# Efficient Concentration Risk Measurement in Credit Portfolios with Haar Wavelets

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# Outline

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# Introduction

- It is very important for a bank to manage the risks originated from its business activities. The credit risk underlying the credit portfolio is often the largest risk in a bank.
- Basel Accords (I, II and III) laid the basis for international minimum capital standards. Banks became subject to **regulatory capital** requirements.
- Basel II is structured in a three Pillar framework:
  - Pillar 1: more risk sensitive minimal capital requirements.
  - Pillar 2: banks are allowed to calculate the **economic capital** (**risk concentration**).
  - Pillar 3: transparency in bank's financial reporting.

# Introduction

- Concentration risks arise from an unequal distribution of loans to single borrowers (**exposure or name concentration**) or different industry or regional sectors (**sector concentration**).
- Merton model: basis of the Basel II IRB approach. Under homogeneity conditions, this model leads to the **ASRF** model. However, this model can **underestimate risks** in the presence of exposure concentration.
- Credit risk managers are interested in:
  - How can concentration risk be quantified?
  - How can risk measures be accurately computed in short times?

# Risk Parameters

- We specify a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We fix a time horizon $T > 0$ (usually one year).
- We consider a credit portfolio consisting of $N$ obligors.
- Any obligor $n$ is characterized by:
  - The **exposure at default** $E_n$: potential exposure measured in currency.
  - The **loss given default** $L_n$: magnitude of likely loss on the exposure as a percentage of the exposure.
  - The **probability of default** $P_n$: likelihood that a loan will not be repaid.

  Each of them can be estimated from empirical default data.

# Risk Measures

Consider an obligor $n$ subject to default in the fixed time horizon $T$.

We introduce $D_n$, the default indicator of obligor $n$,

$$D_n = \begin{cases} 1, & \text{if obligor } n \text{ is in default,} \\ 0, & \text{if obligor } n \text{ is not in default,} \end{cases}$$

where $\mathbb{P}(D_n = 1) = P_n$ and $\mathbb{P}(D_n = 0) = 1 - P_n$.

Let $\mathcal{L}$ be the portfolio loss given by,

$$\mathcal{L} = \sum_{n=1}^{N} \mathcal{L}_n,$$

where $\mathcal{L}_n = E_n \cdot L_n \cdot D_n$.

# Risk Measures

Credit risk can split in **Expected Losses** EL (which can be forecasted) and **Unexpected Losses** UL (more difficult to quantify).

**Assumption 1.1**

*The exposure at default $E_n$, the loss given default $L_n$ and the default indicator $D_n$ of an obligor $n$ are independent.*

Denote by $EL_n$ the expectation value of $L_n$, therefore,

$$\mathrm{EL} = \mathbb{E}(\mathcal{L}) = \sum_{n=1}^{N} E_n \cdot EL_n \cdot P_n.$$

Holding the $\mathrm{UL} = \sqrt{\mathbb{V}(\mathcal{L})}$ as a risk capital for cases of financial distress might not be appropriate (peak losses can be very large when they occur).

# Risk Measures

Let $\alpha \in (0,1)$ be a given confidence level, the $\alpha$-quantile of the loss distribution of $\mathcal{L}$ in this context is called **Value at Risk** (VaR). Thus,

$$\text{VaR}_\alpha = \inf\{l \in \mathbb{R} : \mathbb{P}(\mathcal{L} \leq l) \geq \alpha\} = \inf\{l \in \mathbb{R} : F_{\mathcal{L}}(l) \geq \alpha\},$$

where $F_{\mathcal{L}}$ is the cumulative distribution function of the loss variable $\mathcal{L}$.

VaR is the measure chosen in the Basel II Accord ($\alpha = 0.999$) for the computation of capital requirement.

Another important risk measure is the so called economic capital $\text{EC}_\alpha$ for a given confidence level $\alpha$,

$$\text{EC}_\alpha = \text{VaR}_\alpha - \text{EL}.$$

# The One-Factor Merton Model

Let us consider the asset returns $r_n$,

$$r_n = \sqrt{\rho_n} Y + \sqrt{1-\rho_n} \epsilon_n,$$

where $Y$ and $\epsilon_n, \forall n$ are independent and standard normally distributed. $Y$ usually denotes the **business cycle** and $\epsilon_n$ the **idiosyncratic shock**. $\sqrt{\rho_n}$ represents the borrower $n$'s sensitivity to systematic risk $Y$.

$$D_n = \chi_{\{r_n < t_n\}} \sim B(1, \mathbb{P}(r_n < t_n)),$$

We have $P_n = \mathbb{P}(r_n < t_n)$, $t_n = \Phi^{-1}(P_n)$ and,
$P_n(y) \equiv \mathbb{P}(r_n < t_n | Y = y) = \Phi\left(\frac{t_n - \sqrt{\rho_n} y}{\sqrt{1-\rho_n}}\right)$, cond. default probability.

# Portfolio Loss

**Purpose:** find an expression for the portfolio loss variable $\mathcal{L}$.

Assuming a constant loss given default equal to $L_n$ for obligor $n$, the portfolio loss distribution can then be derived as,

$$\mathbb{P}(\mathcal{L} \leq l) = \sum_{\substack{(d_1,\ldots,d_N)\in\{0,1\}^N \\ \sum_{n=1}^N s_n \cdot L_n \cdot d_n \leq l}} \left( \sum_{n=1}^{N} s_n \cdot L_n \cdot d_n \right) \cdot \mathbb{P}(D_1 = d_1, \ldots, D_N = d_N).$$

**Impractical** from a computational point of view for realistic portfolios (for instance $N = 1000$).

## Remark 1.1

*We present an analytical approximation for the $\alpha^{th}$ percentile of the loss distribution in the one-factor framework, under the assumption that portfolios are infinitely fine-grained such that the idiosyncratic risk is completely diversified.*

# The ASRF Model

The Asymptotic Single Risk Factor Model (ASRF) is the model chosen in Basel II to calculate regulatory capital. It is based on the one-factor Merton model and it mainly relies in the following assumptions,

## Assumption 1.2

1. *Portfolios are infinitely fine-grained, i.e. no exposure accounts for more than an arbitrarily small share of total portfolio exposure.*
2. *Dependence across exposures is driven by a single systematic risk factor $Y$. Default indicators are mutually independent conditional on $Y$.*

## Assumption 1.3

1. $\sum_{n=1}^{N} E_n \uparrow \infty$.
2. *There exist a positive $\zeta$ such that the largest exposure share is of order $\mathcal{O}(N^{-(\frac{1}{2}+\zeta)})$.*

## Theorem 1.1

*Let us denote the exposure share of obligor n by* $s_n = \frac{E_n}{\sum_{n=1}^N E_n}$. *Then, under assumptions 1.2 and 1.3 the portfolio loss ratio* $\mathcal{L} = \sum_{n=1}^N s_n \cdot L_n \cdot D_n$ *conditional on any realization y of the systematic risk factor Y satisfies,*

$$\mathcal{L} - \mathbb{E}(\mathcal{L}|Y = y) \to 0 \text{ almost surely as } N \to \infty.$$

Under one-factor Merton model and assuming $L_n$ to be deterministic,

$$\mathbb{E}(\mathcal{L}|Y = y) = \sum_{n=1}^{N} s_n \cdot L_n \cdot \Phi\left(\frac{t_n - \sqrt{\rho_n} y}{\sqrt{1-\rho_n}}\right). \quad (1)$$

By Theorem 1.1: $\mathrm{VaR}_\alpha(\mathcal{L}) - \mathbb{E}(\mathcal{L}|Y = l_{1-\alpha}(Y)) \to 0$ a.s. as $N \to \infty$.
Finally,

$\mathrm{VaR}_\alpha^A = \sum_{n=1}^N s_n \cdot L_n \cdot \Phi\left(\frac{t_n + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right)$ and,

$\mathrm{VaRC}_{\alpha,n}^A = s_n \cdot \frac{\partial \mathrm{VaR}_\alpha^A}{\partial s_n} = s_n \cdot L_n \cdot \Phi\left(\frac{t_n + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right)$.

# Concentration Risk

**However:**

- Real world portfolios are not perfectly fine-grained.
- The ASRF model might be approximately valid for huge portfolios but less satisfactory for portfolios of smaller institutions (or more specialized).
- The formula can **underestimate the required economic capital**.
- Does not allow the measurement of sector concentration risk.

**In practice:** Monte Carlo simulations (robust but computationally intensive).

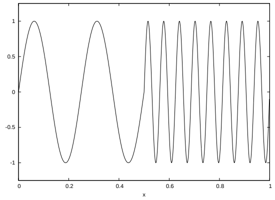
**Proposal:** A new method based on wavelets to overcome the computational complexity.

**References:** E. Lütkebohmert (2009). Concentration Risk in Credit Portfolios. Springer.

# Outline

# Introduction

- Multi-scale methods: signal analysis, statistics, image processing and numerical analysis.
- Approximations $(f_j)_{j\geq 0}$ to the unknown function $f$ at various resolution levels indexed by $j$.
- **Wavelet**: a "little wave" with remarkable approximation properties.
- Fourier basis: composed of waves (approximation in frequency domain).
- Wavelet basis: composed of wavelets (approximation in frequency and time domain).

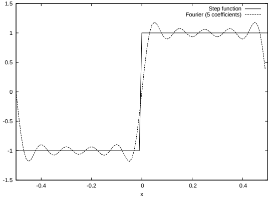


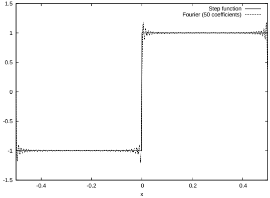
# The Haar System

**Example:** the step function

$$f(x) = \begin{cases} -1, & x \in [-\frac{1}{2}, 0], \\ 1, & x \in (0, \frac{1}{2}], \\ 0, & \text{otherwise.} \end{cases}$$

This function is poorly approximated by its Fourier series:




Wavelets are more flexible: $f(x) = \frac{\sqrt{2}}{2}\phi_{1,0}(x) - \frac{\sqrt{2}}{2}\phi_{1,-1}(x)$.

**References:** I. Daubechies (1992). Ten Lectures on Wavelets. SIAM.

# The WA Method

Let $f$ be a function in $L^2([0,1])$.

$$f(x)=\sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x)+\sum_{j=m}^{+\infty}\sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x),$$

or alternatively,

$f(x)=\lim_{m\to+\infty} f_m(x), \quad f_m(x)=\sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x)$, where,

$$c_{m,k}=\int_{\frac{k}{2^m}}^{\frac{k+1}{2^m}} f(x)\phi_{m,k}(x)dx, \quad d_{j,k}=\int_{\frac{k}{2^m}}^{\frac{k+1}{2^m}} f(x)\psi_{j,k}(x)dx,$$

$k=0,\ldots,2^m-1, j\geq m, k=0,\cdots,2^j-1$ and $\{\phi_{m,k}\}_{k=0,\ldots,2^m-1}\cup\{\psi_{j,k}\}_{j\geq m,k=0,\cdots,2^j-1}$ is the Haar basis system in $L^2([0,1])$.

# The WA Method

Consider the Laplace Transform of $f$:

$\widetilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x)dx$, (assume $f(x) = 0, \forall x \notin [0,1]$).

Wavelet Approximation (WA) method: **approximate** $\widetilde{f}$ by $\widetilde{f}_m$ and **compute** the coefficients $c_{m,k}$.

$$\widetilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x)dx \simeq \int_0^{+\infty} e^{-sx} f_m(x)dx =$$
$$= \frac{2^{m/2}}{s}\left(1 - e^{-s\frac{1}{2^m}}\right)\sum_{k=0}^{2^m-1} c_{m,k} e^{-s\frac{k}{2^m}}.$$

**Change of variable** $z = e^{-s\frac{1}{2^m}}$: $\sum_{k=0}^{2^m-1} c_{m,k} z^k \simeq \overline{Q}_m(z)$.

# The WA Method

We obtain the coefficients $c_{m,k}$ by means of the Cauchy's integral formula,

$$c_{m,k} \simeq \frac{1}{2\pi i} \int_\gamma \frac{Q_m(z)}{z^{k+1}} dz, \quad k = 0, \dots, 2^m - 1,$$

where $\gamma$ denotes a circle of radius $r$, $0 < r < 1$, about the origin.

Considering now the change of variable $z = re^{iu}$, $0 < r < 1$ we have,

$$c_{m,k} \simeq \frac{2}{\pi r^k} \int_0^\pi \Re(Q_m(re^{iu})) \cos(ku) du, \quad k = 0, \dots, 2^m - 1.$$

Finally, the integral can be evaluated by means of the **trapezoidal rule**.

# Outline

# The Model

Focus on the **one-factor Merton** model.

**Assume:** $L_n = 100\%$ and $\sum_{n=1}^{N} E_n = 1$.

Let $F$ be the CDF of $\mathcal{L}$ and $f_{\mathcal{L}}$ its PDF.

$r_n = \sqrt{\rho} Y + \sqrt{1-\rho}\epsilon_n$ ($Y$, $\epsilon_n$ i.i.d. $N(0,1)$).

Conditional default probabilities, $P_n(y) \equiv \Phi\left(\frac{t_n - \sqrt{\rho} y}{\sqrt{1-\rho}}\right)$, $t_n = \Phi^{-1}(P_n)$.

**References:**

- Granularity Adjustment: Gordy and Lütkebohmert (2007).
- Recursive Approximation: Andersen et al. (2003).
- Normal Approximation: Martin (2004).
- Saddle Point: Martin et al. (2001), Huang and Oosterlee (2007) (**cond. framework**).
- Poisson Method: Glasserman (2007).

# The Approximation

Consider,

$$F(x) = \begin{cases} \overline{F}(x), & \text{if } 0 \leqslant x \leqslant 1, \\ 1, & \text{if } x > 1, \end{cases}$$

Define unconditional MGF: $\tilde{M}_{\mathcal{L}}(s) \equiv \mathbb{E}(e^{-s\mathcal{L}})$.

**Assumption 3.1**

**Conditional Independence Framework**. *If the systematic factor $Y$ is fixed, defaults occur independently because the only remaining uncertainty is the idiosyncratic risk.*

Define conditional MGF:
$\tilde{\tilde{M}}_{\mathcal{L}}(s;y) \equiv \mathbb{E}(e^{-s\mathcal{L}} \mid Y = y) = \prod_{n=1}^{N} \left[1 - P_n(y) + P_n(y)e^{-sE_n}\right]$.

Then:
$\tilde{M}_{\mathcal{L}}(s) = \mathbb{E}(\tilde{\tilde{M}}_{\mathcal{L}}(s;y)) = \int_{\mathbb{R}} \prod_{n=1}^{N} \left[1 - P_n(y) + P_n(y)e^{-sE_n}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$.

# The Approximation

**Since:** $\overline{F} \in L^2([0,1])$ then:

$$\overline{F}(x) \simeq \overline{F}_m(x), \quad \overline{F}_m(x) = \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x),$$
$$\overline{F}(x) = \lim_{m\to+\infty} \overline{F}_m(x).$$

**Observe:** $\tilde{M}_{\mathcal{L}}(s) = \int_0^{+\infty} e^{-sx}F'(x)dx = e^{-s} + s\int_0^1 e^{-sx}\overline{F}(x)dx.$

**Then:** $\left(\tilde{M}_{\mathcal{L}}(s) - e^{-s}\right)/s$ is the Laplace transform of $\overline{F}$.

**Apply the WA** method,

**Compute:** $c_{m,k} \simeq \frac{2}{\pi r^k}\int_0^{\pi} \Re(Q_m(re^{iu}))\cos(ku)du, \quad k = 0,\ldots,2^m-1,$
by means of the trapezoidal rule.

**Parameters:** $m = 10, m_T = 2^m, l = 20$. MC with $5 \times 10^6$ scenarios.

## Portfolio 3.1

*We consider* $N = 102$ *obligors, with* $P_n = 0.1\%$, $E_n = 1, n = 1, \dots, 100$, $E_{101} = E_{102} = 20$, $\rho = 0.3$ *and* $L_n = 1$.

| Method | $\text{VaR}_{0.999}$ | Relative Error |
|---|---|---|
| Monte Carlo | 0.1500 | |
| ASRF | 0.0474 | −68.39% |
| Saddle Point | 0.1270 | −15.37% |
| Wavelet Approximation | 0.1490 | −0.69% |

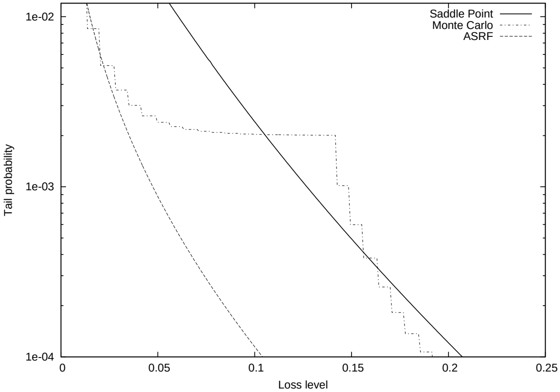



Figure: Tail probability approximation of a heterogeneous portfolio with severe name concentration.

| Portfolio | $N$ | $P_n$ | $E_n$ | $\rho$ | HHI | $\frac{1}{N}$ |
|---|---|---|---|---|---|---|
| P1 | 100 | 0.21% | $\frac{C}{n}$ | 0.15 | 0.0608 | 0.0100 |
| P2 | 1000 | 1.00% | $\frac{C}{n}$ | 0.15 | 0.0293 | 0.0010 |
| P3 | 1000 | 0.30% | $\frac{C}{n}$ | 0.15 | 0.0293 | 0.0010 |
| P4 | 10000 | 1.00% | $\frac{C}{n}$ | 0.15 | 0.0172 | 0.0001 |
| P5 | 20 | 1.00% | $\frac{1}{N}$ | 0.5 | 0.0500 | 0.0500 |
| P6 | 10 | 0.21% | $\frac{C}{n}$ | 0.5 | 0.1806 | 0.1000 |

| Portfolio | $\text{VaR}_{0.999}^{W(8)}$ | $\overline{\text{RE}}(0.999, 8)$ | $\text{VaR}_{0.999}^{W(9)}$ | $\overline{\text{RE}}(0.999, 9)$ | $\text{VaR}_{0.999}^{W(10)}$ | $\overline{\text{RE}}(0.999, 10)$ | $\text{VaR}_{0.999}^{M}$ |
|---|---|---|---|---|---|---|---|
| P1 | 0.1934 | −0.19% | 0.1963 | 1.32% | 0.1938 | 0.06% | 0.1937 |
| P2 | 0.1934 | 1.01% | 0.1924 | 0.50% | 0.1919 | 0.25% | 0.1914 |
| P3 | 0.1426 | 1.46% | 0.1416 | 0.77% | 0.1411 | 0.42% | 0.1405 |
| P4 | 0.1621 | 0.24% | 0.1611 | −0.36% | 0.1616 | −0.06% | 0.1617 |

| Portfolio | $\text{VaR}^{W(8)}_{0.999}$ | $\text{VaR}^{W(9)}_{0.999}$ | $\text{VaR}^{W(10)}_{0.999}$ | $\text{VaR}^{M}_{0.999}$ |
|---|---|---|---|---|
| P1 | 0.2 | 0.4 | 0.7 | 58.3 |
| P2 | 1.8 | 3.6 | 7.2 | 571.6 |
| P3 | 1.8 | 3.6 | 7.2 | 567.6 |
| P4 | 18.2 | 36.1 | 72.4 | 1379.1 |

Table: CPU time (in seconds).

| | | $2^{10}$ | | |
|---|---|---|---|---|
| Portfolio | $\text{VaR}^{W(10)}_{0.9999}$ | $\overline{\text{RE}}(0.9999, 10)$ | $\text{VaR}^{W(10)}_{0.99999}$ | $\overline{\text{RE}}(0.99999, 10)$ |
| P1 | 0.2251 | −0.07% | 0.2935 | −1.70% |
| P2 | 0.2622 | −0.46% | 0.3325 | −1.80% |
| P3 | 0.1812 | −0.10% | 0.2290 | −1.88% |
| P4 | 0.2261 | −0.25% | 0.2935 | −1.30% |
| | | $2^{11}$ | | |
| Portfolio | $\text{VaR}^{W(10)}_{0.9999}$ | $\overline{\text{RE}}(0.9999, 10)$ | $\text{VaR}^{W(10)}_{0.99999}$ | $\overline{\text{RE}}(0.99999, 10)$ |
| P1 | 0.2251 | −0.07% | 0.2935 | −1.70% |
| P2 | 0.2622 | −0.46% | 0.3325 | −1.80% |
| P3 | 0.1812 | −0.10% | 0.2290 | −1.88% |
| P4 | 0.2261 | −0.25% | 0.2935 | −1.30% |
| | | MC | | |
| Portfolio | $\text{VaR}^{M}_{0.9999}$ | | $\text{VaR}^{M}_{0.99999}$ | |
| P1 | 0.2253 | | 0.2985 | |
| P2 | 0.2634 | | 0.3386 | |
| P3 | 0.1813 | | 0.2334 | |
| P4 | 0.2267 | | 0.2973 | |

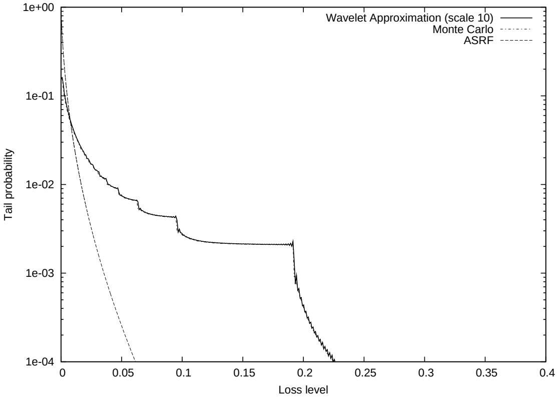



Figure: Tail probability approximation of Portfolios P1 at scale $m = 10$.

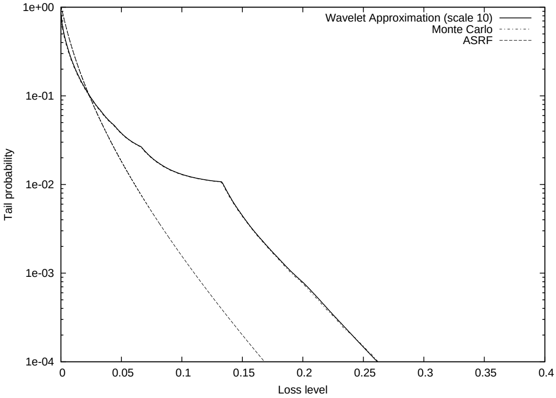



Figure: Tail probability approximation of Portfolios P2 at scale $m = 10$.

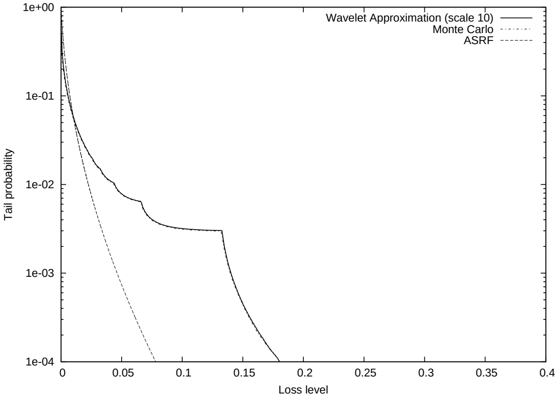



Figure: Tail probability approximation of Portfolios P3 at scale $m = 10$.

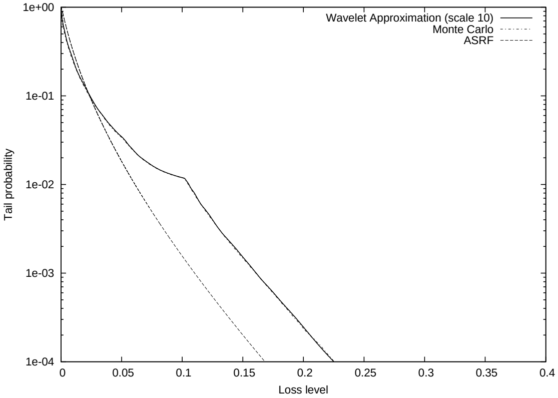



Figure: Tail probability approximation of Portfolios P4 at scale $m = 10$.

# Outline

# European Options

We consider the risk-neutral valuation formula,

$$v(x,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left(v(y,T)|x\right) = e^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)dy, \quad (2)$$

Whereas $f$ is typically not known, the characteristic function of the log-asset price is often known.
We represent the payoff as a function of the log-asset price, and denote the log-asset prices by,

$$x = \log(S_0/K) \quad \text{and} \quad y = \log(S_T/K),$$

with $S_t$ the underlying price at time $t$ and $K$ the strike price. The payoff $v(y,T)$ for European options in log-asset price then reads,

$$v(y,T) = \left[\alpha \cdot K\left(e^y - 1\right)\right]^+, \quad \text{with,} \quad \alpha = \begin{cases} 1, & \text{for a call,} \\ -1, & \text{for a put.} \end{cases}$$

# Outline

# Conclusions

- New method for Laplace Transform inversion based on Haar wavelets. Particularly well suited for stepped-shape functions, often arising in discrete probability models.
- Computation of the VaR risk measure under the one-factor Merton model. Very accurate and fast: **MC/WA** $\simeq 300$, even in the presence of severe name concentration. **Rel. err.** $< 1\%$.
- The WA method computes the entire distribution of losses without extra computational time (for instance CDO pricing).
- The WA method can be extended to compute the Expected Shortfall and the Risk Contributions to VaR and ES.

# References

- J. J. Masdemont and L. Ortiz-Gracia (2011). Haar wavelets-based approach for quantifying credit portfolio losses. *Quantitative Finance*, DOI: 10.1080/14697688.2011.595731.
- L. Ortiz-Gracia and J. J. Masdemont (2012). Credit risk contributions under the Vasicek one-factor model: a fast wavelet expansion approximation. To appear in *Journal of Computational Finance*.
- L. Ortiz-Gracia and C. W. Oosterlee (2013). Robust pricing of European options with wavelets and the characteristic function. Submitted.

Thank you for your attention