Mortality Risk Modeling: Applications to Insurance Securitization

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MORTALITY RISK MODELING: APPLICATIONS TO INSURANCE SECURITIZATION

ABSTRACT. This paper proposes a stochastic mortality model featuring both permanent longevity jump and temporary mortality jump processes. A trend reduction component describes unexpected mortality improvement over an extended period of time. The model also captures the uneven effect of mortality events on different ages and the correlations among them. The model will be useful in analyzing future mortality dependent cash flows of life insurance portfolios, annuity portfolios, and portfolios of mortality derivatives. We show how to apply the model to analyze and price a longevity security.

1. INTRODUCTION

Over the past half century, and especially in the most recent decades, remarkable mortality improvements have led to the growth of the population of older people (Bourguignon and Morrisson 2002, Lakdawalla and Philipson 2002, Vaupel 1998). To the extent that this progress is unexpected, it has a negative impact on pension plans and annuity providers. In the US, private defined benefit pension plans currently have close to $6 trillion in liabilities for future benefits. In addition, US life insurers hold approximately $2 trillion in annuity reserves (Salou and Hu 2006, ACLI 2006). Uncertainty of longevity improvements increases risk for pension funds and annuity insurers since annuity benefits may need to be paid longer than expected. In a recent study of pension liabilities of the companies in the UK's FTSE100 index, Cowling and Dales (2008) found overly optimistic longevity assumptions for pension valuations reported at the end of 2007. These authors believe that companies underestimate future life expectancy by one to three years and, therefore, understate the aggregate pension deficit of these companies by as much £40 billion.

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In contrast, in modeling life insurance, one naturally looks at scenarios which are pessimistic about mortality improvement and often include threats such as epidemics. Several recent articles focus on management such mortality risks, including Hardy (2005), Rogers (2002), Goss et al. (1998), Cox et al. (2006), Cox and Lin (2007) and Lin and Cox (2008). Genetic analysis has confirmed that the virus of "Spanish flu" which killed 40 to 100 million people in 1918 developed in birds and was similar to today's "bird flu" (Juckett 2006). Some public health experts think that a pandemic is overdue and another will inevitably occur (Dowdle 2006). Should a pandemic occur, a life insurer will suffer financially since it will pay more death benefits than expected when the policies were issued. It seems reasonable that a major pandemic event could trigger turbulence in the life insurance industry. Toole (2007) studied this issue in detail and concluded that the industry as a whole can withstand a severe pandemic, as severe as the 1918 pandemic, with a loss of about $64 billion relative to aggregate risk-based capital (RBC) of about $256 billion in 2005. While the industry as a whole can sustain a severe flu pandemic, it would nevertheless be disruptive. Companies holding less than 100 percent of RBC (only 14 companies in 2005) may become insolvent. As Toole (2007) notes, if a severe pandemic were to occur when the financial markets are weak (*e. g.* when asset values such as mortgage-backed securities are depressed), then the financial impact could be much worse and the number of companies near insolvency could be much greater. Thus, it seems clear that including pandemic effects is an important issue in modeling mortality for life insurance liabilities.

In light of the above discussion, the terms *mortality risk* and *longevity risk* are opposite in this paper. Mortality risk is the risk of more deaths than expected, or the risk that observed death probabilities are higher than expected. We are usually thinking that severe, short-termed events such as pandemics underlie mortality risk. On the other hand, more lives may survive than expected or observed death rates may be lower than expected. Unanticipated improvements in medicine and health technology may give rise to longevity risk, leading to a surge in mortality improvement in a short period (e.g. one or two years) or excess improvement over a long period (e.g. more than ten years). For longevity events with a long-term effect, the slope of the mortality curve may deviate from that of the base trend with normal deviations, as evident, for example, in the US population mortality dynamics in the 1970's. Actual mortality has been improving so parametric models estimated with actual data will reflect improving mortality. Models that have a

random deviation from the expected mortality may reflect some longevity risk, but this is not a reflection of a fundamental change in the trend (as we have in mind). Our model, for which we will provide a detailed discussion later, allows for such fundamental changes.

Marocco and Pitacco (1998), Milevsky and Promislow (2001), Dahl (2004), Miltersen and Persson (2005), Cairns et al. (2006), Dahl and Møller (2006), Gründl et al. (2006), Ballotta and Haberman (2006), Biffis and Millossovich (2006) and Bauer and Kramer (2007) focus on mortality risk as deviations from a trend. In those papers, the base trend may reflect mortality improvements, but longevity risk is not modeled explicitly. On the other hand, pension and annuity research has focused on longevity risk. Loise and Serant (2007) model longevity (and mortality) risk using a stationary Gaussian process. However, this model may not be appropriate for modeling longevity (and mortality) shocks which may not follow a stationary Gaussian process. To describe the stochastic longevity trend, Hári et al. (2008) extend the Lee-Carter model with a time-varying, stochastic drift. Biffis (2005) captures mortality random departures around a time-varying target with a longevity compound Poisson process, but that model cannot guarantee a nonnegative force of mortality.

US mortality data has only two extreme events, 1918 flu pandemic and the change in the rate of improvement of mortality around 1970. However, experts conjecture that we may experience extreme events of both types in the future. This means that in investigating the impact of possible future events, we cannot simply calibrate our models to the experience. Therefore, we offer a flexible, although somewhat complex, method of including expert opinion in forming future mortality scenarios.

Specifically, we propose a new approach by introducing a trend reduction jump component to describe longevity risk. Unexpected longevity improvement, in general, may be less dramatic than that of a mortality death shock but in the long run longevity risk may be just as important. Most of longevity risk events in the past seem to have a pattern: unexpected survival gains often extended over a long period of time, leading to a steeper downward sloping force of mortality curve. The traditional one-time jump models such as the model that combines a geometric Brownian process and a compound Poisson jump process, can not provide this kind of longevity risk.

Moreover, to obtain relevant results, a stochastic mortality model must reflect three major features of the current mortality universe: (a) both mortality improvement and deterioration jump factors, (b) correlation among different ages and over time, and (c) uneven effects of a mortality jumps across different ages. Our model attempts to address these issues. In particular, we are explicitly combining both mortality and longevity risk in a single, comprehensive model in order to make a more realistic assessment of future survivor dependent cash flows. Finally, we show how to determine a parsimonious version with historical data.

As an application to mortality risk management, Blake et al. (2006) examine a wide variety of longevity bonds. We continue in the same vein, but with explicit mortality derivatives such as options which we think can be important tools in managing mortality and longevity risk. Derivatives can be written on indices based on publicly available data like the LifeMetrics index (Coughlan et al. 2007) offered by Goldman-Sachs[1]and the Credit Suisse index[2].

Our paper is organized as follows. In Section 2 we describe mortality dynamics as a combination of a Lee-Carter diffusion process, a permanent longevity jump process and a temporary mortality jump process. In Section 3 we discuss capital market solutions for mortality and longevity risk, including new structures with our proposed longevity index respectively. We show how to price those securities with the indifference pricing method in Section 4. We conclude the article in Section 5.

2. Dynamic Mortality Model

Stochastic mortality models usually start with the assumption that there is an "initial" curve for the force of mortality. For a life (x) at time 0, the remaining life time random variable is denoted $T(x)$. The force of mortality at time $t > 0$ is denoted $\mu(x+t,t)$. In effect, the force acting during the life of a person age x at time $t = 0$ is a stochastic process $\{\mu(x+t,t) : t \geq 0\}$ evolving from an initial curve $\mu(x,0) = \mu(x)$. Mathematically speaking this is very complex. In order to reduce the complexity, the model specifies the force of mortality as a parametric function of age and time, with a small number of parameters.

Statistical methods applied in actuarial practice use this same approach. The expected value, or forecast value, of the force of mortality $\mu(x+t,t)$ acting on

[1] http://www.qxx-index.com

[2] http://www.credit-suisse.com/ib/en/fixed_income/longevity_index.html

$(x+t)$ at time t, given $T(x) > t$, is used to price life insurance and annuity polices. However, actual mortality experience can change in a nondeterministic way and deviate from expected mortality rates.

Here is how we will model these events. The aggregate effect on (x) over the period $(0, h)$ is a random variable since it is the path integral over the sample path $\{\mu(x+t, t) : 0 \leq s \leq h\}$ of the force of mortality:

$$\int_0^h \mu(x+t, t)\, dt.$$

The probability $S(x, h)$ that (x) will attain age $x+h$ is random at time 0, because it depends on the sample path of the force of mortality:

$$S(x, h) = \exp\left(-\int_0^h \mu(x+s, s)\, ds\right).$$

The probability that (x) will attain age $x+h$ is the expected value of the probability:

$$\Pr\left[T(x) > h\right] = \mathrm{E}\left[S(x, h)\right]$$

Among the reasons for changes in the force of mortality are the following:

- general trends in health and mortality resulting in gradual improvements to mortality,
- deviations from the general trends,
- relatively rare advances in medical technology resulting in sudden and permanent improvements in mortality, and
- sudden catastrophic surges in mortality from events such as pandemics resulting in temporary increases in mortality.

Therefore, it is natural to develop a model for mortality changes based on the evolution of the force of mortality with general trend, diffusion process and jump components. A number of recent studies have sought to model mortality trend involving both age-dependent and time-dependent terms (Lee and Carter 1992, Renshaw et al. 1996, Lee 2000, Sithole et al. 2000, Milevsky and Promislow 2001, Olivieri and Pitacco 2002, Dahl 2004, Cairns et al. 2006). Our model is inspired by and closely related to this stream of research. As an extension, we explicitly model both mortality and longevity jump processes in a single model (See Section 2.2).

2.1. **Modeling General Mortality Trends.** We adopt the well-known Lee and Carter (1992) mortality model as a basis. This model captures the evolution of mortality in mutually exclusive age cohorts, while at the same time a time-series common risk

factor, $k(t)$, links all cohorts together. The force of mortality $\mu^{LC}(x,t)$ is modeled as[3]

$$\mu^{LC}(x,t) = \exp\left[a(x) + b(x)k(t)\right]. \tag{2.1}$$

The parameters $a(x)$ and $b(x)$ are age-specific while $k(t)$ is time varying.

We fit the model to the US male data for ages $x = 0, 1, \ldots, 103$ and times $t = 1901, 1902, \ldots, 2005$, using the technique suggested by Lee and Carter (1992). Tables for years 1901 to 1999 are from the Human Life Table Database.[4] Tables for 2000 to 2005 are from the Human Mortality Database.[5] The estimates of $a(x)$ and $b(x)$ are in Table 1 and the estimated $k(t)$ is shown in Figure 1. There are several generalizations of Lee and Carter (1992) model in the literature. We believe that our ideas illustrated below could be applied to them as well by using the original Lee-Carter model.

Following Lee and Carter (1992), the mortality index $k(t)$ evolves as

$$k(t+1) = k(t) + g_1 + g_2 \times \text{Flu} + \sigma z(t), \qquad t = 1901, 1902, \ldots, 2005 \tag{2.2}$$

where g_1, g_2, and σ are constants and $z(1901), z(1902), \ldots, z(2005)$ are independent standard normal random variables. The flu in 1918 is identified by a dummy variable, Flu, in equation (2.2). Specifically, Flu $= 1$ if $t = 1918$, and 0 otherwise. Then we obtain the estimated $g_1 = -0.2244$, $g_2 = 2.2065$, and $\sigma = 0.6123$. Simulating mortality scenarios with the Lee-Carter model, involves simulating future mortality index values $k(t)$ for $t > 2005$. For convenience we now re-label the years so that year 2005 corresponds to $t = 0$. For a person age x in 2005, we can generate Lee-Carter model scenarios for future values of the force of mortality as follows: For $x = 0, 1, \ldots, 103$,

$$\begin{aligned} &\mu^{LC}(x,0) = \mu(x) \text{ observed in 2005} \\ &\mu^{LC}(x,t) = \exp\left[a(x) + b(x)k(t)\right] \qquad t > 0. \end{aligned} \tag{2.3}$$

[3]Lee and Carter use the central death rate rather than the force of mortality. For reasonable assumptions about the distribution of deaths between age x and $x+1$, the two measures are the same or very close. For example, if the force of mortality is constant between integral ages, the central death rate at age x and the force of mortality at age x are equal. The data base has values of $q(x,t)$. Under the same assumption, $q(x,t) = 1 - \mathrm{e}^{-\mu(x,t)}$ so we can easily change from $q(x,t)$ to $\mu(x,t)$.

[4]Data source: Human Life Table Database. Max Planck Institute for Demographic Research (Germany), University of California, Berkeley (USA) and the Institut national d'études démographiques (France). Available at `www.lifetable.de` (data downloaded on June 8, 2008).

[5]Data source: Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at `www.mortality.org` or `www.humanmortality.de` (data downloaded on June 8, 2008).

TABLE 1. The estimates of $a(x)$ and $b(x)$ based on the US Male Population data from 1901 to 2005.

Age	$a(x)$	$b(x)$	Age	$a(x)$	$b(x)$	Age	$a(x)$	$b(x)$
0	-3.4039	0.1736	35	-5.5990	0.1061	70	-3.0020	0.0392
1	-5.6955	0.2612	36	-5.5442	0.1043	71	-2.9207	0.0398
2	-6.2000	0.2374	37	-5.4851	0.1017	72	-2.8391	0.0400
3	-6.4963	0.2287	38	-5.4259	0.0990	73	-2.7575	0.0396
4	-6.7038	0.2269	39	-5.3661	0.0966	74	-2.6766	0.0393
5	-6.8571	0.2171	40	-5.3015	0.0934	75	-2.5950	0.0384
6	-6.9863	0.2029	41	-5.2354	0.0903	76	-2.5132	0.0378
7	-7.1111	0.1908	42	-5.1666	0.0872	77	-2.4290	0.0377
8	-7.2240	0.1820	43	-5.0960	0.0841	78	-2.3429	0.0381
9	-7.3290	0.1820	44	-5.0242	0.0812	79	-2.2492	0.0371
10	-7.3986	0.1878	45	-4.9509	0.0784	80	-2.1687	0.0391
11	-7.3761	0.1883	46	-4.8750	0.0752	81	-2.0834	0.0391
12	-7.2273	0.1737	47	-4.7995	0.0721	82	-2.0022	0.0384
13	-6.9989	0.1495	48	-4.7227	0.0687	83	-1.9251	0.0370
14	-6.7715	0.1278	49	-4.6471	0.0655	84	-1.8509	0.0352
15	-6.5665	0.1113	50	-4.5699	0.0621	85	-1.7774	0.0335
16	-6.3902	0.0992	51	-4.4928	0.0593	86	-1.7025	0.0319
17	-6.2527	0.0934	52	-4.4151	0.0569	87	-1.6262	0.0309
18	-6.1422	0.0903	53	-4.3380	0.0554	88	-1.5482	0.0303
19	-6.0618	0.0911	54	-4.2612	0.0544	89	-1.4695	0.0298
20	-5.9908	0.0944	55	-4.1827	0.0532	90	-1.3906	0.0295
21	-5.9233	0.0959	56	-4.1041	0.0521	91	-1.3116	0.0290
22	-5.8849	0.0986	57	-4.0249	0.0506	92	-1.2336	0.0282
23	-5.8732	0.1006	58	-3.9452	0.0488	93	-1.1572	0.0274
24	-5.8802	0.1021	59	-3.8663	0.0471	94	-1.0828	0.0265
25	-5.8953	0.1039	60	-3.7859	0.0453	95	-1.0142	0.0269
26	-5.9028	0.1050	61	-3.7063	0.0440	96	-0.9457	0.0265
27	-5.9019	0.1067	62	-3.6267	0.0423	97	-0.8779	0.0265
28	-5.8855	0.1082	63	-3.5503	0.0410	98	-0.8122	0.0267
29	-5.8537	0.1092	64	-3.4739	0.0395	99	-0.7502	0.0268
30	-5.8191	0.1101	65	-3.3970	0.0383	100	-0.6867	0.0270
31	-5.7830	0.1100	66	-3.3189	0.0375	101	-0.6228	0.0277
32	-5.7435	0.1099	67	-3.2396	0.0370	102	-0.5587	0.0262
33	-5.6993	0.1091	68	-3.1614	0.0376	103	-0.5230	0.0222
34	-5.6519	0.1079	69	-3.0822	0.0383			

When we use the model to simulate future values of the force of mortality applying to a life age x at $t = 0$ over a period of T years

$$\mu^{LC}(x, 0), \mu^{LC}(x + 1, 1), \mu^{LC}(x + 2, t + 2), \ldots, \mu^{LC}(x + T, T),$$

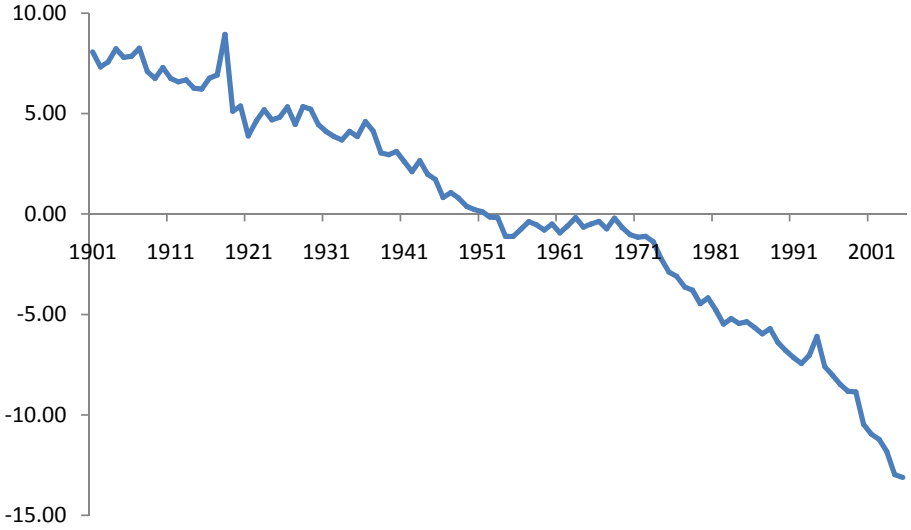

FIGURE 1. Estimated time-series common risk factor $k(t)$ shown in the vertical axis for year $t = 1901, 1902, ..., 2004, 2005$ in the horizontal axis.

we need to provide for the cases for which $x + j > 103$ and $j > 0$. In other words, the way we use the model requires extending equation (2.3) to ages $x > 103$ after the first year. A natural approach is to simply extend both functions $a(x)$ and $b(x)$ linearly using the values for ages 90 to 103 as a basis for the extrapolation. This approach results in the linear extrapolation of $a(x)$ and $b(x)$,

$$a(x) = -7.43 + 0.067x$$
$$b(x) = 0.055 - 0.0003x$$

extending equation (2.3) to $x > 103$ for $t > 0$. This is consistent with models with linearly changing mortality, as described by Schoen (2006). In applications for which "oldest-of-old mortality" is critical other approaches might be necessary.

2.2. **Modeling Mortality Jumps.** Our goal is a stochastic mortality model suitable for dynamic financial analysis, taking into account the complexity of observed mortality dynamics. We consider two types of mortality shocks: permanent jump $G(x,t)$ and temporary jump $H(x,t)$.

The unexpected mortality improvement that results from genetic, environmental, behavioral, bio-reliability, and/or heterogeneity forces and constraints, often has a long-term effect on future mortality rates. In contrast, many catastrophe death events, like the 1918 worldwide flu and the 2008 earthquake in China, have

a more severe but transitory effect on death rates. In many cases, these short-term events drive up the death rate just for a couple of years, but the mortality falls back to the normal level afterwards. Accordingly, we view sharp decreases in mortality as "permanent" events, for instance, associated with aforementioned medical advances while dramatic increases as "temporary" events such as pandemics.

When we add these two types of mortality jump events $G(x,t)$ and $H(x,t)$ to the Lee-Carter force of mortality $\mu^{LC}(x,t)$, the initial version of our model takes the form

$$\mu(x,t) = \mu^{LC}(x,t) \times \exp\left(-G(x,t) + H(x,t)\right), \tag{2.4}$$

where $\mu^{LC}(x,t)$ is the force of mortality from the Lee-Carter model (2.3).

2.2.1. *Permanent Longevity Jump $G(x,t)$.* The unexpected mortality improvement process $G(x,t)$ consists of a jump reduction component $K(x,t)$ and a trend reduction component $D(x,t)$. That is,

$$G(x,t) = K(x,t) + D(x,t).$$

The jump reduction component $K(x,t)$ is the one-time longevity jump induced by a surge in survival rates *over a short period* (for example, a year) with a permanent effect on longevity. It is defined as

$$K(x,t) = \sum_{s=1}^{\infty} y_s A_s(x) \mathbf{1}_{\{t \geq \eta_s\}}, \tag{2.5}$$

where η_s is the time of jump reduction event s. The time η_s can be modeled as the arrival time of the s-th event of a point process, as described in Daley and Vere-Jones (2003). The simplest version would be the arrival times of a Poisson process, in which case the times would have gamma distributions. A second feasible choice would be a Hawkes process. The primary difference between these two approaches lies in the ability of a Hawkes process to create event clustering.

The positive variable y_s is the *maximum* mortality improvement of all ages in medical advancement event s. The effect of y_s is transferred to the mortality rates through a function $A_s(x)$ for age x with $0 \leq A_s(x) \leq 1$. When $A_s(x) = 0$, medical advancement s has no effect on age x. In contrast, $A_s(x) = 1$ means age x enjoys the biggest mortality improvement (y_s) among all ages. When $0 < A_s(x) < 1$, it captures the effects that fall between the above two extremes. That is, the function $A_s(x)$ differentially spreads the mortality improvement across x's because, when a

medical advancement occurs, the benefits of the technology usually vary by age. For example, compared to demography changes of the whole population in the 1970's, the annual improvement in mortality of old ages in the same period is much more dramatic than that of young ages. In fact, Cutler and Richardson (1998) find that improvement for the elderly was greater than for the young in the 1970's as a result of a decrease in cardiovascular disease deaths. Since cardiovascular disease is more prominent late in life than earlier, the life expectancy gain is greater for the elderly than for the young. Mortality improvements due to a change in a cause of death may vary by age.

There is no definitive data set to say what the function $A_s(x)$ should look like. Practitioners will use a conservative choice, depending on the products being modeled. If you think the shape shown in Figure 2 may be more realistic in the future, you must believe that there is little mortality improvement for the very young and very old.

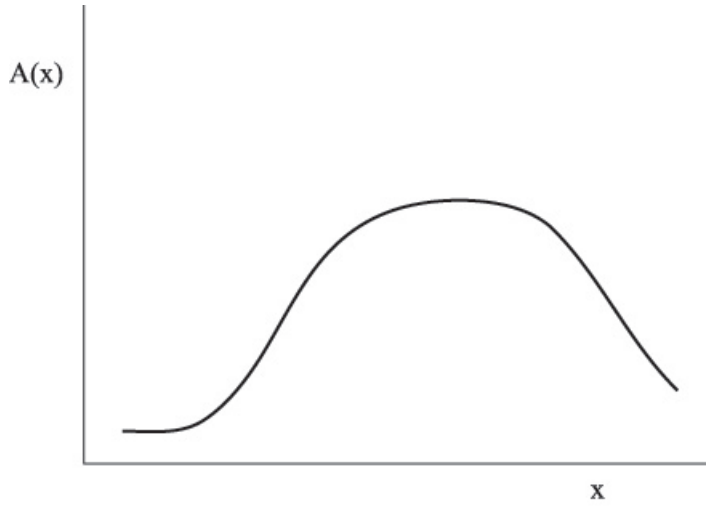

FIGURE 2. Medical advancement by age.

Furthermore, the annual mortality rate *over a long period* (e.g. twenty years) may continue to improve at a much higher rate than that in other periods, leading to a steeper downward-sloping force of mortality curve. For example, the *annual* percentage decrease in death rates for the US population aged 55 and above in the 1970's, on average, is about three times that of the same age cohort in other periods. This type of mortality improvement, in a certain sense, is more significant than the one-time longevity jump $K(x,t)$. The data shows no one-time longevity

surge across ages (like a 20 percent decrease in $\mu(x,t)$ during one year), but that does not mean there was no longevity risk. The cumulative effect of unanticipated mortality improvement over an unexpectedly long period of time obviously has a devastative effect on the financial stability of pensions plans and annuity markets.

Therefore, we model this long-term decay in mortality rates by introducing a trend reduction component $D(x,t)$ defined as follows:

$$D(x,t) = \sum_{i=1}^{\infty} \zeta_i (t - \upsilon_i) F_i(x) \exp(-\xi_i (t - \upsilon_i)) \mathbf{1}_{\{t \geq \upsilon_i\}}, \tag{2.6}$$

where υ_i is the time of trend reduction event i. The time of the trend reduction event υ_i can be modeled like the time η_s, which we discussed after equation (2.5). The percentage change in $\mu(x,t)$ is the log μ-ratio, $\log \dfrac{\mu(x,t+1)}{\mu(x,t)}$. When a trend reduction event occurs, we calculate the difference between its average percentage change in the trend reduction period and its average level in the whole observation period. In equation (2.6), ζ_i represents the *maximum* excess change across ages given $\zeta_i > 0$. For instance, consider male mortality in the 1970's: age $x = 0$ had the biggest annual excess longevity gain 0.0211 above its average improvement from 1901 to 2005, so $\zeta_i = 0.0211$.

Like $A_i(x)$, the function $F_i(x)$, with $0 \leq F_i(x) \leq 1$, distributes age effects. Given $\zeta_i = 0.0211$, we have $F_i(0) = 1$ in the 1970's. Based on the historical data, we can obtain other $F_i(x)$'s. For example, $F_i(75) = 0.3723$ for age 75. This means $\mu(75,t)$ decreases by an excess $0.0211 \times 0.3723 \times 100 = 0.79$ percent per year during this trend reduction period. Moreover, the factor $(t - \upsilon_i)$ provides cumulative mortality improvement as t increases beyond the jump event time υ_i.

How long the aforementioned mortality trend reduction will last *in the future* is a topic for debate: ten years, twenty years or even longer? The parameter $\xi_i > 0$ specifies the length of trend reduction event i. Should one believe that history will reflect the future, when $\zeta_i = 0.0211$ and $F_i(75) = 0.3723$, for example, one might choose $\xi_i = 0.01$. In this case, when an unexpected medical advancement causes an annual excess decrease in $\mu(75,t)$, the force of mortality curve for age 75 will have a notable steeper downward slope for about 35–40 years, the pattern similar to the curve for age 75 after year 1970 in Figure 3.

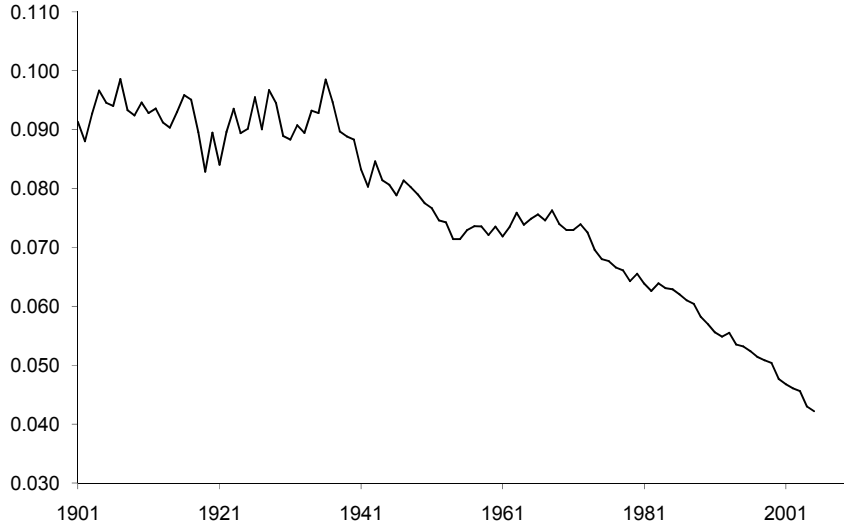

FIGURE 3. Actual force of mortality $\mu(75,t)$ for age 75 in year t. The vertical axis stands for force of mortality and the horizontal axis represents year.

2.2.2. *Temporary Adverse Mortality Jump $H(x,t)$.* The transitory mortality jump process $H(x,t)$ is defined as follows:

$$H(x,t) = \sum_{j=1}^{\infty} b_j B_j(x) \exp(-\kappa_j(t-\tau_j))\mathbf{1}_{\{t\geq\tau_j\}}, \tag{2.7}$$

where τ_j is the time of adverse mortality event j. The time of the adverse mortality event can be modeled like the time η_s, which we discussed after equation (2.5). The basic effect of a pandemic is modeled using a Poisson process with a jump size $b_j > 0$, where b_j is the maximum severity of jump event j. The function $B_j(x)$, with $0 \leq B_j(x) \leq 1$, distributes mortality jump impact across ages. $B_j(x)$ can be random to reflect various age effects for different types of events or it can also be deterministic. For example, the specification of $B_j(x)$ in Figure 4 would apply to a pandemic for which the bulk of the mortality spike is under a certain age (for example, under age 50 as is the case for deaths in the 1918 worldwide flu).

Since modeling transitory nature of mortality jumps is important in practice, we introduce a nonnegative deterministic function $\exp(-\kappa_j(t-\tau_j))$ in equation (2.7) with $\kappa_j > 0$. The higher κ_j the faster the jump effect will die out.[6] We can estimate

[6]In general, κ_j is much larger than ξ_i because of the temporary effect of catastrophe mortality event and the long-term impact of unexpected mortality improvement.

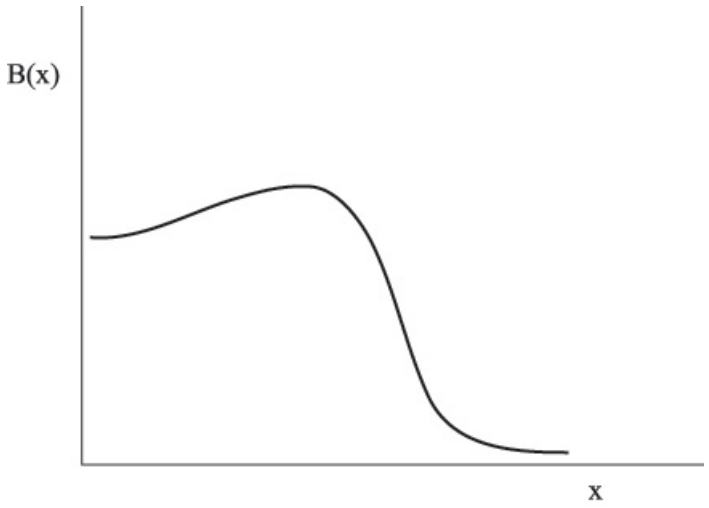

FIGURE 4. Pandemic effect by age.

κ_j to fit the historical data and make the mortality jump only have the impact over a reasonable short period of time.

2.3. **Comment on Estimation Issues.** The primary motivation for the model that we have presented is to provide a flexible and realistic model that can be used to capture adverse mortality and longevity risks in a business application. In order to allow for these effects in a manner that can be compared to the historical data, a fair number of parameters needs to be included. This poses significant estimation challenges. Our method of estimation/calibration can best be described as follows:

(1) Estimate the original Lee-Carter model based on the available data;
(2) Calibrate the A, B and F functions to approximate past history. If the primary focus is on pandemics one would look at how the mortality effects were distributed across ages and calibrate accordingly;
(3) Calibrate the jump frequency for the point processes driving mortality changes in a range that is consistent with history. For example, one can use the historical frequency of pandemics as a guide to setting the jump frequency.

In financial and economic applications of jump-diffusion type models, one issue that emerges is how to appropriately disentangle the diffusion from the jump dynamics. The problem is acute in the modelling of stock market returns because one generally introduces jumps into the model with the idea of picking up "crashes" such as 1929, 1987 or 2008 but what can happen is that the estimation of such a model produces a preponderance of high frequency low severity jumps rather

than the intuitively pleasing outcome of fairly smooth diffusion behavior punctuated by occasional severe jumps. In our model we are focused on mortality rather than equity returns but the same question can be asked. While we do not have an estimation procedure that simultaneously sorts out the diffusive and jump parameters of our model, we do feel that the infrequent occurrence of severe adverse mortality events makes our imposed reconciliation of the diffusion and adverse mortality jumps reasonable. In the case of possible jump components to mortality improvement we are not able to reliably measure the extent to which jump behavior is present. Insofar as the stochastic features of the model can be ascertained for a given parameterizations, the user of the model can at the very least perform risk management functions for a range of life insurance products on plausible mortality scenarios.

In sum, while our model can generate a rich set of scenarios, it does pose a serious estimation challenge. This is especially so because of the relative paucity of extreme events in mortality data sets. Indeed our goal is to allow for scenarios reflecting expert judgment about these events, which may not be the same as the data. For example our approach allows one to set the frequency of a severe 1918-like flu so that it is a 1 in 25 year event, even though such a flu has been observed only once in 100 years. Therefore, "estimation" of this model requires a combination of traditional estimation techniques and expert judgement.

2.4. **Parsimonious Model.** For the Lee-Carter base model substituting in equation (2.3) for $\mu^{LC}(x,t)$, equation (2.4) becomes

$$
\begin{aligned}
\mu(x,t) &= \exp\left[a(x) + b(t)k(t) - G(x,t) + H(x,t)\right] \\
&= \exp\left[a(x) + b(t)k(t)\right] \\
&\times \exp\left\{-\sum_{s=1}^{\infty} y_s A_s(x)\mathbf{1}_{\{t\geq\eta_s\}}\right\} \\
&\times \exp\left\{-\sum_{i=1}^{\infty} \zeta_i(t-\upsilon_i)F_i(x)\exp(-\xi_i(t-\upsilon_i))\mathbf{1}_{\{t\geq\upsilon_i\}}\right\} \\
&\times \exp\left\{\sum_{j=1}^{\infty} b_j B_j(x)\exp(-\kappa_j(t-\tau_j))\mathbf{1}_{\{t\geq\tau_j\}}\right\}.
\end{aligned} \tag{2.8}
$$

A permanent longevity jump can be driven by the one-time component $K(x,t)$ that substantially reduces $\mu(x,t)$ within a short period of time, but it can also come from the trend reduction component $D(x,t)$ that accumulates the excess mortality

improvement over a long time. As discussed in Section 2.2.1, of these two components, $D(x,t)$ seems to play a more important role, especially for ages above 65 — the age range we are keenly interested in for longevity risk. Accordingly, we provide a parsimonious version of our model,

$$\begin{aligned}\mu(x,t) &= \exp\left[a(x)+b(t)k(t)\right] \\ &\times \exp\left\{-\sum_{i=1}^{M(t)}\zeta_i(t-\upsilon_i)F_i(x)\exp(-\xi_i(t-\upsilon_i))\mathbf{1}_{\{t\geq\upsilon_i\}}\right\} \\ &\times \exp\left\{\sum_{i=1}^{N(t)}b_jB_j(x)\exp(-\kappa_j(t-\tau_j))\mathbf{1}_{\{t\geq\tau_j\}}\right\}.\end{aligned} \tag{2.9}$$

In equation (2.9), the permanent longevity jump is only a function of the trend reduction component. The process M counts mortality trend reduction jumps; and $M(t)$ is the number of such jumps by time t. Similarly, the process $N(t)$ counts the number of catastrophe death events by time t. Evidently, equation (2.9) can be further simplified to this:

$$\begin{aligned}\mu(x,t) &= \exp\left[a(x)+b(x)k(t)\right] \\ &\times \exp\left\{-\zeta F(x)\sum_{i=1}^{M(t)}(t-\upsilon_i)\exp(-\xi(t-\upsilon_i))\mathbf{1}_{\{t\geq\upsilon_i\}}\right\} \\ &\times \exp\left\{bB(x)\sum_{i=1}^{N(t)}\exp(-\kappa(t-\tau_j))\mathbf{1}_{\{t\geq\tau_j\}}\right\},\end{aligned} \tag{2.10}$$

assuming constant mortality and longevity jump effects for age x.

2.5. **Example.** Below we show how to apply the parsimonious equation (2.10) to model mortality dynamic process. All of our estimates are based on the aforementioned US male population mortality tables from 1901 to 2005. As we noted earlier, we begin by using the data to estimate the Lee-Carter model.

The importance of the impact of medical advancement on pension plans and annuity insurers should not be underestimated. Trend reduction factors in our model provide for such mortality improvements. The 1970's were the years of accelerating decline of mortality for most ages, so our trend reduction parameters are estimated from this period. The left graph of Figure 5 shows our estimated $F(x)$ for different ages based on the data from 1970 to 1979. As mentioned in

Section 2.2.1, age 0 had the highest average annual excess mortality decrease rate $\zeta = 0.0211$, so $F(0) = 1$. However, we can observe mortality declines at high excess rates mostly at the older ages. On the other hand, during the 1970s, mortality increased for some male young ages, perhaps due to an increase in homicides, suicides, and accidents. If an age does not have an excess mortality decrease rate, we set their $F(x) = 0$. Since the annuity benefits in general are distributed mostly in the older age ranges, their unexpected improvement in life expectancy will have a great bearing on the overall impact of the longevity event. Furthermore, based on the historical data, we specify the trend reduction event arrival rate $\lambda_\nu = 0.01$ and the duration parameter $\xi = 0.01$.

The 1918 worldwide flu is considered the most severe flu epidemic. Naturally, it serves as the basis for the estimation of our temporary adverse mortality jump parameters. The right graph of Figure 5 presents $B(x)$, the normalized log μ-ratio in 1918 relative to 1917 for different ages. The 1918 flu increased the one-year force of mortality for different ages very unevenly. The flu struck ages 0-50, especially ages 20-40, seriously while there were no excess deaths at the ages above 53. According to Taubenberger and Morens (2006), these older people survived earlier flu epidemics and may have acquired immunity to the flu virus. Among all ages, age 28 has the highest log μ-ratio, which is the value for b. Specifically, $b = 0.9647$ and $F(28) = 1$. Since the 1918 flu is approximately a one-in-one-hundred-year event, $\lambda_\tau = 0.01$. Moreover, this pandemic only lasted a couple of years so $\kappa = 1$ is reasonable.

With the observed force of mortality for age x in year $t = 2005$ as the base case, sample paths of $\mu(x, t)$ for ages 30, 40, 50 and 60 in one simulation iteration are in

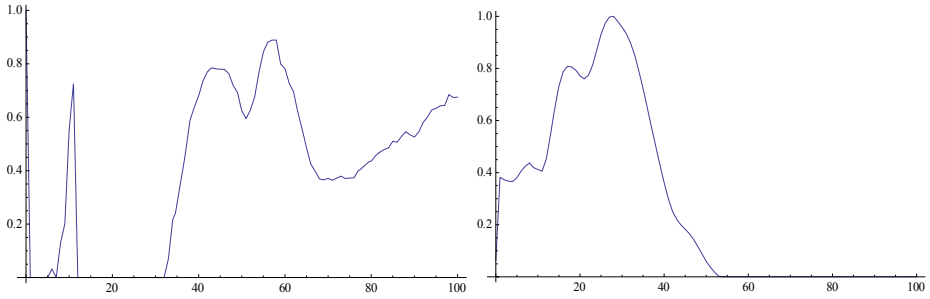

FIGURE 5. The left figure is the function $F(x)$, the normalized annual excess percentage decrease in $\mu(x, t)$ for different ages in the 1970's. The right figure is the the function $B(x)$, giving the impact of the 1918 worldwide flu on $\mu(x, t)$ across ages.

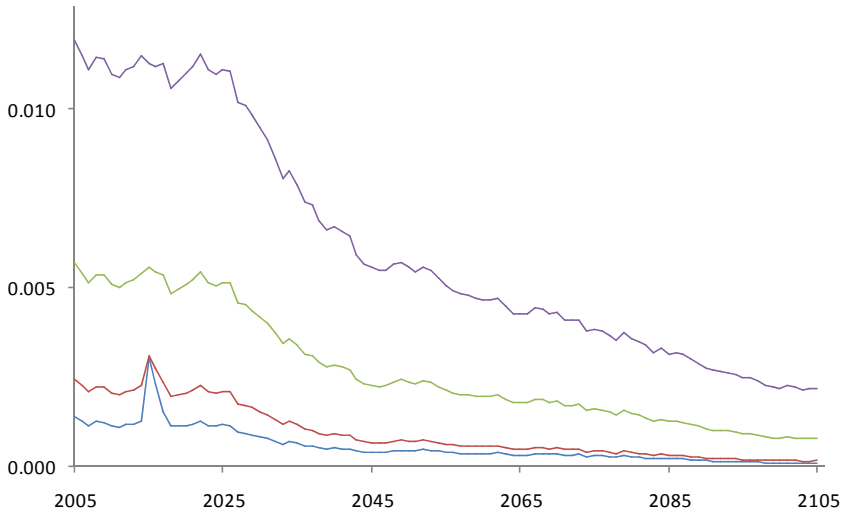

FIGURE 6. These are simulated sample paths of the force of mortality, based on equation (2.10) and the parameters described earlier. The top path is the simulated force of mortality $\mu(60,t)$, the one just below it is $\mu(50,t)$, then $\mu(40,t)$ and $\mu(30,t)$.

Figure 6. They highlight three major features of our model: First, effects of both mortality improvement and deterioration jump factors are clear. For example, the curve of $\mu(40,t)$ has a steeper downward slope that starts in year $t = 2027$ and lasts for about 20 years (its trend reduction longevity jump $D(40,t)$ illustrated in the left graph of Figure 7). Moreover, age 40 has a sharp increase in $\mu(40,t)$ in year $t = 2014$ but $\mu(40,t)$ falls back to the normal level afterwards (its temporary jump factor $H(40,t)$ shown in the right graph of Figure 7). Second, the movement of different $\mu(x,t)$'s are correlated among different ages and over time. This is partially attributed to the functions $F(x)$ and $B(x)$. Third, $F(x)$ and $B(x)$ also distribute uneven effects of a mortality jump across different ages. Based on the historical data, in our model, the adverse mortality jumps mainly strike younger ages while the older ages enjoy more excess survival gains.

All parameters of the example shown in the above example are estimated from the historical data. This is a problem that we will want to carefully consider, as opinions vary widely regarding future health care system and pandemic. The shape of mortality curve often changes in different mortality or longevity jump events. Toole (2007) concludes that the typical distribution of excess death rates for seasonal influenza is U-shaped; that is, "excess deaths are heaped at age 0, quickly decrease to close to zero until they start to increase again at older ages

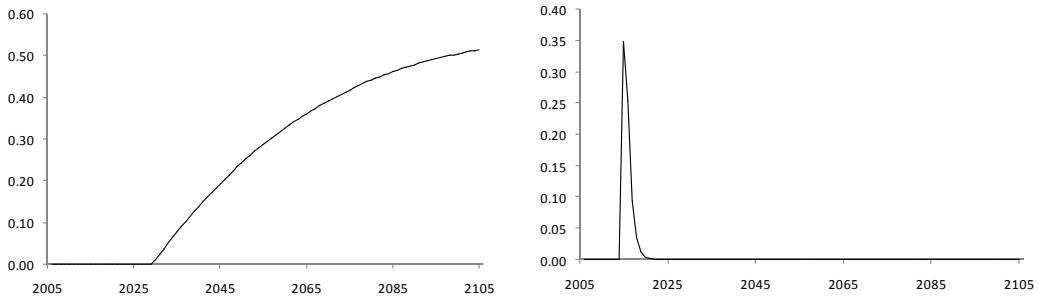

FIGURE 7. This shows the trend reduction longevity jump $D(40,t)$ for age 40 (left) based on equation (2.6) and temporary adverse mortality jump $H(40,t)$ (right) based on equation (2.7). These are the ones used in the simulated paths shown in Figure 6.

(typically age 65) with a rapid rise at ages 85 and older." However, the 1957 pandemic showed a spike in excess death rates at ages 65 and older but there were no excess death at age 0 (Luk et al. 2001). Another exception is the 1918 worldwide flu as mentioned earlier: there were no excess deaths at ages above 65. It is important to note our model can flexibly estimate the impact of a particular jump pattern by adjusting $F(x)$ and $B(x)$ based on different opinions or updated information.

The parsimonious model (2.10) does not include the jump reduction component $K(x,t)$ since there is no clear historical evidence to guide us on its pattern. As long as the estimates of y, $A(x)$ as well as the jump intensity are developed based on judgment, it is not difficult to include $K(x,t)$ in a richer model. An illustration is provided in the next section.

2.6. **A Simple Model.** Although we envision the primary role of our model as risk management tool for the generation of a robust set of plausible mortality scenarios, it is possible to calibrate the model to behave in a manner that is qualitatively similar to the basic Lee-Carter model. This can be done using jump events in longevity and adverse mortality and a fixed base mortality rate resulting in a simple model of the form

$$\mu(x,t) = \mu(x,0)\exp\left(-K(x,t)+H(x,t)\right). \tag{2.11}$$

We calibrated this model by taking $\mu(x,0)$ as the force of mortality as of 2005 from the data that we have discussed above. We then used the forces of mortality for the period 1965 through 2005 to estimate average rates of mortality improvement

by age. We calibrated the adverse mortality effects as was done in the previous section. This led to the functions $A(x)$ and $B(x)$ shown in Figure 8.

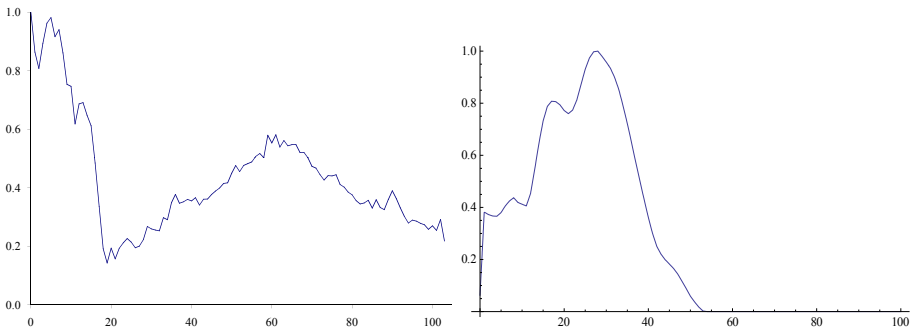

FIGURE 8. The functions $A(x)$ and $B(x)$ for a very simple calibration process. The left figure is the function $A(x)$, the normalized annual average rates of mortality improvement in $\mu(x,t)$ for different ages. The right figure is the the function $B(x)$, giving the impact of the 1918 worldwide flu on $\mu(x,t)$ across ages.

One can use the improvement in mortality to identify average rates but it is not clear how one can identify the frequency with which longevity events occur. For the purposes of this simple example, we use a Poisson process with intensity 0.50 to model longevity jumps and a jump size $y = 0.06506$. Although we know what the product of these two values ought to be, we cannot identify them individually. Consequently, one must judge the parameters by the qualitative features of the model.

Figure 9 shows simulated mortality trajectories for the model parameters that we have described. If one compares this with the trajectories from the basic Lee-Carter model they are similar in their longevity behavior.

3. SECURITIZATION OF MORTALITY AND LONGEVITY RISKS

3.1. **Market Development.** Financial innovation has led to the development of several classes of mortality securities. The Swiss Re bond, the first pure death-risk linked deal, was issued in December 2003 (MorganStanley 2003, Swiss Re 2003, The Actuary 2004). After successfully issuing the first-ever pure death-linked security, Swiss Re sold two new mortality bonds with different tranches in April and December 2006 (Lane 2006). Following Swiss Re, some life insurers started to reduce their extreme mortality exposures through financial markets. For example,

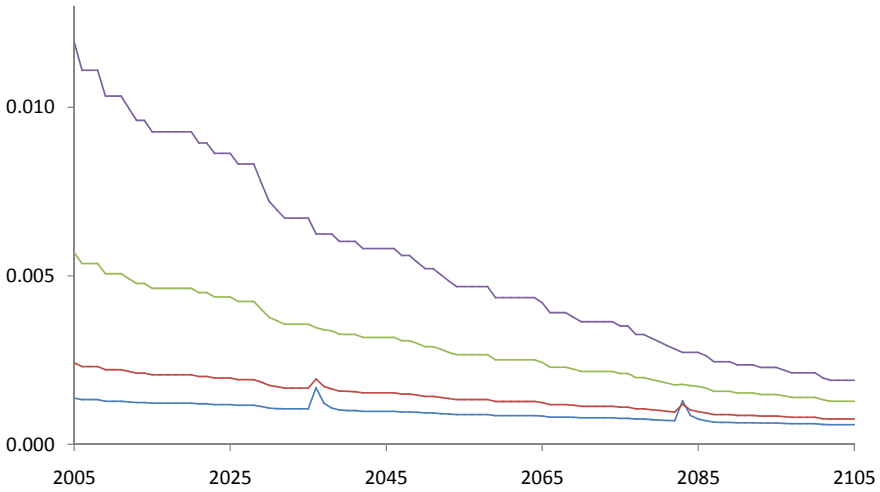

FIGURE 9. Evolution of mortality for a simple model, based on equation (2.11) and the parameters described earlier. The top path is the simulated force of mortality $\mu(60,t)$, the one just below it is $\mu(50,t)$, then $\mu(40,t)$ and $\mu(30,t)$.

in May 2006 Scottish Re sold a mortality bond with two tranches via a special purpose vehicle called Tartan Capital Ltd., and in November 2006, AXA issued its first catastrophe mortality deal — the Osiris bond (Lane and Beckwith 2007). See Bauer and Kramer (2007) for more details.

Capital market solutions for unanticipated longevity risk have been explored relatively recently, first appearing in articles by Blake and Burrows (2001), Milevsky and Promislow (2001), Lin and Cox (2005) and others. Possibly inspired by the successful securitization of catastrophe mortality risks, in November 2004, the European Investment Bank (EIB) offered the first longevity bond to provide a solution for pension plans to hedge their long-term systematic longevity risks. Unlike the Swiss Re mortality bond, the EIB longevity bond did not sell. The design of the EIB bond may be problematic. The EIB bond provides "ground up" protection, covering the entire survival payment. Since the plan can predict the number of survivors to some extent, especially in the early contract years, a more satisfactory solution may be a hedge using, for example, J.P. Morgan q-forwards with a settlement date 10 or more years in the future, rather than hedging the whole benefit liability.

3.2. **J. P. Morgan q-Forwards.** A q-forward contract requires at maturity an exchange of a fixed amount based on the predetermined mortality rate, in return for

TABLE 2. An illustration for a q-forward to hedge longevity risk for a pension plan, described by Coughlan et al. (2007)

Notional Amount	GBP 50,000,000
Trade Date	December 31, 2006
Effective Date	December 31, 2006
Maturity Date	December 31, 2016
Reference Year	2015
Fixed Rate	1.2%
Fixed Amount Payer	J.P. Morgan
Fixed Amount	GBP $60,000,000 = 0.012 \times 50,000,000$
Reference Rate	LifeMetrics male q_{65} in the reference year for England & Wales national population
Floating Amount Payer	XYZ
Floating Amount	Notional Amount $\times$ Reference Rate $\times$ 100
Settlement	Net settlement = Fixed Amount $-$ Floating Amount

a variable amount based on a realized mortality rate published at a specified future date by J.P. Morgan, in the LifeMetrics system.[7] Coughlan et al. (2007) describe an example with a 10-year q-forward written on the one-year death rate q_{65} for males age $x = 65$ in England & Wales. See Table 2 for a summary. The example provides a longevity hedge for the XYZ pension plan. The q-forward was opened on December 31, 2006 with a notional amount of £50,000,000 and a fixed death rate $q_{65} = 1.2\%$. The floating-rate payer XYZ pays an amount proportional to the mortality rate at maturity on December 31, 2016, determined by the reference rate in 2015.

At the maturity, the contract is settled at the net amount which is the difference between the fixed amount and the floating amount. For example, if the reference rate is only 1%, then XYZ Pension gets $£10,000,000$ at settlement of the q-forward contract, calculated as the net settlement amount:

$$\begin{aligned}\text{Net Settlement} &= \text{Fixed Amount} - \text{Floating Amount} \\ &= 50,000,000 \times 1.2\% \times 100 - 50,000,000 \times 1\% \times 100 \\ &= £10,000,000.\end{aligned}$$

Better mortality means fewer than 1.2% of those age 65 in 2015 die within the year, and XYZ will have more than expected pension benefits to pay if it has similar experience. If the reference rate is higher than the fixed rate 1.2%, the settlement is

[7] Available at `www.lifemetrics.com`.

negative and XYZ pays J.P. Morgan. In this case, mortality increases and it is likely that the pension plan pays lower pension benefits.

Life insurers can use q-forwards to hedge mortality risk in a similar way. In summary, insurers or pension plans can take a position (as fixed amount payer or floating amount payer) in q-forwards to hedge their liability to pay benefits to their clients.

The q-forward contract could be modified to create q-calls and q-puts. There are two practical problems with the q-forward (or analogous options). When the q-forward is opened in 2005, XYZ has to estimate the number of survivors to age 65 in 2015 in order to determine the size of the hedge. While XYZ may be more concerned about rates 10 years in the future, they also have longevity risk during the 10 year contract period beginning in 2005. The example only hedges survivor risk from age 65 to age 66.

The other problem is that it is difficult to extend the q-forward hedge. The example covers survivor risk only for one year. While XYZ could open a portfolio of q-forwards on the indices $q_{64+k,2014+k}$ for $k = 1, 2, \ldots$, it would also have to forecast the number of survivors to each age in order to determine the size of each q-forward contract. This gets more difficult as the hedge is extended because of the problem of estimating the number of survivors, exactly the risk the q-forward is supposed to hedge.

3.3. **Securities Based on a Longevity Index.** Consider the XYZ plan again. As before, XYZ has a liability to make payments to a plan participants who survive to age 66 in 2016. When the longevity security is opened in 2005, those participants are age 55. XYZ knows how many participants, their gender, and projected benefits so there is no problem in determining the size of the hedge. Let us consider one participant with 1 unit of annual benefit paid at age 66 in 2016.

In its funding calculations, XYZ will have an estimate of the probability that a participant now age 55 survives to age 66, denoted ${}_{11}p_{55}^{\text{XYZ}}$. The index allows XYZ to hedge underestimating this value (and underfunding its liabilities).

The index for this hedge is based on the government or Lifemetrics series of future death probabilities $q_{55,2005}, q_{56,2006}, q_{57,2007}, \ldots, q_{65,2015}$, published each year from 2006 to 2016. The first index value is $p_{55,2005} = 1 - q_{55,2005}$, published in 2006. The second index value is

$$ {}_2p_{55,2005} = (1 - q_{56,2006})\, p_{55,2005} $$

which is published in 2007. The subsequent index values are calculated analogously. The index value for hedging the payment in 2016 is

$$ {}_{11}p_{55,2005} = (1 - q_{55,2005})\,(1 - q_{56,2006})\,(1 - q_{57,2007}) \cdots (1 - q_{65,2015}) $$

and it is based on the national population experience during the hedge period from 2005 to 2015. XYZ can use the index directly to hedge its payments to the cohort of plan participants age 55 in 2005. Forwards or options can be written on this index just as easily as they might be written on $q_{65,2015}$ but with no problem in determining the hedge size. Moreover, XYZ could just as easily extend the hedge to the second benefit payment with an option on the next index value ${}_{12}p_{55,2005}$.

For example, a call option could pay XYZ when the realized population rate ${}_{11}p_{55,2005}$, published in 2016, exceeds a strike rate such as ${}_{11}p_{55}^{\mathrm{XYZ}}$ (which the plan uses in 2005). The population index is likely to be in the money in the same circumstances that the plan has experienced unexpected increases in longevity, since the two groups are subject to some of the same forces of mortality.

Using the population index, based on LifeMetrics or government data, reduces moral hazard since the index is transparent to all investors. However, basis risk may be a problem. Generally pension plan participants have higher survival rates than the population as a whole. However, XYZ could estimate this relationship statistically and use this relation to adjust its hedge. We show how to price longevity options in the next section.

4. Mortality Securitization Modeling

Insurance–linked securitization, as an alternative risk management method and a new investment opportunity, has gained more and more attention from both scholars and practitioners. Accordingly, to develop this emerging market, actuaries and financial economists have begun to make considerable efforts to improve its structure and take on the challenges to connect financial and insurance pricing theories. The well-known capital asset pricing model (CAPM) lacks the flexibility to be applied appropriately to heavy-tailed insurance risks. On the other hand, many insurance principles lack the ability to produce arbitrage–free prices. To address these problems, we use the indifference pricing method to price mortality and longevity securities. To illustrate the idea, we combine the indifference pricing method and our proposed stochastic mortality model. We show how to estimate

the risk aversion parameter, using annuity market data. Then, we use the estimated parameter to price a longevity option. The same techniques can be used to price other mortality securities.

4.1. **Indifference Pricing Method.** We consider longevity risk in a life-only single premium immediate annuity (SPIA) for age x. We further assume the annuity benefit is $\bar{s}$, paid at the end of each period. Thus, the present value of future annuity benefits for an annuity written on M lives (x_i) at time 0, for $i = 1, 2, \ldots, M$ is

$$X_i = \bar{s} \sum_k \mathrm{e}^{-kr_k} \mathbb{I}_{\{T(x_i) \geq k\}}, \tag{4.1}$$

where r_k is the default-free interest yield from time 0 to time k and k runs over the payment times.[8] The aggregate present value is

$$Y = \sum_{i=1}^{M} X_i. \tag{4.2}$$

The premium P per policy is to be determined from the equation:

$$\mathrm{E}[u(w + MP - Y)] = u(w), \tag{4.3}$$

where w is the wealth of the insurer prior to accepting the risk to be priced. We use the exponential utility function,

$$u(w) = \frac{1 - \mathrm{e}^{-\alpha w}}{\alpha}, \tag{4.4}$$

where α is the risk aversion parameter of the insurer. The random variable Y depends on the random path of future force of mortality values $\{\mu(x+t, t) : t \geq 0\}$, which can be simulated with our model, using equation (2.10). The risk aversion parameter α in equation (4.3) can be approximated using known insurance market prices. Under our exponential utility function, the premium implied by (4.3) may be expressed as

$$P = \frac{1}{M\alpha} \log \mathrm{E}\left[\mathrm{e}^{\alpha Y}\right]. \tag{4.5}$$

When calibrating to market data, we need to determine α so that equation (4.5) is satisfied.

[8]The notation $\mathbb{I}_A$ is the indicator function taking the value 1 if the event A occurs and the value 0 if not.

In the annuity pricing application we are considering, we are pricing an aggregate annuity exposure for which the component risks are identical but not necessarily independent. As we have noted, the indifference premium under exponential utility is given by (4.5). An approximation for P, which can be found in Gerber and Pafumi (1998), can be made as

$$P \approx \frac{1}{M\alpha}\left[\alpha \mathrm{E}[Y] + \frac{1}{2}\alpha^2 \mathrm{Var}(Y)\right].$$

Since the aggregate loss is composed of M identical risks, $\mathrm{E}[Y] = M\mathrm{E}[X]$ and thus

$$P \approx \mathrm{E}[X] + \frac{\alpha}{2M}\mathrm{Var}(Y). \tag{4.6}$$

The mortality is uncertain but it is reasonable to assume that, given a mortality scenario Θ, the annuitant lives are independent. In this case, the variance of Y is the sum of two terms:

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{E}[\mathrm{Var}(Y|\Theta)] + \mathrm{Var}[\mathrm{E}(Y|\Theta)] \\ &= \mathrm{E}[M\mathrm{Var}(X|\Theta)] + \mathrm{Var}[M\mathrm{E}(X|\Theta)] \\ &= M\mathrm{E}[\mathrm{Var}(X|\Theta)] + M^2\mathrm{Var}[\mathrm{E}(X|\Theta)].\end{aligned}$$

The approximation becomes

$$P \approx \mathrm{E}[X] + \frac{\alpha}{2}\mathrm{E}[\mathrm{Var}(X|\Theta)] + \frac{M\alpha}{2}\mathrm{Var}[\mathrm{E}(X|\Theta)]. \tag{4.7}$$

When the mortality scenario is known (as in traditional actuarial calculations with static tables), there is no variance in $\mathrm{E}(X|\Theta)$, the third term is zero, and the premium P per policy does not depend on the number of polices M. However, in general mortality scenarios are uncertain and the premium P will depend on the portfolio size M as indicated by (4.6). Moreover, even when the lives are conditionally independent, given the mortality table, the premium still depends on the size of the portfolio as indicated by (4.7).

4.2. **Estimating α from annuity prices.** The Genworth Financial Group sold around \$560 million of single premium individual annuities in the US in 2005 with a monthly payout rate of \$6.40 per \$1,000 premium (Stern 2008). We assume that the policies are identical, issued to males age 65 with a gross premium of \$250,000. Since aggregate premium is \$560 million, then the number of policies is $M = 560 \text{ million}/250,000 = 2,240$.

We assume the underlying expense factor is 12.25 percent, which is around the industry average, so the net premium per policy to the insurer is

$$\begin{aligned} P &= 250,000 \times (1 - 0.1225) \\ &= \$219,375. \end{aligned}$$

Equation (4.1) gives the present value of benefits for one life with $\bar{s} = 250 \times 6.40 = \$1,600$ per month.

We estimate the risk aversion parameter α from equation (4.5) by simulation, with a "Monte Carlo simulation within a Monte Carlo simulation". We start with a trial value of $\alpha = \alpha_0$ and estimate the corresponding value of $\mathrm{E}(\mathrm{e}^{\alpha_0 Y}) = \mathrm{E}[\mathrm{E}(\mathrm{e}^{\alpha_0 Y}|\Theta)]$. First we generate 10,000 of scenarios Θ_j. Then we simulate $M = 2,240$ values of $\mathrm{e}^{\alpha_0 Y}|\Theta_j$, for each j, and estimate $\mathrm{E}[\mathrm{e}^{\alpha_0 Y}|\Theta_j]$ with the sample mean. The estimate of $\mathrm{E}\left(\mathrm{e}^{\alpha_0 Y}\right)$ is the mean of these sample means. This is a very fast calculation. The corresponding model price, from equation (4.1), is

$$P_0 = \frac{1}{M\alpha_0} \log \mathrm{E}\left(\mathrm{e}^{\alpha_0 Y}\right).$$

We repeat the calculation with another trial value α_1, then obtaining P_1. Then we use the secant method to find a third approximation,

$$\alpha_2 = \alpha_1 - \frac{\alpha_1 - \alpha_0}{P_1 - P_0}(P_1 - P).$$

This process converges rapidly to $\alpha = 4.8 \times 10^{-7}$. For the value of α that was determined above, the unscaled utility function for Genworth is shown in Figure 10.

In this example, we used the population mortality tables rather than annuity tables because we do not have a time series of annuity tables. On a static basis, population mortality tables will tend to underestimate the longevity risk in annuities relative to the applicable set of annuity tables, thereby tending to create an overestimate of α. However, we estimated α using Lee-Carter projections for mortality fluctuations which embeds longevity risk. As we do not know the mortality assumptions that Genworth used in setting their annuity rates, this exercise must be viewed as an illustrative approximation. The reason that we calibrated α using Lee-Carter projections rather than the full model is that the full model induces significantly thicker longevity tail risk than Lee-Carter projections alone. Indeed, if α is estimated under the full model a significantly lower value of α is obtained because of the greater tail risk and the Genworth annuity premiums do not appear

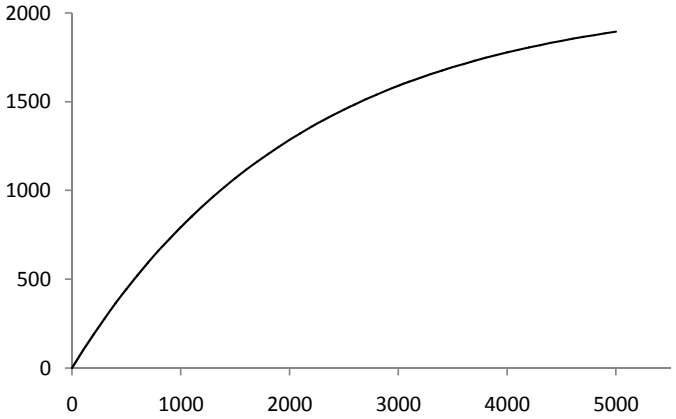

FIGURE 10. Genworth's estimated utility function under the exponential utility model for $\alpha = 4.8 \times 10^{-7}$. The vertical axis represents utility and the horizontal axis represents wealth, both in thousands.

to contain a material charge for this tail risk. It seems reasonable to believe that our estimated $\alpha = 4.8 \times 10^{-7}$ can serve as an upper bound for Genworth's risk aversion to annuity longevity risk. Although we show how to determine α with an annuity market quote, the same technique can be applied to estimate life insurance risk aversion parameters. We will use this value in the next section to estimate what Genworth would pay for a longevity option.

4.3. **Longevity Option Price.** Let us now illustrate the pricing of longevity options using indifference pricing with the pricing parameter α that we have previously derived from annuity prices. Consider a 10-year call option issued in December 2005 based on our longevity index ${}_{10}p_{65,2005}$ as defined in Section 3.3 with a notional amount of $\$100,000,000$ and a strike price of p. The dollar payoff from this longevity call at maturity in 2015 may be expressed as

$$\mathsf{C} = \begin{cases} 100,000,000\,({}_{10}p_{65,2005} - p) & \text{if } {}_{10}p_{65,2005} > p \\ 0 & \text{otherwise} \end{cases}. \tag{4.8}$$

Although one can attribute a "cohort" to the payoff amount, in practice the owner of the option is most likely concerned with the total dollar exposure as it relates to a pension fund or annuity pool liability. We apply equation (4.3) to price this longevity call option for a mortality data generating process driven by our mortality model and by the Lee-Carter model alone. In all cases we assume that the market risk aversion parameter is $\alpha = 4.8 \times 10^{-7}$, in line with the estimate that

TABLE 3. 10-year longevity call option prices with data generating process driven by our model.

Strike rate p	0.775	0.780	0.785	0.790	0.795
Call price	\$1,121,364	\$924,288	\$772,136	\$652,506	\$553,345

TABLE 4. 10-year longevity call option prices with data generating process driven by the Lee-Carter model alone.

Strike rate p	0.775	0.780	0.785	0.790	0.795
Call price	\$191,962	\$77,070	\$23,864	\$5,522	\$937

was made for the Genworth Financial Group. Table 3 shows longevity call option prices for a range of strikes for mortality driven by our model. Table 4 shows longevity call option prices for a range of strikes for mortality driven by the Lee-Carter model alone. For example, given the strike level $p = 0.775$ and the US treasury discount factors for the 2005 calendar year,[9] our 10-year longevity call option price equals \$1,121,364 if the mortality data generating process is driven by our model.

Evidently, the nature of the mortality data generating process if critical in assessing the risk in a longevity derivative contract. In the trading of standardized equity and interest rate derivatives, one discovers that market prices reflect a "smile". In simple terms, one finds that is it far more expensive to purchase "lottery tickets" on equity indices or interest rates than basic theory suggests. If the market for longevity derivatives should evolve to a level of reasonable liquidity, it would not be surprising of similar effects were observed. While no one can be certain of the true nature of longevity risk, our model has shown that then presence of relatively infrequent mortality improvements poses a significant risk. Figure 11 shows a histogram of simulated values of ${}_{10}p_{65,2005}$ for 1 million draws using the full model. Figure 12 shows a histogram of simulated values of ${}_{10}p_{65,2005}$ for 1 million draws using the Lee-Carter model alone. Evidently, it is the right tail behavior that is driving the differences between the prices of the longevity calls across the two mortality data generating processes.

[9]To be precise, we require discount factors that are representative of interest rates for 2005. We take the average US treasury coupon-curve yields for 2005 as published in the Federal Reserve's H15 data set and strip them to produce zero-coupon data. We then fit a Nelson-Siegel yield curve to this zero-coupon data which can be extrapolated beyond the last observation. At this point we have a set of discount factors that is representative of market conditions in 2005.

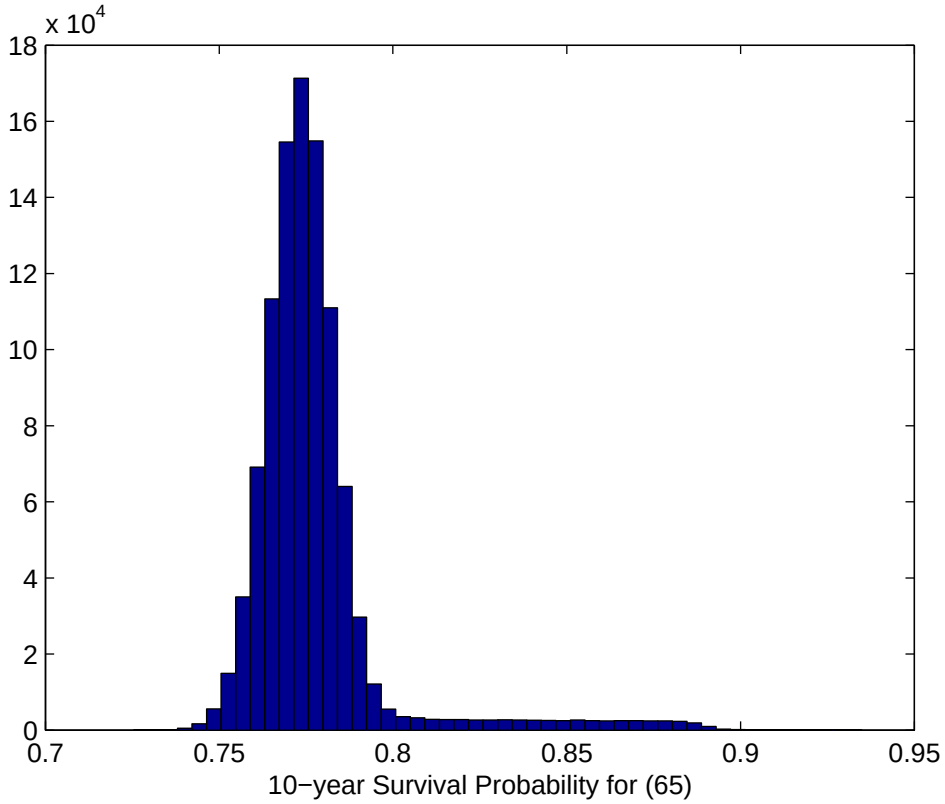

FIGURE 11. Range of outcomes for ${}_{10}p_{65,2005}$ under the full model.

Reinsurance contracts typically have a cap on the reinsurer's liability. It is reasonable to anticipate that mortality traders may follow this tradition. The typical reinsurance contract is analogous to a call spread with two calls having different strike prices. Consider a call spread written on the survivor index ${}_{10}p_{65,2005}$, with lower strike level, "attachment point", p_1 and an upper strike level, or "detachment point", p_2 and a notional amount of $\$100,000,000$. The dollar payoff from this longevity call spread at maturity in 2015 may be expressed as

$$\mathsf{B} = \begin{cases} 100,000,000\,(p_2 - p_1) & \text{if } {}_{10}p_{65,2005} \geq p_2 \\ 100,000,000\,({}_{10}p_{65,2005} - p_1) & \text{if } p_1 <_{10} p_{65,2005} < p_2 \\ 0 & \text{if } {}_{10}p_{65,2005} \leq p_1 \end{cases}. \tag{4.9}$$

Table 5 shows the longevity call spread option prices for a range of strikes for mortality driven by our model. Note that the spreads in Table 5 have different, decreasing seniorities due to decreasing trigger and exhaustion levels. In particluar,

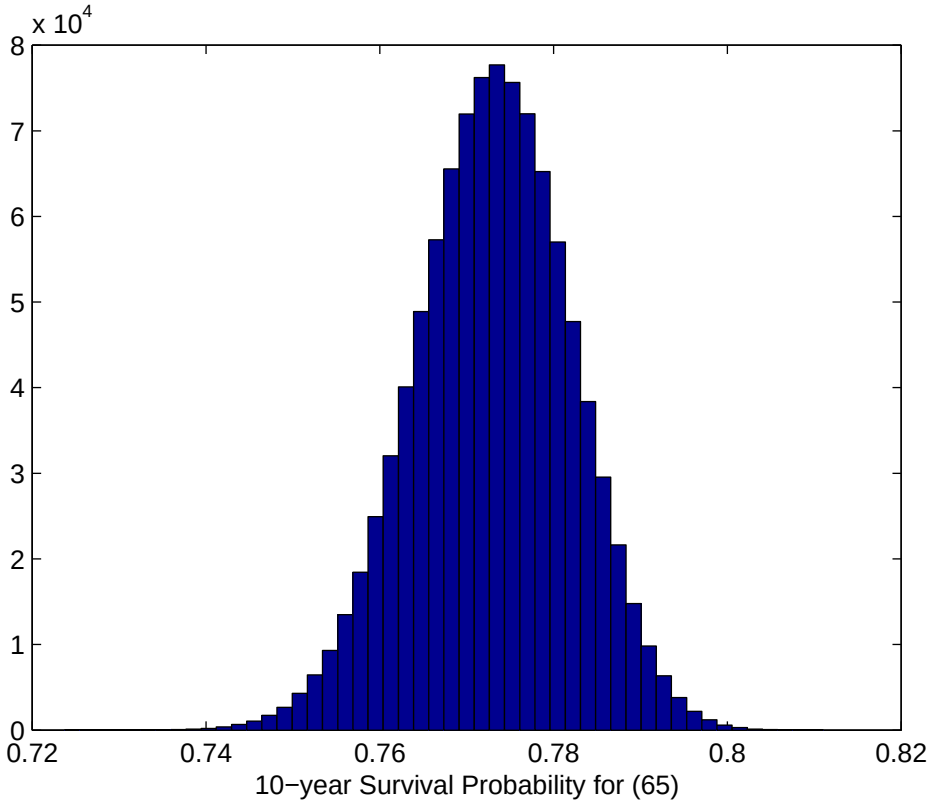

FIGURE 12. Range of outcomes for ${}_{10}p_{65,2005}$ under the Lee-Carter model alone.

TABLE 5. Prices for 10-year longevity call spread options.

	Spread A	Spread B	Spread C	Spread D
Attachment point p_1	0.775	0.780	0.785	0.790
Detachment point p_2	0.780	0.785	0.790	0.795
Price	\$122,985	\$70,405	\$39,337	\$25,835

spread A has the lower attachment and detachment points relative to the other spread options and thus it demands a higher price.

5. SUMMARY

We have described a mortality model which provides a rich and realistic space of sample paths of future mortality rates. The model evolves as a dynamic process that combines a general mortality trend, a diffusion process, a permanent longevity jump process, and a temporary mortality jump process. Furthermore, we describe a parsimonious version of the model based on judgement and historical mortality

data available through the Human Life Table Database and the Human Mortality Database. Another novel aspect is that our model includes uneven effects of a mortality or longevity shock to different ages. In addition, the model is tractable enough to allow path-by-path simulation which can be combined with pricing techniques, as we illustrate with the indifference pricing method. The model is well-suited to risk management application where there is a need to test extreme but plausible mortality scenarios.

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