{SZ}^2$  are defined in (8), and (9). Next, Substitute (D-6) into

(D-5), we have

$$I_1 = F(0,U) \exp \left[ -\delta T + \mu_{BASIS}(0,T) \right] N(d_1^{BASIS}) \quad (D-7)$$

Where  $d_1^{BASIS}$  is defined in (15).

For the second term,  $I_2$ , on the RHS of (D-1):

$$\begin{aligned} I_2 &= ke^{-rT} E_Q [F(T,U) > k] \\ &= Ke^{-rT} P_Q \left[ \ln F(0,U) + (r - \delta - \frac{1}{2} \sigma_S^2)T + \frac{Z(0,U)T}{U} + \int_0^T \sigma(v) dW^Q(v) \geq \ln K \right] \\ &= ke^{-rT} N(d_2^{BASIS}) \end{aligned} \quad (D-8)$$

Where  $d_2^{BASIS}$  is defined in (16). Substitute (D-7) and (D-8) into (D-1), The Eq.(14) in Theorem III is completed.

## Appendix E

Using (14) and chain rule, the delta is as follows:

$$\begin{aligned} \text{Delta} &= \frac{\partial C}{\partial F} = \exp \left[ -\delta T + \mu_{BASIS} (0, T) \right] N \left( d_1^{BASIS} \right) + \\ &F(0, U) \exp \left[ \left( -\delta - \frac{1}{2} \sigma_S^2 \right) T + \mu_{BASIS} (0, T) \right] \frac{\partial N \left( d_1^{BASIS} \right)}{\partial S} - k e^{-rT} \frac{\partial N \left( d_2^{BASIS} \right)}{\partial S} \quad (\text{E-1}) \end{aligned}$$

Where

$$\frac{\partial N \left( d_1^{BASIS} \right)}{\partial F} = \frac{\partial N \left( d_1^{BASIS} \right)}{\partial d_1^{BASIS}} \times \frac{\partial d_1^{BASIS}}{\partial F} = \frac{1}{F(0, U) \sqrt{2\pi\sigma_{SZ}(T, U)^2}} \times e^{\frac{-1}{2} \left( d_1^{BASIS} \right)^2}$$

According to (16), we have

$$d_1^{BASIS} = d_2^{BASIS} - \sqrt{\sigma_{SZ}(T, U)^2}$$

From previous derivations, the second term on the RHS of (E-1) can be expressed as

follow:

$$\begin{aligned} &F(0, U) \exp \left[ \left( -\delta - \frac{1}{2} \sigma_S^2 \right) T + \mu_{BASIS} (0, T) \right] \frac{1}{F(0, U) \sqrt{2\pi\sigma_{SZ}(T, U)^2}} \times e^{\frac{-1}{2} \left( d_2^{BASIS} - \sqrt{\sigma_{SZ}(T, U)^2} \right)^2} \\ &= k e^{-rT} \frac{1}{F(0, U) \sqrt{2\pi\sigma_{SZ}(T, U)^2}} \times e^{\frac{-1}{2} \left( d_2^{BASIS} \right)^2} \quad (\text{E-2}) \end{aligned}$$

We use a similar trick to derive the solution of the third term on the RHS of (E-1).

The solution is as follow:

$$k e^{-rT} \frac{\partial N \left( d_2^{BASIS} \right)}{\partial F} = k e^{-rT} \frac{1}{F(0, U) \sqrt{2\pi\sigma_{SZ}(T, U)^2}} \times e^{\frac{-1}{2} \left( d_2^{BASIS} \right)^2} \quad (\text{E-3})$$

We found that (E-2) is just the same with (E-3). Therefore, the Eq. (17) in Theorem IV is obtained.

Since we have the solution of delta, we obtain

$$\begin{aligned}
Gamma &= \frac{\partial^2 C}{\partial F^2} = \frac{\partial Delta}{\partial S} \\
&= \exp[-\delta T + \mu_{BASIS}(0, T)] \frac{\partial N(d_1^{BASIS})}{\partial F} \\
&= \frac{\exp[-\delta T + \mu_{BASIS}(0, T)]}{F(0, U) \sqrt{2\pi\sigma_{SZ}(T, U)^2}} e^{-\frac{1}{2}(d_1^{BASIS})^2}
\end{aligned}$$

## Appendix F

### Proof of Put-Call parity in Theorem V

Firstly, we introduce two notations: indicator  $1_D$  stands for  $F(T, U) \geq K$ , and indicator  $1_E$  means for  $K \geq F(T, U)$ . Under spot martingale measure  $Q$ , a futures European put options price at time  $t$  is as follows:

$$\begin{aligned}
P_t &= E_Q \left( \frac{P_T}{B_T} \mid f_t \right) = E_Q \left( \frac{k1_E - F(T, U)1_E}{B_T} \mid f_t \right) \\
&= E_Q \left( \frac{k - F(T, U) + F(T, U)1_D}{B_T} \mid f_t \right) \\
&= E_Q \left( \frac{B_t k}{B_T} \mid f_t \right) - E_Q \left( \frac{B_t F(T, U)}{B_T} \mid f_t \right) + E_Q \left( \frac{B_t C_T}{B_T} \mid f_t \right) \\
&= ke^{-r(T-t)} - F(t, U) \exp[-\delta(T-t) + \mu_{BASIS}(t, T)] + C_t
\end{aligned}$$

Therefore, the put-call parity in Eq.(19) is obtained. By the derived put-call parity, a futures put option is derived.

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