

# On the Pricing of Longevity-Linked Securities\*

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## Abstract

For annuity providers, longevity risk, i.e. the risk that future mortality trends differ from those anticipated, constitutes an important risk factor. In order to manage this risk, new financial products, so-called *longevity derivatives*, may be needed, even though a first attempt to issue a longevity bond in 2004 was not successful.

While different methods of how to price such securities have been proposed in recent literature, no consensus has been reached. This paper reviews, compares and comments on these different approaches. In particular, we use data from the United Kingdom to derive prices for the proposed first longevity bond and an alternative security design based on the different methods.

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## 1 Introduction

Longevity risk, i.e. the risk that the realized future mortality trend exceeds current assumptions, constitutes an important risk factor for annuity providers and pension funds. This risk is increased by the current problems of state-run pay-as-you-go pension schemes in many countries: The reduction of future benefits from public pension systems and tax incentives for annuitization of private wealth implemented by many governments may lead to an increasing demand for annuities (cf. Kling et al. (2008)). One prevalent way of managing this risk is securitization, i.e. isolating the cash flows that are linked to longevity risk and repackaging them into cash flows that are traded in capital markets (see Cowley and Cummins (2005) for an overview of securitization in life insurance).

In academic literature, several supporting instruments such as so-called longevity or survivor bonds (cf. Blake and Burrows (2001)) or survivor swaps (cf. Dowd et al. (2006)) have been proposed (see Blake et al. (2006a) for an overview). However, a first attempt to issue a longevity-linked security in 2004 failed (see Section 4 for more information). Nevertheless, the general consensus amongst practitioners appears to be that whilst a breakthrough is yet to come, “*betting on the time of death is set*”<sup>1</sup>; and several investment banks such as *JPMorgan* or *Goldman Sachs* have installed trading desks for longevity risk.

Aside from the question of how to appropriately engineer such longevity-linked securities or *longevity derivatives*, there is an ongoing debate of how to price a given instrument. This debate is twofold: On one hand, there is a quest for actuarial or economic methods of how to derive prices “conceptually” and, on the other hand, there is the problem of how to derive prices given a certain methodology – i.e. what data to use for calibration purposes.

In this paper, we address all of these issues. First, we review and compare different pricing methods proposed in recent literature. Subsequently, we empirically compare different approaches based on the established methods using data from the United Kingdom (UK). Moreover, we discuss the financial engineering aspect of longevity securitization and, based on this discussion, introduce and analyze a longevity derivative with option-type payoff. Thus, in addition to a long-needed theoretical and empirical survey on proposed pricing approaches for longevity securities, this text provides several new ideas which could be helpful in building a market in longevity risk.

The pricing of longevity derivatives is of course closely related to modeling the stochastic evolution of mortality. So far, several stochastic mortality models have been proposed – for a detailed overview and a categorization see Cairns et al. (2006a). We rely on a continuous-time stochastic modeling framework for mortality. Milevsky and Promislow (2001) were among the first to propose a stochastic hazard rate or force of mortality, Dahl (2004) presents a general stochastic model for the mortality intensity, and in Biffis (2005), affine jump-diffusion processes are employed to model both, financial and demographic risk factors. However, while in these articles the spot force of mortality is modeled, we adopt the forward modeling approach for mortality (see Cairns et al. (2006a), Miltersen and Persson (2005), Bauer (2007), or Bauer et al. (2007)).

The remainder of the paper is organized as follows. In Section 2, we review pricing approaches presented

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<sup>1</sup>The Business, 08/15/2007, “*Betting on the time of death is set*”, by P. Thornton.

in recent literature, and formalize as well as compare them in a common setting in Section 3. In particular, we point out technical problems with some of these approaches. Subsequently, we present an empirical comparison based on the first announced – but never issued – longevity bond, the so-called EIB/BNP-Bond, in Section 4. Aside from providing valuable insights on the adequacy of the different methods, our results reveal why different authors have come to different appraisals of “how good of a deal” the EIB/BNP-Bond was. However, beyond the pricing, the financial engineering of the Bond may also have been a reason for its failure. We discuss this matter and, in Section 5, introduce and analyze an alternative security design with an option-type payoff structure. Finally, Section 6 concludes.

## 2 Different Approaches for Pricing Longevity-Linked Securities

The price a party is willing to pay for some longevity derivative depends on both, the estimate of uncertain future mortality trends and the level of uncertainty.<sup>2</sup> This uncertainty is likely to induce a *mortality risk premium* that should be priced by the market (cf. Milevsky et al. (2005)). However, so far, if at all, there are no liquidly traded securities. Therefore, it is not possible to rely on market data for pricing purposes.<sup>3</sup> As a consequence, different methods for pricing mortality risk have been proposed.

Friedberg and Webb (2007) apply the *Capital Asset Pricing Model* (CAPM) and the *Consumption Capital Asset Pricing Model* (CCAPM) to quantify risk premiums for potential investors in longevity bonds, which turn out to be very low. However, the authors acknowledge that there is likely to exist a “mortality premium puzzle” similar to the well-known “equity premium puzzle” (cf. Mehra and Prescott (1985)) implying higher mortality risk premiums than these economic models would suggest. Thus, their quantitative results seem to be of limited applicability and we will not further discuss this approach.

Milevsky et al. (2005) and Bayraktar et al. (2008) develop a theory for pricing non-diversifiable mortality risk in an incomplete market: They postulate that an issuer of a life contingency requires compensation for this risk according to a pre-specified *instantaneous Sharpe ratio* (see also Bayraktar and Young (2008)). They show that their pricing approach has several appealing properties, which are particularly intuitive in a discrete-time setting (cf. Milevsky et al. (2006)). The basic idea is that an additional return in excess of the risk free rate is paid on the insurer’s asset portfolio, which is determined as a multiple  $\lambda$  – the instantaneous Sharpe ratio – of the remaining standard deviation after all diversifiable risk is hedged away. While for a finite number of insureds this approach leads to a non-linear partial differential equation for the price of an insurance contract, the pricing rule becomes linear as the portfolio size approaches infinity – as it is clearly the case for an index-linked longevity derivative. In this case, it can be shown that their approach coincides with a change of the probability measure from the objective measure  $P$  to a “risk-adjusted” measure  $\mathbb{Q}$  invoked by a constant “market price of risk”-process (cf. Bauer (2007), Sec. 3.2.1).

Similarly, other structures of the market price of mortality risk could be considered, too; basically, we may choose any adapted, predictable process. However, as pointed out by Blake et al. (2006b) “one can argue

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<sup>2</sup>In order to keep our presentation concise, we turn our attention to pricing approaches for index-linked longevity derivatives. In particular, we do not consider indemnity securitization transactions, where small sample risk may need to be taken into account.

<sup>3</sup>Cairns et al. (2006a) calibrate their 2-factor model to published pricing data of the EIB/BNP-Bond. However, we believe that relying on the published data is problematic since pricing may have been an issue for its failure.

that more sophisticated assumptions about the dynamics of the market price of longevity risk are pointless given that, at the time, there was [is] just a single item of price data available for a single date (and even that is no longer valid).” However, even if a constant market price of risk is chosen, the question of how to calibrate it remains open, even though Milevsky et al. (2006) state that the “*Sharpe ratio from stocks as an asset class*” is approximately equal to  $\lambda^{S\&P} = 0.25$  (Loeys et al. (2007) also report a level of 25%).

In order to account for a risk premium, Lin and Cox (2005, 2006) apply the so-called 1- and 2-factor Wang transform, respectively, to adjust the best estimate cumulative distribution function of the future lifetime of an individual aged  $x$ . In order to find a suitable transform for pricing particular longevity securities, i.e. to calibrate the transform parameters, the authors rely on market prices of annuity contracts.

However, as pointed out by Pelsser (2008), the Wang transform does not provide “*a universal framework for pricing financial and insurance risks*” (cf. Wang (2002)), so the adequacy is questionable<sup>4</sup>. Moreover, it is not clear whether annuity prices offer an adequate starting point when pricing longevity derivatives, and it is unclear “*how different transforms for different cohorts and terms to maturity relate to one another and form a coherent whole*” (cf. Cairns et al. (2006a)).

### 3 Methodological Comparison of the Approaches

In the previous section, we introduced pricing approaches for longevity-linked securities. In the current section, we establish them in a common framework: We adopt the setup from Bauer et al. (2007), which is summarized in the first subsection. The second subsection then presents pricing formulae for simple longevity bonds under the different approaches. Finally, the last subsection points out potential shortcomings of the transform-based approach from Lin and Cox (2005, 2006).

#### 3.1 The Forward Mortality Framework

For the remainder of this paper, we fix a time horizon  $T^*$  and a filtered probability space  $(\Omega, \mathcal{F}, \mathbf{F}, P)$ , where  $\mathbf{F} = (\mathcal{F})_{0 \leq t \leq T^*}$  is assumed to satisfy the usual conditions. Furthermore, we fix a (large) underlying population of individuals, where each age cohort is denoted by the age  $x_0$  at time zero. We assume that the *best estimate forward force* of mortality with maturity  $T$  as from time  $t$ ,

$$\hat{\mu}_t(T, x_0) := -\frac{\partial}{\partial T} \log \left\{ E_P \left[ T p_{x_0}^{(T)} \mid \mathcal{F}_t \right] \right\} \stackrel{T \geq t}{=} -\frac{\partial}{\partial T} \log \left\{ E_P \left[ T-t p_{x_0+t}^{(T)} \mid \mathcal{F}_t \right] \right\},$$

is well defined, where  $T-t p_{x_0+t}^{(T)}$  denotes the proportion of  $(x_0 + t)$ -year olds at time  $t$  who are still alive at time  $T$ , i.e. the survival rate or the “realized survival probability”. Moreover, we assume that  $(\hat{\mu}_t(T, x_0))_{0 \leq t \leq T}$  satisfies the stochastic differential equation

$$d\hat{\mu}_t(T, x_0) = \hat{\alpha}(t, T, x_0) dt + \hat{\sigma}(t, T, x_0) dW_t, \quad \hat{\mu}_0(T, x_0) > 0, \quad (1)$$

<sup>4</sup>Several other authors also relied on the Wang transform for pricing longevity derivatives. See e.g. Denuit et al. (2007) or Loisel and Serant (2007).

where  $W = (W_t)_{t \geq 0}$  is a  $d$ -dimensional standard Brownian motion independent of the financial market, and  $t \mapsto \hat{\alpha}(t, T, x_0)$  as well as  $t \mapsto \hat{\sigma}(t, T, x_0)$  are continuous functions. Hence, we are in the ‘‘Gaussian case’’, i.e. the forward force of mortality could become negative for extreme scenarios, but we regard this as an acceptable shortcoming for practical considerations. The drift condition (cf. Bauer et al. (2007)) yields

$$\hat{\alpha}(t, T, x_0) = \hat{\sigma}(t, T, x_0) \cdot \int_t^T \hat{\sigma}(t, s, x_0)' ds,$$

and for the  $T$ -year best estimate survival probability for an  $x_0$ -year old at time zero, we have

$${}_T\hat{p}_{x_0} := E_P \left[ {}_T p_{x_0}^{(T)} \right] = e^{-\int_0^T \hat{\mu}_0(s, x_0) ds}.$$

The price of any security is now given as the expected value of its payoff under some equivalent martingale measure (see e.g. Harrison and Kreps (1979) or Duffie and Skiadas (1994)), which in turn is defined by its Radon-Nikodym density, say

$$\left. \frac{\partial Q}{\partial P} \right|_{\mathcal{F}_t} = \exp \left\{ -\int_0^t \lambda(s)' dW_s - \frac{1}{2} \int_0^t \|\lambda(s)\|^2 ds \right\},$$

where – for simplicity – we restrict ourselves to deterministic choices of the ‘‘market price of risk’’ process  $(\lambda(t))_{t \geq 0}$ . Hence, we have (cf. Bauer et al. (2007))

$${}_T\tilde{p}_{x_0} := E_Q \left[ {}_T p_{x_0}^{(T)} \right] = e^{-\int_0^T \int_0^s \hat{\sigma}(u, s, x_0) \lambda(u) du ds} {}_T\hat{p}_{x_0}, \quad (2)$$

and for the ‘‘risk-neutral’’ forward force of mortality with maturity  $T$  as from time  $t$ ,

$$\tilde{\mu}_t(T, x_0) := -\frac{\partial}{\partial T} \log \left\{ E_Q \left[ {}_{T-t} p_{x_0+t}^T \mid \mathcal{F}_t \right] \right\} = \hat{\mu}_t(T, x_0) + \int_t^T \hat{\sigma}(s, T, x_0) \lambda(s) ds, \quad (3)$$

we also have

$$d\tilde{\mu}_t(T, x_0) = \hat{\alpha}(t, T, x_0) dt + \hat{\sigma}(t, T, x_0) d\tilde{W}_t, \quad (4)$$

where  $\tilde{W} = (\tilde{W}_t)_{t \geq 0}$  is a Brownian motion under  $Q$ . In particular, for the time zero ‘‘price’’ of a basic  $(T, x_0)$ -Longevity Bond, which pays  ${}_T p_{x_0}^{(T)}$  at time  $T$  (cf. Cairns et al. (2006a)), we have

$$\Pi_0(T, x_0) := p(0, T) E_Q \left[ {}_T p_{x_0}^{(T)} \right] = e^{-\int_0^T \tilde{\mu}_0(s, x_0) + f(0, s) ds}.$$

where  $p(0, T)$  is the time zero price of a zero-coupon with maturity  $T$  and  $f(0, s)$  is forward force of interest (see e.g. Björk (1999)).

From a practical point of view, this definition could be problematic: In reality, at time  $T$ , the  ${}_T p_{x_0}^{(T)}$  can only be approximated from a finite amount of data and this approximation may not be available until months or even years after time  $T$  (cf. Cairns et al. (2006a)). However, similarly to the EIB/BNP-Bond, payments could be postponed by a deferral period. Moreover, several investment banks are working on establishing real-time longevity indices such as Goldman Sachs' *QxX index* (see <http://www.qxx-index.com/>) or JPMor-

gan's *LifeMetrics Index* (see <http://www.lifemetrics.com>).

It is important to note that we do not assume that  $(T, x_0)$ -Longevity Bonds are liquidly available in the market. In fact, when assuming that a sufficient number of  $(T, x_0)$ -Longevity Bonds are traded, by the same argumentation as for financial bonds in Heath et al. (1992), under weak conditions on the volatility structure, the market would be complete. In particular, the measure  $Q$  would be uniquely determined, such that basically all longevity derivatives would have a well-determined price – and that is clearly unrealistic. Hence, the quantities  $\Pi_0(T, x_0)$  should be interpreted as technical auxiliary definitions, which rather simplify the presentation than denote prices of securities that are actually traded; this is similar to interest rate modeling, where zero-coupon bonds are employed as the basic building blocks.

Their particular usefulness under our specification becomes evident when regarding equation (2): When given best estimate dynamics (1), which can be inferred from historic data, it is sufficient to know the “market-price of risk” process (function)  $(\lambda(t))_{t \geq 0}$  in order to specify the “risk-neutral” (pricing) model (cf. Equations (3) and (4)). By Equation (2), we immediately get

$$\Pi_0(T, x_0) = p(0, T) {}_T\tilde{p}_{x_0} = e^{-\int_0^T \int_0^s \hat{\sigma}(u, s, x_0) \lambda(u) du ds} p(0, T) {}_T\hat{p}_{x_0}. \quad (5)$$

Thus, specifying the “market price of risk” is equivalent to specifying  $\Pi_0(T, x_0)$ , i.e. pricing  $(T, x_0)$ -Longevity Bonds.

### 3.2 Pricing Simple Longevity Bonds

For pricing a  $(T, x_0)$ -Longevity Bond, the idea of Lin and Cox (2005) corresponds to applying the 1-factor Wang transform to the best estimate mortality probability, i.e.

$$1 - {}_T\tilde{p}_{x_0} = \Phi(\Phi^{-1}(1 - {}_T\hat{p}_{x_0}) - \theta),$$

where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution and  $\theta$  is the parameter of the Wang transform. Hence, under this approach we have (cf. Eq. (5))

$$\Pi_0(T, x_0) = p(0, T) (1 - \Phi(\Phi^{-1}(1 - {}_T\hat{p}_{x_0}) - \theta)). \quad (6)$$

A similar equation can be found for the 2-factor Wang transform applied in Lin and Cox (2006).

As pointed out in Section 2, pricing  $(T, x_0)$ -Longevity Bonds using a “pre-specified instantaneous Sharpe ratio”  $\bar{\lambda}$  in the 1-factor model from Milevsky et al. (2005) corresponds to a constant market price of risk. However, in our model setup, we allow for a multi-dimensional Brownian motion – and thus a multi-dimensional market price of risk process. By an application of Itô's product formula, we obtain

$$\lim_{h \rightarrow 0} \sqrt{\frac{1}{h} \text{Var}_P[\hat{\mu}_{t+h}(T, x_0) | \mathcal{F}_t]} = \sqrt{\sum_{i=1}^d (\hat{\sigma}_i(t, T, x_0))^2},$$

where clearly  $\hat{\sigma}_i$  is the  $i^{\text{th}}$  component of the volatility function  $\hat{\sigma}$ . Thus, in the multi-dimensional setting,



this approach yields<sup>5</sup>

$$\sum_{i=1}^d \hat{\sigma}_i(t, T, x_0) \lambda_i(t) = -\bar{\lambda} \sqrt{\sum_{i=1}^d (\hat{\sigma}_i(t, T, x_0))^2}, \quad (7)$$

which in turn yields (cf. Eq. (5))

$$\Pi_0(T, x_0) = p(0, T) e^{\bar{\lambda} \int_0^T \int_0^s \|\hat{\sigma}(u, s, x_0)\| du ds} {}_T\hat{p}_{x_0}. \quad (8)$$

In order to determine  $\bar{\lambda}$ , one could use the Sharpe ratio from equity markets. As aforementioned, Milevsky et al. (2006) and Loeys et al. (2007) identify a level of  $\bar{\lambda} = 25\%$ . However, it is questionable whether relying on equity market data yields a suitable choice. Empirical studies show that the risk premium for stocks is considerably higher than for other securities. Several authors have studied this *equity premium puzzle* (cf. Mehra and Prescott (1985)); for example, relatively recently, Barro (2006) and Weitzman (2007) – using different arguments – have proposed that a heavy-tailed distribution of (log) stock returns yields a coherent explanation. It is rather arguable that the distribution of future mortality is heavy-tailed on the left side of mortality improvements, and hence employing the equity market Sharpe ratio does not seem appropriate. Moreover, traditional financial economic theory attributes (equity) risk premiums to correlation with consumption and, again, it is doubtful that longevity shows similar patterns as stock indices. Hence, we do not share the opinion of, e.g., Loeys et al. (2007) that this approach yields a “*fair compensation for risk*”.

Alternatively, we can calibrate the Sharpe ratio  $\bar{\lambda}$  to a suitable annuity quote, which would be in the spirit of Lin and Cox (2005). However, again the question arises of whether annuity quotes offer a suitable starting point when pricing longevity bonds: In addition to systemic longevity risk, annuity providers are subject to non-systematic types of mortality risk arising from their finite portfolios of insureds. Therefore, the overall “per policy” risk of an annuity provider surmounts the per policy pure longevity risk, so that the risk premium within an annuity policy – if it exists – should be greater than the risk premium for a longevity derivative.

There is strong empirical evidence that market prices of annuities exceed the *actuarially fair price*, i.e. the (best estimate) expected present value. For example, Mitchell et al. (1999) report that in the mid 1990s in the United States (US), the average annuity policy delivers pay-outs of less than 91 cents per unit of annuity premium (Finkelstein and Poterba (2002) make similar observations for the UK). While according to the authors this “*transaction cost*” is primarily due to expenses, profit margins, and contingency funds, Milevsky and Young (2007) do not believe that these fees can be classified “*under the umbrella of transaction costs*”. They rather think the fees are “*inseparably linked to aggregate mortality risk*”. Van de Ven and Weale (2006) come to a similar conclusion: Annuity costs due to aggregate longevity uncertainty arise endogenously in their two period overlapping generation general equilibrium model.

Therefore, we conclude that it is very plausible for life annuities to include a risk premium for longevity risk, and hence, estimates on their bases will at least provide an upper bound for the risk premium to be

<sup>5</sup>Note that the cumulative market price of risk will be negative since we assume that investors require compensation for taking longevity risk, i.e. the “risk-adjusted” forward force of mortality will be smaller than the “best estimate” counterpart.

included in longevity derivative pricing. The question of whether this bound is strict depends on the non-systematic part of mortality risk within annuities, or, more precisely, on the question if annuities include a risk premium for non-systematic risk. While there does not seem to be a distinct answer, we see evidence that this is not the case: On one hand, Brown and Orszag (2006) note that “*insurers are extremely adept at using the LLN (Law of Large Numbers) to essentially eliminate the relevance of idiosyncratic risk facing any one individual that they insure.*” On the other hand, if there were a charge for non-systematic mortality risk, this charge would clearly depend on the number of insureds within an insurer’s portfolio, so that the large insurers would have a significant advantage. This, in turn, would eventually lead to a market with only few, large insurers, which clearly is not the case in many developed markets.

Hence, from our point of view, the idea of Lin and Cox (2005) to rely on annuity market data to derive prices for longevity derivatives seems valid, even though it is not absolutely clear whether the resulting price should not be rather interpreted as an upper bound. However, we still need to address the question of whether their application of the Wang transform is adequate and leads to “*coherent*” prices (cf. Sec. 2).

### 3.3 Shortcomings of the Approach by Lin and Cox (2005, 2006)

As explained above, the pricing approaches of Lin and Cox (2005, 2006) can be interpreted as transforms of best estimate survival probabilities  ${}_T\hat{p}_{x_0}$ ,  $T, x_0 \geq 0$  (or equivalently, the forward force of mortality surface at time zero  $(\hat{\mu}_0(T, x_0))_{T, x_0 \geq 0}$ ) to match observed insurance prices. However, it is not clear if the application of an arbitrary transform is suitable; in particular, the suitability of the Wang transform is of interest.

In order to keep this discussion simple, let us adapt the framework from Section 3.1 with the additional assumption that  $d = 1$ , i.e. that mortality is driven by a one-dimensional Brownian motion. Then, Equation (3) yields

$$\hat{\mu}_0(T, x_0) - \tilde{\mu}_0(T, x_0) = - \int_0^T \sigma(s, T, x_0) \lambda(s) ds. \quad (9)$$

On the other hand, if “risk-neutral” forward forces are derived from best estimate forward forces via some transform, say  $O(\cdot)$ , we have

$$O(\hat{\mu}_0(\cdot, \cdot))(T, x_0) = \tilde{\mu}_0(T, x_0),$$

i.e. we are given the left-hand side of (9), and in order to derive the implied market price of risk we may solve for  $(\lambda(t))_{t \geq 0}$ .

For each fixed  $x_0$ , (9) is a *Volterra integral equation of the first kind*, and under light regularity conditions, there exists a unique solution, say  $(\lambda_{x_0}(t))_{t \geq 0}$  (see e.g. Section 7.2.1.2 in Polyanin and Manzhirov (1999)). Now consider another age-group (cohort), say  $x_1$ . If there exists some  $T$  such that

$$O(\hat{\mu}_0(\cdot, \cdot))(T, x_1) \neq \hat{\mu}_0(T, x_1) + \int_0^T \sigma(s, T, x_1) \lambda_{x_0}(s) ds,$$

then (9) does not have a solution, i.e. the market price of risk process for different ages – but for the same *source of risk* – will be different, which clearly leads to arbitrage opportunities as soon as longevity derivatives on both cohorts are traded. Hence, under the current assumptions, the only *suitable* transforms are of



the form

$$O_{\lambda(\cdot)}(\hat{\mu}_0(\cdot, \cdot))(T, x_0) := \hat{\mu}_0(T, x_0) + \int_0^T \sigma(s, T, x_0) \lambda(s) ds$$

for some given  $\lambda(\cdot)$  or, equivalently, of the form (2) for the best estimate survival survival probabilities. In particular, the Wang transform does not present a suitable choice in this framework as soon as there are at least two different longevity derivatives traded based on at least two cohorts even when allowing for different transform parameters for different cohorts.

While this inconsistency is only shown under the current assumptions, our results indicate drawbacks associated with the methodology from Lin and Cox (2005, 2006). Nevertheless, for practical applications and when only pricing a longevity derivative based on a single cohort, the Wang transform still presents a theoretically valid approach – the question of whether it is adequate, however, prevails.

## 4 Empirical Comparison Based on the EIB/BNP Longevity Bond

In this section, we compare the pricing approaches introduced in Section 2 from an empirical perspective adopting the Gaussian forward mortality model proposed by Bauer et al. (2007). Alongside the Wang transform parameters derived by Lin and Cox (2005, 2006) and the instantaneous Sharpe ratio from equity markets, we use a time series of UK annuity market quotes to derive parameter estimates for different points in time.

### 4.1 “The Volatility of Mortality”

Bauer et al. (2007) propose volatility specifications and, hence, mortality models for the best estimate forward force of mortality  $\hat{\mu}_t(T, x_0)$  in the framework introduced in Subsection 3.1. For details on the volatility structure, resulting characteristics of the model, and the calibration algorithm, we refer to their paper. However, since their estimates are based on US pensioner mortality data, for our objective it is necessary to calibrate the model to UK data. We rely on UK life tables and projections for pension annuities as published on the website of the *Institute of Actuaries and the Faculty of Actuaries*.<sup>6</sup> Details on the employed tables are presented in Appendix A.

According to the notation in Bauer et al. (2007), we deployed the data points

$$(T - t, x_0 + T) \in \{(0, 30), (0, 70), (0, 110), (30, 70), (30, 110), (90, 110)\}.$$

Moreover, we chose the slope parameter of the correction term to be  $a = 0.9$  since this is the average value over all tables of the slope parameter in Gompertz forms fitted to the cohort mortality of a 20-year old in the respective basis years. The resulting values for the volatility parameters are shown in Table 1. Due to the difference in the slope parameter of the correction term, it is impossible to directly compare these values to those obtained by Bauer et al. (2007). However, the standard deviations in absolute value and relative to the forward force of mortality can be compared. In Figure 1, these (spot) standard deviations are plotted

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<sup>6</sup><http://www.actuaries.org.uk>.

Parameters	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$
	0.007	0.287	0.144	0.284	0.021	0.058

Table 1: Estimated parameter values

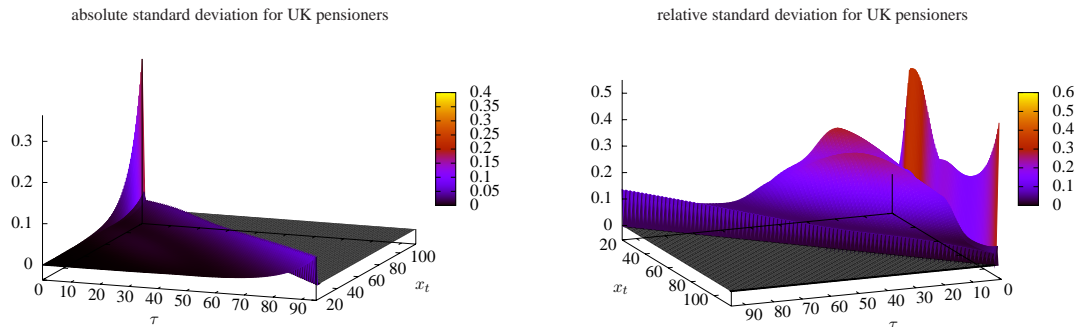


Figure 1: Volatility structure for UK pensioners

using the 1968 generation table scaling the second graph. The most significant differences compared to the US pensioners' model in Bauer et al. (2007) are the stronger short-term effect and the increase in relative volatility for very old ages, where the latter observation is primarily a consequence of the different correction terms.

As explained in Section 3, only a market price of longevity risk  $(\lambda_{(t)})_{t \geq 0}$  is still needed to have a fully specified pricing model at hand. This gap will be filled in the following subsection.

## 4.2 Pricing Methods

In order to compare the pricing approaches by Lin and Cox (2005, 2006) and Milevsky et al. (2005) based on the model introduced in the previous subsection, we derive Sharpe ratios and Wang transform parameters from a monthly time series of UK pension annuity quotes from November 2000 until July 2006<sup>7</sup>. We consider quotes for single premium life annuities payable monthly in advance on 65-year old male lives without guarantee period. For each date, we take the average of 5 available quotes. In order to analyze the pure longevity risk premium, administrative charges have to be eliminated. We assume up-front charges  $\alpha$  between 1.5% and 2% and an investment fee of 7 basis points of the technical reserve p.a. (see also Bauer and Weber (2007)). Moreover, we use Woolhouse's summation formula (cf. Bowers et al. (1986)) to derive the corresponding present values of annuities with annual payments. The resulting present values are then equated with their theoretical counterparts. In case the latter rely on the assumption of a time-constant

<sup>7</sup>Data provided by *Defaqto Ltd.*

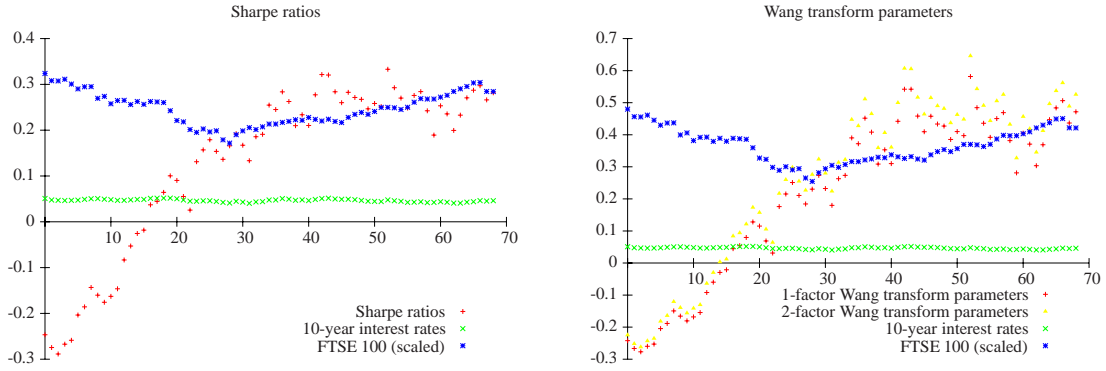


Figure 2: Evolution of the market price of longevity risk

instantaneous Sharpe ratio, we obtain

$$\begin{aligned}
 \mathcal{L}10,000(1 - \alpha) &= R \sum_{k=0}^{\infty} \frac{1}{(1 + i_k - 0.0007)^k} T\tilde{p}_{x_0} \\
 &= R \sum_{k=0}^{\infty} \frac{1}{(1 + i_k - 0.0007)^k} e^{\bar{\lambda} \int_0^k \int_0^s \|\hat{\sigma}(u,s,x_0)\| du ds} T\hat{p}_{x_0}, \quad (10)
 \end{aligned}$$

where  $R$  is the annuity rate,  $i_k$  the annualized interest rate for  $k$  years, and the second equation follows from Equations (2) and (7). The interest rates are taken from the UK government bonds yield curve for the date corresponding to the respective annuity quote as published by the *Bank of England*<sup>8</sup>. Since the yield curve is only given for maturities up to 25 years, we assume a flat yield curve thereafter. The best estimate survival probabilities  $T\hat{p}_{x_0}$  are obtained from the 92 Life Office Pensioners' mortality table. Solving Equation (10) for  $\bar{\lambda}$  for each annuity quote gives a time series of Sharpe ratios. For up-front charges of  $\alpha = 1.5\%$ , this time series is displayed in the left panel of Figure 2, together with the corresponding time series of 10-year interest rates and scaled *FTSE 100* closing prices.<sup>9</sup>

We observe that the Sharpe ratios implied by the first annuity quotes are negative, which means that, at that time, insurers assumed lower survival probabilities than listed in the 92 Life Office Pensioners' mortality table and/or higher expected investment returns than the risk-free interest rate: Until 2000, insurers made large profits from equity investments which may have led them to promise rather high yields to policyholders. However, from November 2000 to March 2003 the *FTSE 100* crashed by about 47%, which then may have forced insurers to lower the offered yields resulting in an increase of the Sharpe ratios. In fact, the correlation between the Sharpe ratio and the *FTSE 100* until March 2003 was -0.908. In the subsequent time period until 2006, on the other hand, the correlation was only 0.383 and, even though the *FTSE 100* index picked up again, the Sharpe ratios remained rather constant. Hence, the insurers seem to have reduced their exposure to equity returns.

We now look at the correlation between the Sharpe ratio and the 10-year interest rate. If insurers never

<sup>8</sup>Downloaded from <http://www.bankofengland.co.uk/statistics/yieldcurve/index.htm> on 01/13/2008.

<sup>9</sup>*FTSE 100* data downloaded from <http://www.livecharts.co.uk/historicaldata.php> on 03/03/2008.

$\alpha$	Sharpe ratios		1-factor Wang parameters		2-factor Wang parameters	
	average	least squares	average	least squares	average	least squares
1.5%	0.2496	0.2496	0.3956	0.3895	0.4488	0.4424
1.75%	0.2445	0.2445	0.3853	0.3792	0.4380	0.4316
2.0%	0.2394	0.2393	0.3750	0.3690	0.4272	0.4209

Table 2: Market prices of longevity risk implied by UK pension annuities

changed their annuity quotes, this correlation would of course be very high. On the other hand, if insurers used pricing rates that move parallel with interest rate movements, this correlation would be very close to 0. The observed correlation between March 2003 and July 2006 was 0.627. This moderate positive correlation indicates that insurers use some “smoothing effects” in determining annuity prices. To some extent, this might be just due to a delayed reaction of the insurers to changing interest rates as insurers usually do not change annuity rates on a daily basis. Another explanation might be drawn from the competition in the annuity market: In a scenario of falling interest rates, no insurer may want to be the first to reduce the annuity rate as the company would loose out on new business. On the other hand, if interest rates increase, the insurer might want to keep a larger profit margin by not immediately increasing its annuity rate. All these effects would result in annuity quotes somewhat smoothing out interest rate movements.

As mentioned above, the Sharpe ratios displayed in Figure 2 have been derived under the assumption of an up-front charge parameter  $\alpha = 1.5\%$ . Assuming larger up-front charges obviously results in smaller Sharpe ratios – however, the overall influence of the parameter  $\alpha$  is rather small. In Table 2, the average values of the Sharpe ratios starting from March 2003 are displayed for different  $\alpha$  together with values of  $\bar{\lambda}$  obtained by a least squares minimization.<sup>10</sup> We find that that increasing  $\alpha$  from 1.5% to 2.0% reduces the Sharpe ratio by only 0.01 and, hence, the assumption on up-front charges is not very decisive. Moreover, we observe that, at least between 2003 and 2006, the average Sharpe ratio implied by UK pension annuities is fairly close to the one of equity markets.

In an analogous fashion to the Sharpe ratio approach, values for the 1-factor and 2-factor Wang transform parameters, i.e.  $\theta_1$  and  $\theta_2$ , can be derived from the UK pension annuity quotes. The respective values are also displayed in Figure 2 and Table 2 and the observations regarding the correlation with the financial market as well as the influence of up-front charges are similar to those based on Sharpe ratios. However, the average transform parameters implied by the UK annuity quotes significantly exceed those from Lin and Cox (2005, 2006) derived from US annuity quotes, i.e. 0.1792 for the 1-factor Wang transform and 0.1524 for the 2-factor Wang transform.

In Section 3, we raised the question of whether or not the Wang transform is adequate for pricing longevity-linked securities. A first indication that the Wang transform may not be adequate is given in Figure 3 where  $T$ -year risk-adjusted survival probabilities for a 65-year old in 2006 are displayed.<sup>11</sup> We observe that the 1-

<sup>10</sup>The least squares expression is the sum of the squared differences of the observed present values and their theoretical counterparts.

<sup>11</sup>The graphs are based on the Sharpe ratio and the Wang transform parameters as implied by the last annuity quote with  $\alpha = 1.5\%$ , i.e.  $\bar{\lambda} = 0.2816$ ,  $\theta_1 = 0.4716$  and  $\theta_2 = 0.5255$ , and the best estimate survival probabilities for 2006 as listed in the 92 Life Office Pensioners’ mortality table.

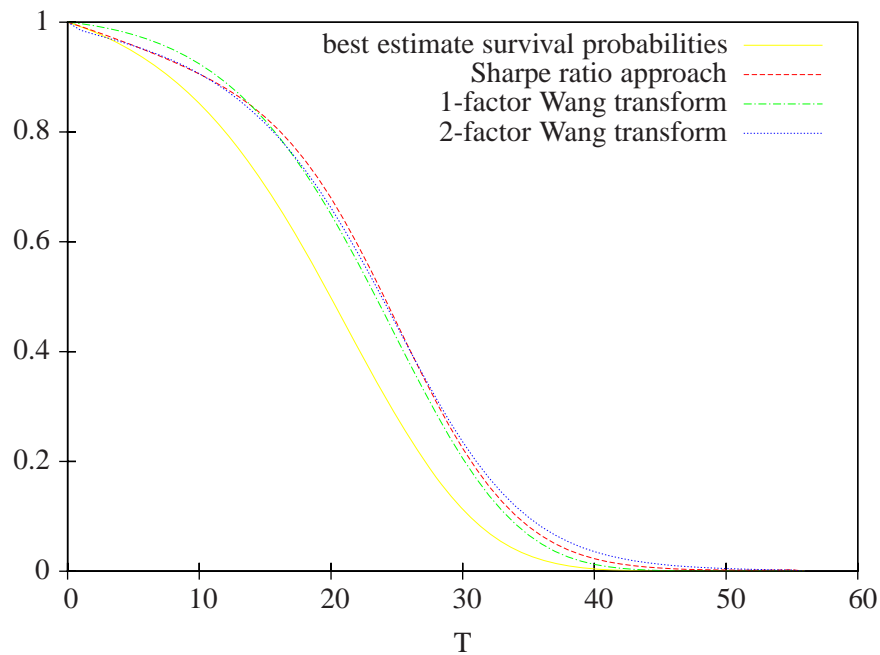


Figure 3: Comparison of risk-adjusted survival probabilities for a 65-year old

factor Wang transform allocates much of the risk premium to shorter maturities and less to longer maturities, whereas for the 2-factor Wang transform, this relation is inverted as much of the risk premium is allocated to longer maturities. In turn, for the first three years, the risk-adjusted survival probabilities are even smaller than the best estimates. For the Sharpe ratio, on the other hand, the allocated risk premium is proportional to the aggregate risk.

In what follows, we consider the different pricing approaches based on the announced – but not issued – EIB/BNP-Bond. By considering both pricing approaches with different parameters, we apply the following six pricing alternatives:

- **SRUK**: Sharpe Ratio approach / from UK Annuity Quotes;
- **SRLOE**: Sharpe Ratio approach / from equity markets (cf. Cairns et al. (2005) and Loeys et al. (2007));
- **1WTUK**: 1-Factor Wang transform / from UK Annuity Quotes;
- **1WTLC**: 1-Factor Wang transform / from Lin and Cox (2005);
- **2WTUK**: 2-Factor Wang transform / from UK Annuity Quotes;
- **2WTLC**: 2-Factor Wang transform / from Lin and Cox (2006).

### 4.3 Comparison of Different Assessments of the EIB/BNP-Bond

In November of 2004, *BNP* announced the issuance of the first publicly traded longevity bond, the so-called EIB/BNP-Bond. While it was withdrawn for redesign in 2005, it still has attracted considerable attention in academia as well as among practitioners.<sup>12</sup>

The notes were to be issued by the *European Investment Bank* (EIB), and the longevity risk was to be taken by the Bahama-based reinsurer *Partner Re*. BNP was the originator and structurer of the deal. The basic design is quite simple: Investors in the bond were entitled to receive variable coupon payments contingent on the mortality experience of English and Welsh males aged 65 in 2003. The coupons  $C(t)$ ,  $t = 1, 2, \dots, 25$ , were set to

$$C(t) = S(t) \text{ £}50\text{mn, where } S(t) = S(t-1) (1 - m(64 + t, 2002 + t)) \text{ and } S(0) = 1.$$

Here,  $m(x, z)$  denotes the central death rate for an  $x$ -year old in year  $z$  as published by the *Office for National Statistics*. Note that there is a time lag of two years between the end of the reference period and the payment date since the mortality experience has to be assessed statistically.

Thus, holding such a security is similar to holding a portfolio of  $(k, 65)$ -longevity bonds as introduced in Section 3 for  $k = 1, \dots, 25$  in 2003, but the payments are delayed by two years and the definition here is based on central death rates rather than mortality probabilities. Now, if more individuals survive than anticipated, coupon payments will be higher; thus, for the next 25 years, the note serves as an almost perfect hedge for an annuity provider whose portfolio of insureds coincides with the reference population.

The total “value” of the issue was £540mn, and the offer price was determined by taking survival rates as projected by the *UK Government Actuary’s Department* and discounting the projected coupon payments at LIBOR minus 35bps. Since, due to its S&P AAA rating, the EIB’s yield curve averages at approximately LIBOR-15, the remaining spread of about 20bps can be interpreted as the fee investors had to pay for the hedge. Lin and Cox (2006) believe that the charged risk premium is very high making the bond unattractive for potential investors. Contrarily, Cairns et al. (2005) figure that this price seems reasonable, even though “it is difficult to judge precisely how good a deal the pension funds are [were] being offered”. It is worth noting that the authors base their conclusions on similar pricing methods to those introduced in Section 2: While Lin and Cox (2006) rely on comparisons of “risk premiums” implied by annuity prices and the EIB/BNP-Bond as parameters in corresponding Wang-transforms, Cairns et al. (2005) take the EIB/BNP risk-premium relative to its “volatility” – i.e. the Sharpe ratio – and compare it to equity risk premiums. Thus, their method is in line with the ideas of Milevsky et al. (2005).

In order to explain these different assessments of the EIB/BNP-Bond, we apply the six pricing methods defined in the previous subsection. In addition, we price two “artificial” bonds of the same structure but assuming that they would have been offered in November 2001 and July 2006, respectively, in order to analyze the influence of different Sharpe ratios and Wang transform parameters on the bond price.<sup>13</sup> The

<sup>12</sup>The following short description of the security is based on Azzopardi (2005), Blake et al. (2006a), and Cairns et al. (2005). See their presentations for more details.

<sup>13</sup>The Sharpe ratios are -0.0834 for November 2001, 0.2674 for November 2004 and 0.2816 for July 2006. The respective Wang transform parameters are -0.0920, 0.4271 and 0.4716 for the 1-factor and -0.0637, 0.4815 and 0.5255 for the 2-factor Wang



	11/2001	11/2004	7/2006
Actual	<i>na</i>	540	<i>na</i>
SRUK	487.56	584.40	605.50
SRLOE	540.42	580.60	597.95
1WTUK	482.19	601.02	618.74
1WTLC	530.87	563.42	576.32
2WTUK	480.03	595.77	612.33
2WTLC	516.91	548.27	560.72

Table 3: Prices of EIB-type bonds using different pricing methods

exact issue dates are assumed to be 16/11/2001, 18/11/2004 and 18/07/2006, the best estimate survival rates are taken from the mortality projections of the *UK Government Actuary's Department* from 2000, 2003 and 2004, i.e. the most recent mortality projection in each case, and the reference cohort is aged 65 in 2000, 2003 and 2005, respectively. The interest rates are obtained from the *commercial bank liability curves* for the respective dates as offered by the Bank of England.<sup>14</sup> The resulting bond prices are shown in Table 3.

We observe that the bond prices for the same pricing method differ considerably between the issue dates. This is due to different interest rates and changes in the mortality projections – as well as to changes in the Sharpe ratio and Wang transform parameters in case these have been derived from the UK annuity quotes. There are also significant differences between the pricing methods for the same issue date which result from the varying risk premium allocations and/or market prices of risk. However, the most striking observation is that, for all six methods, the price of the EIB/BNP-Bond exceeds the £540mn requested by BNP. Hence, our analyzes indicate that the bond was indeed rather “a good deal”. From this observation, two questions arise immediately:

- Why do Lin and Cox (2006) regard the bond as too expensive?
- And why was the bond not successfully placed?

The first question is quite simple to answer as Lin and Cox (2006) used different interest rates (gilt STRIPS) and, in particular, different best estimate survival rates, namely rates based on “*realized mortality rates of English and Welsh males aged 65 and over in 2003*”. Especially in the long term, these survival rates are smaller than those projected by the UK Government Actuary's Department resulting in a lower “fair” price of the bond.

The second question is more difficult to answer. One reason might be the fact that the bond was based on population mortality experience of a specific cohort rather than annuitants' mortality experience and the consequent basis risk may be a reason for its failure (cf. Lin and Cox (2006)). However, as Cairns et al. (2005) point out, differences in mortality improvements between the general and the assured population

transform.

<sup>14</sup>These yield curves, downloaded from <http://www.bankofengland.co.uk/statistics/yieldcurve/index.htm> on 15/03/2008, are derived from yields of LIBOR-linked securities and, thus, already include a premium for default risk. In fact, using the yield curve for 18/11/2004 and a further spread of 20bps for the longevity risk, we obtain a bond value of £541mn which is very close to the £540mn quoted by BNP.

are not very pronounced, even though this observation may differ for a particular insurer's portfolio. Other potential reasons have been pointed out, e.g. that the Bahama based reinsurer Partner Re was not perceived to be a natural holder of UK longevity risk. However, probably the most striking explanations lie in the fixed maturity and the high upfront capital expense (see Cairns et al. (2005)): Due to the fixed maturity of 25 years, insurers and pension funds purchasing the bond would still be stuck with the tail risk, i.e. the longevity risk for high ages in the far future. Moreover, the securitization of the "complete" survivor index takes capital away which may be used to hedge other sources of risk or to speculate in capital markets. After all, insurers are financial service providers meaning that investing in capital markets can be regarded as one of their core competences.

To sum up, aside from basis risk, the financial engineering of the EIB/BNP-Bond may present an important reason for its failure. Thus, in the next section, we propose a differently designed longevity derivative, which overcomes some of the stylized deficiencies.

## 5 An Option-type Longevity Derivative

As detailed out in the previous subsection, the EIB/BNP-Bond may not seem very attractive to insurers. Therefore, we propose a differently designed derivative here: a call option-type longevity derivative with a payoff of the form<sup>15</sup>

$$C_T = \left( {}_T P_{x_0}^{(T)} - K(T) \right)^+,$$

where  $K(T)$ ,  $0 \leq K(T) \leq 1$ , is some threshold or *strike*, for example

$$K(T) = (1 + a) E_P \left[ {}_T P_{x_0}^{(T)} \right], \quad a > 0.$$

This security or a combination of securities of this type overcome several deficiencies of the EIB/BNP-Bond and could thus be more appealing to insurers. For example, the insurer keeps the "equity tranche" of the longevity risk exposure in its own books and only passes over the risk of extreme longevity. In particular, this will significantly decrease the committed capital. We refer to Bauer (2007), Section 5.1, for a more detailed discussion.

Within the setup introduced in 3, such a derivative can be priced by a Black-type formula (see Bauer (2007)). One only needs to provide the volatility of mortality, the best estimate and the risk-adjusted  $T$ -year survival probability for an  $x_0$ -year old today and the price of a "regular" financial bond with maturity  $T$ . In Table 4, prices of this option-type longevity derivative are presented for a 65-year old in July 2006, the six pricing methods defined in the Section 4.1, and various maturities  $T$  and strike parameters  $a$ . The best estimate survival probabilities and the interest rates are the same as in the analysis of the last annuity quote (from 10/07/2006). "Portfolio" refers to a combination of longevity bonds for "each" maturity  $T$ ,  $1 \leq T \leq 55$ , i.e. a portfolio which completely hedges a provider of a life-long annuity for a 65-year old with annual payments against extreme longevity risk.<sup>16</sup>

<sup>15</sup>This derivative has already been introduced in Bauer (2007) and similar payoff structures can also be found in, e.g., Blake et al. (2006a) or Lin and Cox (2005).

<sup>16</sup>Note that we assume a limiting age of 120 as implied by the underlying mortality table.

$a$		$T = 5$	$T = 10$	$T = 15$	$T = 20$	$T = 25$	$T = 30$	Portfolio
2%	SRUK	0.00486	0.03678	0.07483	0.09079	0.07223	0.03827	1.69716
	SRLOE	0.00443	0.03357	0.06752	0.08071	0.06316	0.03277	1.49682
	1WTUK	0.01532	0.04780	0.07357	0.08069	0.06447	0.03430	1.64501
	1WTLC	0.00634	0.02669	0.04235	0.04386	0.03127	0.01442	0.84800
	2WTUK	0.00580	0.03838	0.07034	0.08423	0.07168	0.04237	1.70705
	2WTLC	0.00075	0.01694	0.03545	0.04008	0.03066	0.01771	0.78157
5%	SRUK	0.00049	0.02639	0.06665	0.08573	0.06977	0.03744	1.54034
	SRLOE	0.00043	0.02372	0.05973	0.07589	0.06081	0.03197	1.34823
	1WTUK	0.00342	0.03582	0.06545	0.07587	0.06210	0.03348	1.45405
	1WTLC	0.00076	0.01816	0.03630	0.04030	0.02957	0.01386	0.72076
	2WTUK	0.00066	0.02772	0.06239	0.07933	0.06922	0.04151	1.54745
	2WTLC	0.00003	0.01075	0.03003	0.03671	0.02899	0.01708	0.68892
10%	SRUK	0.00025	0.01366	0.05422	0.07768	0.06578	0.03606	1.34206
	SRLOE	0.00022	0.01195	0.04800	0.06826	0.05703	0.03066	1.16231
	1WTUK	0.00215	0.02013	0.05313	0.06824	0.05829	0.03216	1.23298
	1WTLC	0.00041	0.00855	0.02762	0.03488	0.02693	0.01296	0.57846
	2WTUK	0.00035	0.01454	0.05038	0.07154	0.06525	0.04010	1.34837
	2WTLC	0.00001	0.00445	0.02240	0.03159	0.02637	0.01608	0.57466
15%	SRUK	0.00025	0.00617	0.04336	0.07012	0.06195	0.03473	1.18980
	SRLOE	0.00022	0.00524	0.03790	0.06115	0.05341	0.02940	1.02093
	1WTUK	0.00215	0.00996	0.04241	0.06113	0.05464	0.03087	1.07441
	1WTLC	0.00041	0.00348	0.02062	0.03005	0.02448	0.01212	0.47984
	2WTUK	0.00035	0.00667	0.03998	0.06043	0.06142	0.03872	1.19829
	2WTLC	0.00001	0.00158	0.01638	0.02707	0.02395	0.01514	0.49055

Table 4: Prices of call-option-type payoff using different pricing approaches

Obviously, the prices decrease with increasing strike parameter  $a$ . The equality of prices for  $T = 5$  and  $a = 10\%$  and  $15\%$  are due to the strike being capped at 1. As expected, the bond prices initially increase with maturity as the realized survival probability is more likely to significantly exceed today's expectation in the farther future. However, around  $T = 20$  this trend is reversed since the absolute value of the survival probability becomes very small thereafter. We also observe that, compared to the Sharpe ratio approach, the 1-factor Wang transform leads to significantly higher prices for short maturities and lower prices for longer maturities as a result of the disproportional risk premium allocation (cf. Subsections 3.3 and 4.2). For instance, for  $T = 5$  and using the UK data, the price obtained by the 1-factor Wang transform pricing method is between about three and eight times as high as the price obtained using the Sharpe ratio approach. The differences between the 2-factor Wang transform and the Sharpe ratio approach are not quite as strong but still significant.

In order to assess the question of "how good of a deal" this derivative is from the insurer's perspective, we compare the price of a portfolio of our option-type longevity derivatives with the price of a life-long immediate annuity paying £1 yearly in advance. The average price of the annuity paying £1 in July 2006 was £14.96. Hence, for a rather low strike parameter ( $a = 2\%$ ) and the market price of longevity risk as

implied by UK pension annuities, the insurer would have to pay about 11.3% of the single premium for a full coverage against extreme longevity improvements. In case the insurer is willing to accept a larger portion of the risk, e.g. for  $a = 15\%$ , he can reduce his securitization costs to approximately 8% of the single premium. While, at first glance, such a portfolio may seem rather expensive considering that an insurer would have to pay most of his margin for this coverage against longevity risk (cf. Subs. 3.2). However, for  $a = 2\%$ , the insurer would hardly be exposed to longevity risk while, at the same time, the company would still profit from realized longevity improvements below today's expectation. This justifies a rather high price for the portfolio.<sup>17</sup>

Moreover, an insurer could reduce the cost of longevity coverage by incorporating the company's mortality exposure from the sale of life insurance policies. This idea of compensating longevity risk by mortality risk is often referred to as *natural hedging* (see, e.g., Bayraktar and Young (2007), Cox and Lin (2004), or Wetzel and Zwiesler (2008)). Although such a compensation is only partially partially since the age cohorts exposed to these risks are typically different and the maturities of annuities generally exceed those of life insurance policies, the number of longevity-linked securities necessary to reduce the longevity risk to a desired level can certainly be reduced.

## 6 Conclusion

The current paper analyzes and compares different approaches for pricing longevity-linked securities. Milevsky et al. (2006) postulate that an issuer of such a security should be compensated for taking longevity risk according to a pre-specified instantaneous Sharpe ratio. Lin and Cox (2005, 2006), on the other hand, apply the Wang transform to best estimate death probabilities in order to account for a risk premium.

However, as explained in detail in Section 3, the risk premium implied by the Wang transform is not consistent – arbitrage opportunities may appear as soon as multiple securities based on different cohorts are traded. Even if only one security is considered, the disproportional risk premium allocation contests the adequacy of an application of the Wang transform for pricing longevity derivatives (cf. Sec. 5).

Moreover, we discuss the derivation of an adequate market price of longevity risk, i.e. a Sharpe ratio or Wang transform parameter in our case. Milevsky et al. (2005) do not address this question but Loeys et al. (2007) propose to adopt a Sharpe ratio from equity markets. We believe that this approach is not appropriate since peculiarities of the equity markets may not be present in the longevity market. In contrast, Lin and Cox (2005, 2006) obtain their Wang transform parameters from US annuity quotes which is more sound from our point of view.

We compare the different pricing approaches based on UK data. In particular, we drive a time series for the market price of risk within market annuity quotes and analyze the relationship to interest rates and the stock market. We find considerable correlations indicating that the independence assumption of the risk-adjusted mortality evolution and the development of the financial market may not be adequate.

We then apply the different approaches to assess the first announced – but never issued – longevity bond,

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<sup>17</sup>In our situation, i.e. for the annuity quote from July 2006, the profit margin of the insurer would be approximately 11.6% if mortality evolved as expected.

the so called EIB/BNP-Bond. For each of the considered methods, the price exceeds the £540mn quoted by BNP, i.e. the Bond appears to have been a rather “good deal”. Consequently, there must have been other reasons for its failure, for example the financial engineering. We propose a differently designed and potentially more suitable security, namely an option-type longevity derivative. Such a security allows an insurer to keep the “equity tranche” of the longevity risk in the company’s own books. We derive prices for derivatives of this kind based on the different methods and compare the results. All in all, we believe that such option-type longevity derivatives could become successful tools for securitizing longevity risk.

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## Appendix

### A Underlying Tables for the Calibration of the Mortality Model

- Period table PA(90)m projected backward (until 1968) and forward using the projection applied in constructing this table;<sup>18</sup>
- Basic table PMA80 and projection as published in the “80” Series of mortality tables;
- Basic table PMA92 and projection as published in the “92” Series of mortality tables;
- Basic table PMA92 and the medium cohort projection published in 2002;
- Basic table PNMA00 from the “00” Series of mortality tables and the Lee-Carter projection based on data up to 2003 (part of the *Library of mortality projections*);
- Basic table PNMA00 from the “00” Series of mortality tables and the Lee-Carter projection based on data up to 2004 (part of the *Library of mortality projections*);
- Basic table PNMA00 from the “00” Series of mortality tables and the Lee-Carter projection based on data up to 2005 (part of the *Library of mortality projections*).

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<sup>18</sup>The PA(90)m table is a projected version of a period table from 1968. Hence, the best estimate mortality forecast in 1968 in form of a generation table would have been based on the period table for 1968 and the aforementioned mortality projection.