INSURANCE RISK CAPITAL FOR THE SPARRE ANDERSEN MODEL WITH GEOMETRIC LÉVY STOCHASTIC RETURNS

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Abstract

Some multi-period insurance risk economic capital models that include the effects of heavy-tail claims and random returns are considered. They are based on the Sparre Andersen risk model with geometric Lévy stochastic returns. The random accumulated surplus over an arbitrary finite time horizon is decomposed into insurance risk, market risk and future profit components. A protection against the solvency risk of the policyholders is obtained by applying the VaR (CVaR) measure to the insurance risk component and defines a multi-period insurance risk VaR (CVaR) economic capital. A classical asymptotic result by Resnick and Willekens (1991) on the tail probability of moving averages with random coefficients is applied to the accumulated aggregate claims random variable for claim size distributions with regularly varying tail to derive asymptotic formulas for these multi-period insurance risk economic capitals. Numerical examples with a Pareto claim size distribution reveal interesting features and differences between these two solvency rules. Since the preceding results exclude the log-normal and the heavy-tailed Weibull claim size distributions, we consider also an extension to sub-exponential claim sizes for the compound Poisson model with constant force of interest, which is based on Hao and Tang (2008). The obtained results are compared with the standard Solvency II specification of the non-life insurance risk.

Key words

Sparre Andersen risk model, geometric Lévy stochastic returns, VaR, CVaR, regularly varying claim size, sub-exponential claim size, asymptotic approximation, Solvency II

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1. Introduction

An insurance company needs capital in order to be able to take risks from its policyholders. According to the standard Solvency II specification the Solvency Capital Requirement (SCR) is the proxy for risk capital under normal circumstances. Its calculation is based on the value-at-risk (VaR) measure over a one-year insurance period taken at the confidence level 99.5%. Other solvency systems like the Swiss Solvency Test (SST) prescribe similarly a one-year Economic Capital (EC), which is based on the conditional value-at-risk (CVaR) measure to the reduced confidence level 99%. However, in the Solvency II and SST projects multi-period economic capital models have scarcely been discussed, and the effects of heavy-tail claims and random returns have not been treated so far. The present contribution offers new insight into these open issues and is organized as follows.

Section 2 recalls first the Sparre Andersen model with geometric Lévy stochastic returns. Then, the random accumulated surplus over an arbitrary finite time horizon is decomposed into insurance risk, market risk and future profit components. A protection against the solvency risk of the policyholders is obtained by applying the VaR (CVaR) measure to the insurance risk component and defines a multi-period insurance risk VaR (CVaR) economic capital. In Section 3 we use a well-known asymptotic result by Resnick and Willekens (1991) on the tail probability of moving averages with random coefficients to derive asymptotic formulas for these multi-period insurance risk VaR and CVaR economic capitals in case the accumulated aggregate claims random variable has a claim size distribution with regularly varying tail. Numerical examples with a Pareto claim size distribution reveal interesting features and differences between these two solvency rules. Since the preceding results exclude the log-normal and the heavy-tailed Weibull claim size distributions, we consider in Section 4 an extension to sub-exponential claim sizes for the compound Poisson model with constant force of interest, which is based on Hao and Tang (2008). The obtained results are compared with the standard Solvency II specification of the non-life insurance risk. We argue that the number of claims is a main driving factor of the risk process. Indeed, due to the law of large numbers the larger this number the less risk capital is actually required. A comparison of the asymptotic VaR formula with the current standard VaR specification shows that the effect of this risk factor is underestimated in the following sense. Measured in units of the standard deviation of aggregate claims, the relative reduction of the asymptotic VaR economic capital factor with respect to the standard SCR one increases with the number of claims. Section 5 summarizes and concludes with open issues for further investigation.

2. Sparre Andersen model with geometric Lévy returns and solvency risk capital

We assume that claim flows can be described by the classical *Sparre Andersen model* where inter-claim times are assumed to form a renewal process (e.g. Rolski et al. (1999)). We suppose that the insurer makes risk-free and risky investments whose accumulated returns follow a *geometric Lévy process* (e.g. Cont and Tankov (2004)).

Consider the stochastic sequence $T_1, T_2, ...$ of the *inter-claim times* (T_1 is the moment when the first claim occurs). If $X_1, X_2, ...$ is the sequence of corresponding *claim sizes*, and $N_t = \max\{k : T_1 + ... + T_k \le t\}$, for $t \ge 0$, is the *number of claims* up to time t, then the *aggregate claims* over that time period is

$$S_t = \sum_{k=1}^{N_t} X_k, \quad t \ge 0, \tag{2.1}$$

with $S_t = 0$ if $N_t = 0$. The moment when the k-th claim occurs, called *arrival time*, is given by

$$M_k = T_1 + T_2 + ... + T_k, \quad k = 1,2,... \tag{2.2}$$

For later convenience set $M_0 = 0$. The *accumulated return process* of the investment portfolio is described by a geometric stochastic process $\{e^{Y_t}, t \ge 0\}$ and $\{Y_t, t \ge 0\}$ is the associated *logarithmic return process*.

The described claim flow model with stochastic investment return is called a *Sparre Andersen model with geometric Lévy return* provided the following assumptions are fulfilled:

(A1) The sources of randomness $\{X_1, X_2, ...\}$, $\{N_t, t \ge 0\}$ and $\{Y_t, t \ge 0\}$ are mutually independent.

(A2) For $k = 1,2,...$ the random variables X_k and T_k are both identically distributed with finite means and variances (in reinsurance it is sometimes assumed that the variance of X_k does not exist, see e.g. Theorem 3.2) and have distribution functions F_X and G_T respectively.

(A3) The number of claims $\{N_t, t \ge 0\}$ is an ordinary renewal counting process.

(A4) The logarithmic return process $\{Y_t, t \ge 0\}$ is a Lévy process, which starts at time zero, has independent and stationary increments, and is stochastically continuous. The stochastic process $\{R_t, t \ge 0\}$ defined by $R_t = e^{Y_t}$ is called *geometric Lévy return process*.

The geometric Lévy return prototype is a geometric Brownian process with drift, which is related to the so-called Black-Scholes-Merton return model used in option pricing theory. However, the empirical evidence of non-normality of returns is easily confirmed using the Bera-Jarque(1987) statistic (e.g. Sheikh and Qiao (2010)). In fact, the empirical observations exhibit fat tails, skewness and excess kurtosis. Additional dynamic features include time-varying volatility, short- and long-range dependence. There exist three main classes of general (competing) distributions, which are able to capture in a "realistic" way the relevant features, namely the stable distributions, the extreme value distributions and the generalized hyperbolic distributions (see

Hürlimann (2009a) for a recent empirical study related to financial returns). Since the latter distributions are known to be infinitely divisible, every member of this class generates a Lévy process and henceforth a geometric Lévy return process (see Eberlein (2001) for an excellent paper on the application of generalized hyperbolic Lévy motions in finance). The class of generalized hyperbolic distributions has been introduced by Barndorff-Nielsen (1977) in connection with the "sand project" (investigation of the physics of wind-blown sand). In the context of finance, it includes many attractive families, namely the hyperbolic distributions introduced by Eberlein and Keller (1995), the normal inverse Gaussian distributions introduced by Barndorff-Nielsen (1998), and the normal inverse gamma distribution, first suggested by Praetz (1972), and studied by the author (e.g. Hürlimann (2004)).

The *compound Poisson model with geometric Lévy return* is the special case for which the inter-claim times are exponentially distributed with $G_T(t) = 1 - e^{-\lambda t}$. The number of claims $\{N_t, t \geq 0\}$ is then Poisson distributed with mean λt.

Consider now the solvency risk. Let $t = n \geq 1$ be a variable integer time horizon over which the insurance risk is an on-going concern. Besides claim flows the risk capital process over the time horizon $[0, n]$ depends upon the premium flows. We assume that the *earned premium* over a time period $[M_{k-1}, M_k)$, $k = 1,..., N_n$, has a random value EP_k at time M_k defined by

$$EP_k = E[X_k] + \Theta_k, \quad k = 1,..., N_n, \tag{2.3}$$

with $\Theta_k > 0$ the *premium loading* included in the earned premium. The latter certainly depends upon the return on investments in each period and on the aggregate claims amount and should be viewed as a random quantity. Concerning investment we assume that earned premiums are fully invested on the financial market in a pool of assets, whose logarithmic rate of return follows a Lévy process. This supposes that the asset mix is continuously rebalanced. Such an investment strategy is not reasonable in practice. For example, if the insurer's surplus is low, the insurer does not want to "gamble" any of its money with risky assets. In case the required solvency capital is invested at the risk-free rate, it is always available to protect the insurance business, which partially resolves the preceding concern. The consideration of more appropriate investment strategies is left open to further studies (for a dampened alternative consult for example Lechkar and Van Welie (2008)). Let C_0 denote the *available capital* (=market value of assets minus market value of liabilities) at initial time zero. We assume that it is invested at the risk-free rate of return. Let r_f denote the annual accumulation factor for risk-free return on investment, and $v_f = r_f^{-1}$ the *risk-free discount rate*. To simplify notation, consider the stochastic process $R_{s,t} = e^{Y_t - Y_s}$, $0 < s < t \leq n$, which represents the random *accumulation factor* over the time period $[s, t)$. A calculation shows that the random surplus at time n, denoted by U_n, is given by

$$U_n = C_0 \cdot r_f^n + P_n^a - S_n^a, \tag{2.4}$$

with $P_n^a = \sum_{k=1}^{N_n} R_{M_k, n} EP_k$ being the accumulated *earned premium income* and $S_n^a = \sum_{k=1}^{N_n} R_{M_k, n} X_k$ being the accumulated *aggregate claims*, both taken over the time horizon $[0, n]$. Rewrite (2.4) as $U_n = C_0 \cdot r_f^n - TL_n$, where TL_n represents the *total accumulated underwriting losses* by the end of the time period $[0,n]$. This quantity can be decomposed as follows:

$$TL_n = \left(S_n^a - E\left[S_n^a\right]\right) + \left(E\left[S_n^a\right] - P_n^a\right). \tag{2.5}$$

The first component, abbreviated $TL_n^I = S_n^a - E\left[S_n^a\right]$, represents the increase of the accumulated aggregate claims with respect to the mean over the period $[0,n]$ and is called total *insurance risk* at time n. The second component in (2.5), which is equal to the difference between the expected accumulated value of the insurance claims and the random accumulated value of the earned premiums, can be rewritten as

$$E\left[S_n^a\right] - P_n^a = E\left[P_n^a\right] - P_n^a - E\left[P_n^a - S_n^a\right] = \left(E\left[P_n^a\right] - P_n^a\right) - E\left[\sum_{k=1}^{N_n} R_{M_k,n}\Theta_k\right]. \tag{2.6}$$

As a justification it is easy to see that $E\left[S_n^a\right] = E\left[\sum_{k=1}^{N_n} R_{M_k,n}\right] \cdot E\left[X_k\right]$ as well as $E[P_n^a] = E\left[\sum_{k=1}^{N_n} R_{M_k,n}\right] \cdot E\left[X_k\right] + E\left[\sum_{k=1}^{N_n} R_{M_k,n} \cdot \Theta_k\right]$, which implies (2.6). The difference of the first two terms in (2.6) represents the decrease in random invested accumulated premiums with respect to the mean over the period $[0,n]$, while the third term is the expected accumulated *future profit* at time n, denoted by FP_n. The difference $TL_n^M = \left(E\left[P_n^a\right] - P_n^a\right) - FP_n$ is called total *market risk* at time n. The total loss decomposition $TL_n = TL_n^I + TL_n^M$ is meaningful from an economic point of view. If one supposes that the future profit belongs to the stakeholders of the insurance company, then the latter have to share the market risk component TL_n^M. Consequently, the insurance risk component TL_n^I represents the solvency risk related to the policyholders. To protect both components separately, one considers besides the overall initial available capital C_0 the initial *insurance risk* related *available capital* C_0^I (allocated to the insurance risk) and the initial *market risk* related *available capital* C_0^M (allocated to the market risk) such that $C_0 = C_0^I + C_0^M$. Again, we assume that these initial amounts are invested at the risk-free rate. It follows that the random values at time n of the insurance risk surplus, resp. market risk surplus, denoted by U_n^I, resp. U_n^M, are given by

$$U_n^I = C_0^I \cdot r_f^n - TL_n^I, \quad U_n^M = C_0^M \cdot r_f^n - TL_n^M. \tag{2.7}$$

The (total) required *initial solvency capital* over the time horizon $[0,n]$, also called (total) *economic capital* and denoted EC_n, is defined to be the minimum amount of capital required at initial time in order to satisfy the probability criterion $\Pr(U_n < 0) \le \varepsilon$ that avoids financial bankruptcy at the $100 \cdot \alpha = 100 \cdot (1-\varepsilon)$ confidence level.

Lemma 2.1. Assume that the available capital and the required initial solvency capital are invested at the risk-free rate. Then, the initial solvency capital is necessarily given by

$$EC_n = C_0 - v_f^n \cdot VaR_\varepsilon[U_n] = v_f^n \cdot VaR_\alpha[TL_n], \tag{2.8}$$

and satisfies the solvency condition $\Pr(U_n < 0) \le \varepsilon$.

Remarks 2.1. The quantity defined by $IC_n = -v_f^n \cdot VaR_\varepsilon[U_n]$, read *injected capital*, can be interpreted as the amount of capital to be injected (released) at the initial time in order to guarantee the solvency condition $\Pr(U_n < 0) \le \varepsilon$ (see e.g. Devineau and Loisel (2009), p.192-193). On the other hand, the use of the multi-period solvency capital formula (2.8) reduces in the Solvency II situation, i.e. a one-year time horizon $n=1$ and a confidence level $\alpha = 99.5\%$ to the so-called Solvency Capital Requirement (SCR).

Proof. The assumption implies that the injected capital IC_n is invested (disinvested) at the risk-free rate. Let $\tilde{U}_n = U_n + IC_n \cdot r_f^n = U_n - VaR_\varepsilon[U_n]$ denote the surplus at time n that results from adding at time zero the injected capital to the initial available capital. One has $\Pr(\tilde{U}_n < 0) = \Pr(U_n < VaR_\varepsilon[U_n]) = \varepsilon$, hence the probability criterion for the surplus is fulfilled. From $U_n = C_0 \cdot r_f^n - TL_n$, one obtains $VaR_\varepsilon[U_n] = C_0 \cdot r_f^n - VaR_\alpha[TL_n]$, which shows the last equality in (2.8). ◊

In the same way we define the *insurance risk economic capital* EC_n^I and the *market risk economic capital* EC_n^M by requiring that the insurance and market risk surplus in (2.7) satisfy the conditions $\Pr(U_n^I < 0) \le \varepsilon$ and $\Pr(U_n^M < 0) \le \varepsilon$. Lemma 2.1 implies the formulas

$$EC_n^{VaR_\alpha} = v_f^n \cdot VaR_\alpha[TL_n], \quad EC_n^{I,VaR_\alpha} = v_f^n \cdot VaR_\alpha[TL_n^I], \quad EC_n^{M,VaR_\alpha} = v_f^n \cdot VaR_\alpha[TL_n^M]. \tag{2.9}$$

The chosen notation emphasizes the fact that the economic capital quantities depend upon the value-at-risk (VaR) measure. It is common to use other risk measures like the popular conditional value-at-risk (CVaR) measure to some given confidence level α. Similarly to (2.9) one defines the *total CVaR economic capital*, the *insurance risk CVaR economic capital*, and the *market risk CVaR* economic capital:

$$EC_n^{CVaR_\alpha} = v_f^n \cdot CVaR_\alpha[TL_n], \quad EC_n^{I,CVaR_\alpha} = v_f^n \cdot CVaR_\alpha[TL_n^I], \quad EC_n^{M,CVaR_\alpha} = v_f^n \cdot CVaR_\alpha[TL_n^M]. \tag{2.10}$$

One notes that the two sources of risk, namely the insurance and market risk, depend on the same random rates of return and are therefore not stochastically independent. This is in alignment with Geman (2005) stating that "as a general rule, one can safely state that two sources of risk in the economy are never independent". Let us recall the decomposition $TL_n = TL_n^I + TL_n^M$. From the sub-additive property of the VaR and CVaR measures one gets the relationships

$$VaR_\alpha[TL_n] \le VaR_\alpha[TL_n^I] + VaR_\alpha[TL_n^M], \quad CVaR_\alpha[TL_n] \le CVaR_\alpha[TL_n^I] + CVaR_\alpha[TL_n^M]. \quad (2.11)$$

It is therefore possible to measure the total *diversification* effect at time n between these two risk categories through the non-negative difference defined and denoted by

$$DIV_n^\bullet = EC_n^{I,\bullet} + EC_n^{M,\bullet} - EC_n^\bullet . \quad (2.12)$$

The placeholder $\bullet$ stands for the VaR or CVaR measure and distinguishes between *VaR and CVaR diversification effects*.

3. <u>Heavy-tail claims and geometric Lévy return process</u>

We use an asymptotic result by Resnick and Willekens (1991) on the tail probability of moving averages with random coefficients to derive asymptotic formulas for the multi-period insurance risk VaR and CVaR measures in (2.9) and (2.10).

Assume that the Lévy process $\{Y_t, t \ge 0\}$ is right continuous with left limit and *Lévy-Khintchine triplet* $(\delta, \sigma^2, \upsilon)$, where $-\infty < \delta < \infty, \sigma \ge 0$ are two constants (the drift and diffusion component) and υ is the *Lévy measure* on $(-\infty, \infty)$ satisfying the properties $\upsilon(\{0\}) = 0$ and $\int_{-\infty}^{\infty} \min\{y^2, 1\} \cdot \upsilon(dy) < \infty$ (the jump component). Let $E[Y_1] > 0$ so that Y_t drifts to ∞ almost surely as $t \to \infty$. The *Lévy exponent* is the function defined by $\psi(z) = \ln E[e^{zY_1}], z \in (-\infty, \infty)$. If $\psi(z)$ is finite one has the *Lévy-Khintchine representation*

$$\psi(z) = \tfrac{1}{2}\sigma^2 z^2 + \delta z + \int_{-\infty}^{\infty} \left(e^{zy} - 1 - zy 1_{(-1,1)}(y)\right) \cdot \upsilon(dy), \quad (3.1)$$

and by Hammel's theorem one has

$$E[e^{zY_t}] = \exp\{t\psi(z)\} < \infty, \quad t \ge 0 . \quad (3.2)$$

On the other hand, the *renewal function* of the counting process $\{N_t, t \ge 0\}$ is defined by

$$\lambda_t = E[N_t] = \sum_{k=1}^{\infty} P(M_k \le t). \quad (3.3)$$

Denote by Λ the set of all $t \ge 0$ for which $0 < \lambda_t \le \infty$. Furthermore, we assume that the claim size distribution $F_X(x) = 1 - \overline{F}_X(x) \in R_{-\gamma}$. This means that the right tail is regularly varying in the sense that there exist a constant $\gamma > 0$ and a slowly varying function $L(x)$ such that $\overline{F}_X(x) = x^{-\gamma} L(x), x > 0$. The class $\Re = \cup_{\gamma > 0} R_{-\gamma}$ of all regularly varying claim size distributions contains popular heavy-tail distributions like the Pareto, Burr, log-gamma and t-distributions. The survival distribution of the accumulated aggregate claims satisfies the following *analytical asymptotic approximation*.

Theorem 3.1. Given is the Sparre Andersen risk model with geometric Lévy return process. Suppose that the right tail of the distribution of individual claims $F_X \in R_{-\gamma}$ is regularly varying with index $\gamma > 0$, and assume $\psi(z) < \infty, z \in (-\infty, \infty)$. Then the survival probability of the random accumulated aggregate claims $S_t^a = \sum_{k=1}^{N_t} R_{M_k,t} X_k$ satisfies the asymptotic equivalence as $x \to \infty$ for any fixed $t \in \Lambda$:

$$P(S_t^a > x) \sim \overline{F}_X(x) \cdot \int_{0-}^{t} \exp\{s\,\psi(\gamma)\} \cdot d\lambda_s \; . \qquad (3.4)$$

Recall that for two positive functions $a(x)$ and $b(x)$ the relationship $a(x) \sim b(x)$ in (3.4) means that these functions are *asymptotically equivalent* in the sense that $\lim_{x\to\infty} a(x)/b(x) = 1$.

Proof. To show this we need the following one-dimensional version of Theorem 2.1 in Resnick and Willekens (1991).

Lemma 3.1. Given is the random weighted sum $S = \sum_{k=1}^{\infty} W_k X_k$, where $\{X_1, X_2, ...\}$ is a sequence of i.i.d. non-negative random variables with common distribution function $F_X(x) \in R_{-\gamma}, \gamma > 0$, and let $\{W_1, W_2, ...\}$ be another sequence of non-negative random variables independent of $\{X_1, X_2, ...\}$. Then one has $P(S > x) \sim \overline{F}_X(x) \cdot \sum_{k=1}^{\infty} E[W_k^{\gamma}]$ provided one of the following conditions hold:

(C1) One has $0 < \gamma < 1$ and $\sum_{k=1}^{\infty} E[W_k^{\gamma\pm\varepsilon}] < \infty$ for some $0 < \varepsilon < \min\{\gamma, 1-\gamma\}$.

(C2) One has $\gamma \geq 1$ and $\sum_{k=1}^{\infty} E[W_k^{\gamma\pm\varepsilon}]^{1/(\gamma+\varepsilon)} < \infty$ for some $0 < \varepsilon < \gamma$.

We apply this result to the sum $S_t^a = \sum_{k=1}^{\infty} W_k X_k$, $W_k = 1_{\{M_k \le t\}} \cdot e^{Y_t - Y_{M_k}}$. Using that the laws of $Y_t - Y_{M_k}$ and Y_{t-M_k} are equal, i.e. the time homogeneous property of Lévy processes, one gets

$$\sum_{k=1}^{\infty} E[W_k^{\gamma}] = \sum_{k=1}^{\infty} E\left[1_{\{M_k \le t\}} \cdot e^{\gamma \cdot Y_{t-M_k}}\right] = \int_{0-}^{t} \exp\{(t-u)\cdot\psi(\gamma)\} \cdot d\lambda_u = \int_{0-}^{t} \exp\{s\,\psi(\gamma)\} \cdot d\lambda_s \, ,$$

which proves formula (3.4) provided (C1) or (C2) holds. But the latter is true for a finite Lévy exponent $\psi(z)$. ◊

Remarks 3.1. Theorem 3.1 is very similar to Theorem 2.1 in Tang et al. (2010), which is formulated for a Sparre Andersen risk model with geometric Lévy discounting. Nevertheless, the results must be distinguished. To do so, consider the geometric Lévy discounting process $D_t = e^{-Y_t}$, and let $S_t^d = \sum_{k=1}^{N_t} D_{M_k} X_k$ be the corresponding discounted aggregate claims process. Now, with the different Lévy exponent $\phi(z) = \ln E\left[e^{-zY_1}\right], z \in (-\infty, \infty)$, one has the asymptotic equivalence as $x \to \infty$ (uniformly for all $t \in \Lambda$)

$$P\left(S_t^d > x\right) \sim \overline{F}_X(x) \cdot \int_{0-}^{t} \exp\{s\phi(\gamma)\} \cdot d\lambda_s , \qquad (3.4')$$

provided there exists $\gamma^* > \gamma$ such that $\phi(\gamma^*) < 0$. We note three differences:

(i) Besides finiteness no other condition on the Lévy exponent $\psi(z)$ is required for the validity of (3.4). The condition $\psi(\gamma^*) < 0$ translates the fact that the impact of the insurance claims dominates that of the financial uncertainty, which is reflected in formula (3.4'). Indeed, the claim size survival distribution determines the exact decay rate of the tail aggregate claims probability while the claim frequency and the financial uncertainty influence this through the scaling factor.

(ii) Unlike (3.4) the asymptotic equivalence (3.4') holds uniformly. This additional property has been derived in Tang et al. (2010).

(iii) In contrast to S_t^a, the stochastic present value S_t^d does not have a straightforward economic interpretation. To get a meaningful concept, one should replace S_t^d by a random sum $\widetilde{S}_t^d = \sum_{k=1}^{N_t} \widetilde{D}_{M_k} X_k$, where $\widetilde{D}_t$ is some appropriate (state price) deflator associated to the geometric Lévy return process. Then, the deflated risk process $\widetilde{S}_t^d$ might be used for market-consistent actuarial valuation (see Wüthrich et al. (2010) for a thorough introduction to this topic).

In solvency applications of Theorem 3.1 one must ensure that for a given confidence level α the true VaR and CVaR risk measures are close to those values obtained from the asymptotic approximation (3.4). So far, the author did not make any attempt to quantify the accuracy of the obtained approximations (neither through numerical bounds nor using Monte Carlo simulation). This important problem is one of the numerous open issues in this area (see Section 5). For illustration, consider the special Sparre Andersen model with compound Poisson Pareto aggregate claims and random return following a geometric Lévy process, called here *compound Poisson Pareto Lévy model*. One obtains the following *asymptotic analytical formulas* for the required initial multi-period insurance risk economic capitals.

Theorem 3.2. Given is the compound Poisson Pareto Lévy model with Poisson renewal function $\lambda_t = \lambda \cdot t$, Pareto survival claims $\overline{F}_X(x) = \left(\frac{x}{\beta}\right)^{-\gamma}, x > \beta > 0, \gamma > 1$, and random return described by the finite Lévy exponent $\psi(z)$. Then the multi-period insurance risk VaR and CVaR economic capitals to the confidence level α over the time horizon $[0,n]$ are determined by the following asymptotic formulas as $\alpha \to 1$:

$$\begin{aligned} EC_n^{I,VaR_\alpha} &\sim v_f^n \cdot \mu_n(\lambda,\beta,\gamma,\psi(1)) \cdot \rho_{\alpha,n}^{VaR}(\lambda,\gamma,\psi), \\ EC_n^{I,CVaR_\alpha} &\sim v_f^n \cdot \mu_n(\lambda,\beta,\gamma,\psi(1)) \cdot \rho_{\alpha,n}^{CVaR}(\lambda,\gamma,\psi), \end{aligned} \tag{3.5}$$

where the parametric functions are defined by

$$\begin{aligned} &\mu_n(\lambda,\beta,\gamma,\delta) = \lambda n \cdot \beta \tfrac{\gamma}{\gamma-1} \cdot \bar{\bar{s}}_n(\delta), \quad \rho_{\alpha,n}^{VaR}(\lambda,\gamma,\psi) = (1-\alpha)^{-\frac{1}{\gamma}} \cdot \left(\tfrac{\gamma-1}{\gamma}\right) \cdot \frac{\mu_n\left(\lambda,\frac{\gamma-1}{\gamma},\gamma,\psi(\gamma)\right)^{\frac{1}{\gamma}}}{\mu_n\left(\lambda,\frac{\gamma-1}{\gamma},\gamma,\psi(1)\right)} - 1, \\ &\rho_{\alpha,n}^{CVaR}(\lambda,\gamma,\psi) = (1-\alpha)^{-\frac{1}{\gamma}} \cdot \frac{\mu_n\left(\lambda,\frac{\gamma-1}{\gamma},\gamma,\psi(\gamma)\right)^{\frac{1}{\gamma}}}{\mu_n\left(\lambda,\frac{\gamma-1}{\gamma},\gamma,\psi(1)\right)} - 1, \end{aligned} \tag{3.6}$$

and

$$\bar{\bar{s}}_t(\delta) = \begin{cases} \dfrac{1}{t}\dfrac{e^{\delta t}-1}{\delta}, & \delta > 0, \\ 1, \quad \delta = 0, \end{cases} \tag{3.7}$$

describes the average value of the accumulation function $\exp(\delta \cdot s)$ over the time period $[0,t]$.

Proof. First of all, one notes that the CVaR measure is finite if and only if the mean of the Pareto distribution exists, that is $E[X] = \beta \frac{\gamma}{\gamma-1} < \infty$ or $\gamma > 1$, as assumed. Set $t = n$ and solve for x (the asymptotic value-at-risk) in the equation

$$\bar{F}_X(x) \cdot C_n = 1-\alpha, \quad C_n := \lambda \cdot \int_{0-}^{n} \exp\{s\psi(\gamma)\} \cdot ds,$$

which is taken from the right-hand side of (3.4) as $x \to \infty$ or equivalently $\alpha \to 1$. On gets the asymptotic VaR formula

$$x = \beta \cdot (1-\alpha)^{-\frac{1}{\gamma}} \cdot C_n^{\frac{1}{\gamma}} = E[X] \cdot \tfrac{\gamma-1}{\gamma} \cdot (1-\alpha)^{-\frac{1}{\gamma}} \cdot C_n^{\frac{1}{\gamma}} \sim VaR_\alpha[S_n^a]. \tag{3.8}$$

On the other hand, a calculation of the Pareto stop-loss transform yields the expression $E[(X-d)_+] = \int_d^\infty \bar{F}_X(z)dz = E[X] \cdot \frac{1}{\gamma} \cdot \bar{F}_X(d)^{\frac{\gamma-1}{\gamma}}$. Making use of (3.4) and letting $x \to \infty$ one obtains the asymptotic stop-loss transform approximation of the accumulated aggregate claims

$$E\left[\left(S_n^a - x\right)_+\right] = \int_x^\infty P\left(S_n^a > z\right)dz \sim C_n \cdot \int_x^\infty \overline{F}_X(z)dz = E\left[\left(X-d\right)_+\right]\cdot C_n = E[X]\cdot \tfrac{1}{\gamma}\cdot \overline{F}_X(x)^{\frac{\gamma-1}{\gamma}}\cdot C_n. \quad (3.9)$$

Inserting (3.8) into (3.9) one gets further

$$E\left[\left(S_n^a - VaR_\alpha\left[S_n^a\right]\right)_+\right] \sim E[X]\cdot \tfrac{1}{\gamma}\cdot \left[(1-\alpha)^{-\frac{1}{\gamma}}\cdot C_n^{\frac{1}{\gamma}}\right]^{-\gamma\cdot\left(\frac{\gamma-1}{\gamma}\right)}\cdot C_n = E[X]\cdot \tfrac{1}{\gamma}\cdot (1-\alpha)^{\frac{\gamma-1}{\gamma}}\cdot C_n^{1-\frac{\gamma-1}{\gamma}}.$$

It follows that

$$\begin{aligned} CVaR_\alpha\left[S_n^a\right] &= VaR_\alpha\left[S_n^a\right] + \tfrac{1}{1-\alpha}\cdot E\left[\left(S_n^a - VaR_\alpha\left[S_n^a\right]\right)_+\right] \\ &\sim E[X]\cdot \tfrac{\gamma-1}{\gamma}\cdot (1-\alpha)^{-\frac{1}{\gamma}}\cdot C_n^{\frac{1}{\gamma}} + E[X]\cdot\tfrac{1}{\gamma}\cdot(1-\alpha)^{\frac{\gamma-1}{\gamma}}\cdot C_n^{1-\frac{\gamma-1}{\gamma}} = E[X]\cdot(1-\alpha)^{-\frac{1}{\gamma}}\cdot C_n^{\frac{1}{\gamma}}. \end{aligned} \quad (3.10)$$

Now, let us calculate the expected accumulated aggregate claims of the compound Poisson Pareto Lévy model. From $S_t^a = \sum_{k=1}^{\infty} 1_{\{M_k \le t\}} e^{Y_t - Y_{M_k}} X_k$ and the time homogeneous property of a Lévy process (the laws of $Y_t - Y_{M_k}$ and Y_{t-M_k} are equal) one gets

$$E\left[S_t^a\right] = E[X]\cdot \int_0^t e^{(t-s)\psi(1)} d\lambda_s = \lambda t\cdot \beta\tfrac{\gamma}{\gamma-1}\cdot \overline{\overline{s}}_t\left(\psi(1)\right) = \mu_t\left(\lambda,\beta,\gamma,\psi(1)\right). \quad (3.11)$$

Noting that $C_n = \lambda n\cdot \overline{\overline{s}}_n\left(\psi(\gamma)\right)$ and inserting (3.8) and (3.11) into the defining multi-period VaR economic capital formula (2.9) one obtains

$$\begin{aligned} EC_n^{I,VaR_\alpha} &= v_f^n\cdot VaR_\alpha\left[S_n^a - E\left[S_n^a\right]\right] \sim v_f^n\cdot E[X]\cdot\left\{\tfrac{\gamma-1}{\gamma}\cdot(1-\alpha)^{-\frac{1}{\gamma}}\cdot\left[\lambda n\cdot\overline{\overline{s}}_n\left(\psi(\gamma)\right)\right]^{\frac{1}{\gamma}} - \lambda n\cdot\overline{\overline{s}}_n\left(\psi(1)\right)\right\} \\ &= v_f^n\cdot\left[\lambda n\cdot\beta\tfrac{\gamma}{\gamma-1}\cdot\overline{\overline{s}}_n\left(\psi(1)\right)\right]\cdot\left\{(1-\alpha)^{-\frac{1}{\gamma}}\cdot\tfrac{\gamma-1}{\gamma}\cdot\frac{\left[\lambda n\cdot\overline{\overline{s}}_n\left(\psi(\gamma)\right)\right]^{\frac{1}{\gamma}}}{\lambda n\cdot\overline{\overline{s}}_n\left(\psi(1)\right)} - 1\right\}, \end{aligned}$$

which yields the upper part of (3.5) by the definitions in (3.6). Similarly, inserting (3.10) and (3.11) into (2.10) one obtains

$$\begin{aligned} EC_n^{I,CVaR_\alpha} &= v_f^n\cdot CVaR_\alpha\left[S_n^a - E\left[S_n^a\right]\right] \sim v_f^n\cdot E[X]\cdot\left\{(1-\alpha)^{-\frac{1}{\gamma}}\cdot\left[\lambda n\cdot\overline{\overline{s}}_n\left(\psi(\gamma)\right)\right]^{\frac{1}{\gamma}} - \lambda n\cdot\overline{\overline{s}}_n\left(\psi(1)\right)\right\} \\ &= v_f^n\cdot\left[\lambda n\cdot\beta\tfrac{\gamma}{\gamma-1}\cdot\overline{\overline{s}}_n\left(\psi(1)\right)\right]\cdot\left\{(1-\alpha)^{-\frac{1}{\gamma}}\cdot\frac{\left[\lambda n\cdot\overline{\overline{s}}_n\left(\psi(\gamma)\right)\right]^{\frac{1}{\gamma}}}{\lambda n\cdot\overline{\overline{s}}_n\left(\psi(1)\right)} - 1\right\}, \end{aligned}$$

which yields the lower part of (3.5) by the definitions in (3.6). ◊

Remarks 3.2. Concerning (3.11) one notes that $\mu_t(\lambda,\beta,\gamma,\delta)=\lambda t\cdot\beta\frac{\gamma}{\gamma-1}\cdot\bar{\bar{s}}_t(\delta)$ is the mean of the accumulated compound Poisson Pareto model with constant force of interest δ (e.g. Taylor (1979), formula (16), Willmot (1989), Ross (2003), Example 5.19). More generally, recursive formulas for the moments and the moment generating function of the corresponding discounted model with constant and stochastic forces of interest are found in Léveillé and Garrido (2001a/b), Ren (2008), Léveillé et al. (2009) and Léveillé and Adékambi (2010a/b).

Examples 3.1. For a geometric Brownian process with drift and Lévy exponent $\psi(z)=\frac{1}{2}\sigma^2z^2+\delta z$, the model can be called *compound Poisson Pareto Black-Scholes model*. The formulas (3.5)-(3.6) are functions of the risk-adjusted rates $\delta_\gamma=\psi(\gamma)=\gamma\cdot\left(\delta+\frac{1}{2}\gamma\sigma^2\right)$ and $\delta_1=\psi(1)=\delta+\frac{1}{2}\sigma^2$. By absence of return randomness, that is $\sigma=0$, these formulas coincide with the ones from an accumulated compound Poisson Pareto model with constant force of interest δ, which are derived similarly. Finally, without return, that is $\delta_\gamma=\psi(\gamma)=0$, $\delta_1=\psi(1)=0$, the formulas simplify to

$$\begin{aligned}&EC_n^{I,VaR_\alpha}\sim v_f^n\cdot\mu_n(\lambda,\beta,\gamma)\cdot\rho_{\alpha,n}^{VaR}(\lambda,\gamma),\quad EC_n^{I,CVaR_\alpha}\sim v_f^n\cdot\mu_n(\lambda,\beta,\gamma)\cdot\rho_{\alpha,n}^{CVaR}(\lambda,\gamma),\\&\mu_n(\lambda,\beta,\gamma)=\lambda n\cdot\beta\tfrac{\gamma}{\gamma-1},\\&\rho_{\alpha,n}^{VaR}(\lambda,\gamma)=(1-\alpha)^{-\frac{1}{\gamma}}\left(\tfrac{\gamma-1}{\gamma}\right)(\lambda n)^{-\frac{\gamma-1}{\gamma}}-1,\quad\rho_{\alpha,n}^{CVaR}(\lambda,\gamma)=(1-\alpha)^{-\frac{1}{\gamma}}(\lambda n)^{-\frac{\gamma-1}{\gamma}}-1.\end{aligned}\tag{3.12}$$

A brief analysis of the short and long term properties of the derived asymptotic solvency capital formulas follows. Table 3.1 compares (3.5) for the Black-Scholes return model with (3.12). The parameter set is $(\lambda,\beta,\gamma,\delta,\sigma)=(50,1/3,1.5,4\%,15\%)$, the risk-free discount rate is $v_f=(1+\delta-\frac{1}{2}\sigma^2)^{-1}=1.02875^{-1}$, and the confidence level is $\alpha=99.5\%$ for the VaR criterion and $\alpha=99\%$ for the CVaR criterion over the first 20 years. Concerning time dependence the required VaR economic capital first increases rapidly and importantly in the first 10 years and then remains at a relatively stable level. The dependence upon the return process increases over time and yields an important penalty over the longer time horizons. Similarly, the required CVaR economic capital increases steadily over time beginning at more than double the VaR level and reaches more than 5 times the initial level after 20 years. The return effect is similar, but in contrast to VaR the differences remain relatively stable over the longer time horizons.

On can question whether a constant confidence level over all time horizons is adequate. Alternatively, one might specify a constant initial economic capital independently of the time horizon, the so-called *stability criterion*, for which we refer to Hürlimann (2010a) for a more detailed analysis. According to this criterion, the confidence levels $\alpha(n), n=1,2,3,...$, are determined by the conditions

$$r_f^n\cdot EC_n^{I,VaR_{\alpha(n)}}=r_f\cdot EC_1^{I,VaR_{\alpha(1)}},\ r_f^n\cdot EC_n^{I,CVaR_{\alpha(n)}}=r_f\cdot EC_1^{I,CVaR_{\alpha(1)}},\quad n=2,3,.... \tag{3.13}$$

Table 3.2 is based on the parameters of Table 3.1 and reveals almost stable VaR confidence levels. The CVaR confidence level decreases in the very first years and then remains quite stable.

Table 3.1: Compound Poisson Pareto Black-Scholes versus compound Poisson Pareto

	VaR		CVaR	
n	(3.5)	(3.10)	(3.5)	(3.10)
1	104.9	101.8	242.6	235.6
2	146.3	137.6	365.1	344.1
3	173.6	157.8	460.6	420.9
4	193.1	169.5	541.3	479.3
5	207.6	175.7	612.4	525.0
6	218.6	177.9	676.8	561.4
7	226.9	177.3	736.1	590.4
8	233.2	174.5	791.5	613.5
9	238.0	170.0	844.0	631.6
10	241.5	164.3	894.1	645.6
11	244.0	157.6	942.4	656.2
12	245.7	150.1	989.3	663.7
13	246.8	142.1	1035.2	668.7
14	247.5	133.6	1080.2	671.4
15	247.7	124.9	1124.8	672.3
16	247.7	115.9	1169.1	671.4
17	247.6	106.8	1213.2	669.0
18	247.4	97.7	1257.5	665.4
19	247.1	88.5	1301.9	660.6
20	246.9	79.4	1346.8	654.9

Table 3.2: Comparison of confidence levels under the stability criterion

	α(n) for VaR		α(n) for CVaR	
n	(3.5)	(3.10)	(3.5)	(3.10)
1	99.5%	99.5%	97.4%	99.0%
2	99.3%	99.3%	96.5%	98.4%
3	99.3%	99.3%	96.1%	98.1%
4	99.2%	99.3%	96.0%	97.9%
5	99.2%	99.3%	96.0%	97.7%
6	99.2%	99.3%	96.0%	97.6%
7	99.2%	99.3%	96.1%	97.6%
8	99.3%	99.3%	96.1%	97.5%
9	99.3%	99.3%	96.2%	97.5%
10	99.3%	99.4%	96.2%	97.5%
11	99.3%	99.4%	96.3%	97.5%
12	99.3%	99.4%	96.4%	97.5%
13	99.3%	99.4%	96.4%	97.6%
14	99.3%	99.4%	96.5%	97.6%
15	99.3%	99.4%	96.5%	97.6%
16	99.3%	99.4%	96.5%	97.6%
17	99.3%	99.4%	96.6%	97.6%
18	99.3%	99.5%	96.6%	97.7%
19	99.4%	99.5%	96.7%	97.7%
20	99.4%	99.5%	96.7%	97.7%

4. Sub-exponential tail claims and constant force of interest

The class $\Re$ in Section 3 excludes some important common distributions like the log-normal and the heavy-tailed Weibull. Fortunately, a version of the asymptotic tail equivalence (3.4) has been proved by Hao and Tang (2008) for the discounted compound Poisson model with sub-exponential claim size distribution and constant force of interest. The accumulated return version of this result is re-used here in the context of solvency risk calculations.

Recall that a non-negative claim size distribution is *sub-exponential*, denoted by $F_X \in \boldsymbol{S}$, if $\overline{F}_X(x) = 1 - F_X(x) > 0$ for all $x \geq 0$ and the limiting relationship

$$\lim_{x\to\infty} \overline{F}_X^{*n}(x) / \overline{F}_X(x) = n \tag{4.1}$$

holds for all (or, equivalently, for some) $n = 2,3,...$, where $\overline{F}_X^{*n}$ denotes the n-fold convolution of F_X. Recall that every sub-exponential distribution is long-tailed, that is $F_X \in \boldsymbol{L}$, in the sense that

$$\lim_{x\to\infty} \overline{F}_X(x-y) / \overline{F}_X(x) = 1 \tag{4.2}$$

for all (or, equivalently, for some) $y \neq 0$. The class $\boldsymbol{S}$ contains the class $\boldsymbol{ERV}$ of distributions with extended-regularly-varying tails, for which there exist some constants $0 < a \leq b \leq \infty$, such that

$$v^{-b} \leq \liminf_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} \leq \limsup_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} \leq v^{-a} \tag{4.3}$$

holds for all $v \geq 1$. In the case $a = b$ relation (4.3) defines the class $R_{-a} \subset \Re$ of regularly varying distributions with index $-a$. Another useful class is the sub-class $\boldsymbol{A} \subset \boldsymbol{S}$ introduced by Konstantinides et al. (2002). Note that $F_X \in \boldsymbol{A}$ provided $F_X \in \boldsymbol{S}$ and for some $v > 1$, one has

$$\limsup_{x\to\infty} \frac{\overline{F}_X(vx)}{\overline{F}_X(x)} < 1. \tag{4.4}$$

It has been noted that (4.4) is satisfied by almost all useful distributions with unbounded supports, which implies that $\boldsymbol{A}$ almost coincides with $\boldsymbol{S}$. To summarize, one has the set inclusions

$$\Re \subset \boldsymbol{ERV} \subset \boldsymbol{A} \subset \boldsymbol{S} \subset \boldsymbol{L}. \tag{4.5}$$

For further details on heavy-tailed distributions the reader is referred to Embrechts et al. (1997). As a technical condition we need the *equilibrium distribution* F_X^e of F_X defined by

$$F_X^e(x) = \tfrac{1}{\mu_X} \cdot \int_0^x \overline{F}_X(s)ds, \quad \mu_X = E[X] > 0, \quad x \geq 0. \tag{4.6}$$

Theorem 4.1. Given is the accumulated compound Poisson model with sub-exponential claim size distribution $F_X \in \boldsymbol{S}$ and constant force of interest δ. If $F_X^e \in \boldsymbol{A}$ the asymptotic equivalence as $x \to \infty$

$$P\left(S_t^a > x\right) \sim \lambda \cdot \int_0^t \overline{F}_X(x \cdot e^{\delta \cdot (s-t)}) ds \tag{4.7}$$

holds uniformly for all $t \in (0, \infty]$.

Proof. One notes that $S_t^a = e^{\delta \cdot t} \cdot S_t^d$, where $S_t^d = \sum_{k=1}^{\infty} 1_{\{M_k \le t\}} e^{-\delta \cdot M_k} X_k$ represents the discounted aggregate claims. The results follows from Theorem 2.3 in Hao and Tang (2008), which shows the asymptotic equivalence $P\left(S_t^d > x\right) \sim \lambda \cdot \int_0^t \overline{F}_X(x \cdot e^{\delta \cdot s}) ds$. ◊

As simple illustration let us derive the corresponding asymptotic analytical solvency capital formulas for a log-normal claim size distribution (see e.g. Zuanetti et al. (2006) for motivation).

Theorem 4.2. Given is the accumulated compound Poisson model with log-normal claim size distribution $F_X = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), x \ge 0, \mu, \sigma > 0,$ and constant force of interest δ. Then the multi-period insurance risk VaR and CVaR economic capitals to the confidence level α over the time horizon $[0, n]$ are determined by the following asymptotic formulas as $\alpha \to 1$:

$$\begin{aligned}
&EC_n^{I,VaR_\alpha} \sim v_f^n \cdot \mu_n(\lambda, \mu_X) \cdot \rho_{\alpha,n}^{VaR}(\lambda, \nu_X, \delta), \quad EC_n^{I,CVaR_\alpha} \sim v_f^n \cdot \mu_n(\lambda, \mu_X) \cdot \rho_{\alpha,n}^{CVaR}(\lambda, \nu_X, \delta), \\
&\mu_n(\lambda, \mu_X) = \lambda n \cdot \mu_X, \quad \mu_X = \exp(\mu + \tfrac{1}{2}\sigma^2), \quad \nu_X^2 = \exp(\sigma^2) - 1, \\
&\rho_{\alpha,n}^{VaR}(\lambda, \nu_X, \delta) = \frac{\exp\left\{\delta n + \Phi^{-1}\left(1 - \frac{1-\alpha}{\lambda n}\right) \cdot \sqrt{\ln(1+\nu_X^2)} - \ln(\lambda n)\right\}}{\sqrt{1+\nu_X^2}} - \overline{\overline{s}}_n(\delta), \\
&\rho_{\alpha,n}^{CVaR}(\lambda, \nu_X, \delta) = \exp(\delta n) \cdot \frac{1 - \Phi\left(\Phi^{-1}\left(1 - \frac{1-\alpha}{\lambda n}\right) - \sqrt{\ln(1+\nu_X^2)}\right)}{1-\alpha} - \overline{\overline{s}}_n(\delta).
\end{aligned} \tag{4.8}$$

Proof. Rewrite the integral in (4.7) as $J_t(x) = \int_0^t \overline{\Phi}\left(\frac{\ln x - \mu}{\sigma} + \frac{\delta}{\sigma}(s-t)\right) ds$. Making the change of variables $u = \frac{\ln x - \mu}{\sigma} + \frac{\delta}{\sigma}(s-t)$ one obtains

$$J_t(x) = \frac{\sigma}{\delta} \cdot \int_{\frac{\ln x - \mu}{\sigma} - \frac{\delta}{\sigma} t}^{\frac{\ln x - \mu}{\sigma}} \overline{\Phi}(u) du = \frac{\sigma}{\delta} \cdot \left\{\pi\left(\frac{\ln x - \mu}{\sigma} - \frac{\delta}{\sigma} t\right) - \pi\left(\frac{\ln x - \mu}{\sigma}\right)\right\},$$

with $\pi(z) = \int_z^{\infty} \overline{\Phi}(u) du$ the stop-loss transform of a standard normal random variable. Using that $\pi(z) = \varphi(z) - z \cdot \overline{\Phi}(z), \varphi(z) = \Phi'(z),$ this can be rewritten as

$$J_t(x) = t\cdot\overline{\Phi}\left(\tfrac{\ln x-\mu}{\sigma}-\tfrac{\delta}{\sigma}t\right)+\tfrac{\sigma}{\delta}\cdot\left\{\varphi\left(\tfrac{\ln x-\mu}{\sigma}-\tfrac{\delta}{\sigma}t\right)-\varphi\left(\tfrac{\ln x-\mu}{\sigma}\right)-\tfrac{\ln x-\mu}{\sigma}\cdot\left[\overline{\Phi}\left(\tfrac{\ln x-\mu}{\sigma}-\tfrac{\delta}{\sigma}t\right)-\overline{\Phi}\left(\tfrac{\ln x-\mu}{\sigma}\right)\right]\right\}.$$

This implies that asymptotically as $x\to\infty$ and uniformly for all $t\in(0,\infty]$ one has

$$P\left(S_t^a > x\right)\sim \lambda t\cdot\overline{\Phi}\left(\tfrac{\ln x-(\mu+\delta t)}{\sigma}\right). \tag{4.9}$$

Now, set $t=n$ and solve for x (the asymptotic value-at-risk) in the equation $\lambda n\cdot\overline{\Phi}\left(\tfrac{\ln x-(\mu+\delta n)}{\sigma}\right)=1-\alpha$, which yields $x=\mu_X\cdot\exp\left\{\delta n+\sigma\cdot\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda n}\right)-\tfrac{1}{2}\sigma^2\right\}$. Noting that $\sigma=\sqrt{\ln(1+\nu_X^2)}$ and letting $x\to\infty$, or equivalently $\alpha\to 1$, one obtains the asymptotic VaR formula

$$VaR_\alpha\left[S_n^a\right]\sim\mu_X\cdot\exp\left\{\delta n+\sigma\cdot\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda n}\right)-\tfrac{1}{2}\sigma^2\right\}=\mu_X\cdot\frac{\exp\left\{\delta n+\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda n}\right)\cdot\sqrt{\ln(1+\nu_X^2)}\right\}}{\sqrt{1+\nu_X^2}}. \tag{4.10}$$

On the other side one has the mean formula $E\left[S_n^a\right]=\lambda n\cdot\mu_X\cdot\overline{\overline{s}}_n(\delta)$ (see the Remarks 3.2). By definition of the economic capital formula (2.9) one has $EC_n^{I,VaR_\alpha}=v_f^n\cdot\left\{VaR_\alpha\left[S_n^a\right]-E\left[S_n^a\right]\right\}$. Letting $\alpha\to 1$, inserting the obtained expressions and rearranging, the asymptotic VaR formula (4.8) follows without difficulty. For the CVaR one notes that (4.9) is the tail distribution of a scaled log-normal distribution with parameters $(\mu+\delta t,\sigma)$ such that asymptotically as $x\to\infty$ one gets

$$E\left[\left(S_t^a-x\right)_+\right]\sim\lambda t\cdot\left\{e^{\mu+\delta t+\frac{1}{2}\sigma^2}\cdot\Phi\left(\frac{\mu+\delta t-\ln(x)}{\sigma}+\sigma\right)-x\cdot\Phi\left(\frac{\mu+\delta t-\ln(x)}{\sigma}\right)\right\}. \tag{4.11}$$

Inserting the asymptotic value-at-risk expression $x=\exp\left\{\mu+\delta t+\sigma\cdot\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda t}\right)\right\}\sim VaR_\alpha\left[S_t^a\right]$ into (4.11) one gets further

$$\begin{aligned}&E\left[\left(S_t^a-VaR_\alpha\left[S_t^a\right]\right)_+\right]\sim\lambda t\cdot\left\{\mu_X\cdot e^{\delta t}\cdot\Phi\left(\sigma-\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda t}\right)\right)-\Phi\left(-\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda t}\right)\right)\cdot VaR_\alpha\left[S_t^a\right]\right\}\\&=\lambda t\cdot\mu_X\cdot e^{\delta t}\cdot\left\{1-\Phi\left(\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda t}\right)-\sigma\right)\right\}-(1-\alpha)\cdot VaR_\alpha\left[S_t^a\right],\end{aligned}$$

hence

$$CVaR_\alpha\left[S_n^a\right]=VaR_\alpha\left[S_n^a\right]+\tfrac{1}{1-\alpha}\cdot E\left[\left(S_n^a-VaR_\alpha\left[S_n^a\right]\right)_+\right]\sim\lambda t\cdot\mu_X\cdot e^{\delta t}\cdot\left\{1-\Phi\left(\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda t}\right)-\sigma\right)\right\}.$$

Inserting into (2.10) one gets the desired economic capital CVaR expression in (4.8). ◊

Examples 4.1. Let us compare the obtained asymptotic results with the standard Solvency II SCR specification of the non-life insurance risk. We argue that the number of claims is a main driving factor of the risk process. Indeed, due to the law of large numbers the larger this number the less economic capital is actually needed. A comparison of the asymptotic VaR formula with the current standard VaR specification in Table 4.1 shows that the effect of this risk factor is underestimated in the following sense. Measured in units of the standard deviation $\sigma_S = \sqrt{Var[S_1]}$ of aggregate claims, the relative reduction of the asymptotic VaR economic capital factor with respect to the standard SCR one increases with the expected number of claims. Of course, this qualitative result holds in a strict quantitative sense provided the asymptotic VaR formula is sufficiently accurate to validate this statement. Unfortunately, the question of accuracy is one of the main open issues (see Section 5, issue (I1)). For the practical evaluation, recall the QIS5 (2010) standard SCR specification of the non-life insurance risk, which is obtained under the assumptions of a log-normal distribution of the one-year aggregate claims random variable S_1 with vanishing constant force of interest $\delta = 0$ for the confidence level $\alpha_0 = 99.5\%$ (e.g. Hürlimann (2010b), formula (3.4)):

$$r_f \cdot EC_1^{I,S2,VaR_{\alpha_0}} = VaR_{\alpha_0}[S_1] - E[S_1] = \rho_{\alpha_0}^{S2,VaR}(\nu_S)\cdot E[S_1],$$
$$\rho_{\alpha_0}^{S2,VaR}(\nu_S) = \frac{\exp\left\{\Phi^{-1}(\alpha_0)\cdot\sqrt{\ln(1+\nu_S^2)}\right\}}{\sqrt{1+\nu_S^2}} - 1, \quad \nu_S^2 = \frac{Var[S_1]}{E[S_1]^2} = \frac{1+\nu_X^2}{\lambda}. \tag{4.12}$$

For the sake of comparison, the one-year asymptotic VaR formula in (4.8) with $\delta = 0$ can be rewritten as

$$r_f \cdot EC_1^{I,as,VaR_\alpha} \sim \rho_\alpha^{as,VaR}(\lambda,\nu_X)\cdot E[S_1],$$
$$\rho_\alpha^{as,VaR}(\lambda,\nu_X) = \frac{\exp\left\{\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda}\right)\cdot\sqrt{\ln(1+\nu_X^2) - \ln\lambda}\right\}}{\sqrt{1+\nu_X^2}} - 1. \tag{4.13}$$

A standard CVaR SCR is obtained under the same assumptions as for the VaR SCR but with a reduced confidence level $\alpha_0 = 98.675\%$ (Hürlimann (2009b), formula (13.9)):

$$r_f \cdot EC_1^{I,S2,CVaR_{\alpha_0}} = CVaR_{\alpha_0}[S_1] - E[S_1] = \rho_{\alpha_0}^{S2,CVaR}(\nu_S)\cdot E[S_1],$$
$$\rho_{\alpha_0}^{S2,CVaR}(\nu_S) = \frac{1-\Phi\left(\Phi^{-1}(\alpha_0) - \sqrt{\ln(1+\nu_S^2)}\right)}{1-\alpha_0} - 1. \tag{4.14}$$

For the sake of comparison, the one-year asymptotic CVaR formula in (4.8) with $\delta = 0$ can be rewritten as

$$r_f \cdot EC_1^{I,as,CVaR_\alpha} \sim \rho_\alpha^{as,CVaR}(\lambda,\nu_X)\cdot E[S_1],$$
$$\rho_\alpha^{as,CVaR}(\lambda,\nu_X) = \frac{1-\Phi\left(\Phi^{-1}\left(1-\tfrac{1-\alpha}{\lambda}\right) - \sqrt{\ln(1+\nu_X^2)}\right)}{1-\alpha} - 1. \tag{4.15}$$

In the Tables below we set $\alpha_0 = 99.5\%, \alpha = 99.5\%, 99.75\%, 99.8\%, \nu_X = 9$ in (4.12)-(4.13) and $\alpha_0 = 98.675\%, \alpha = 98.675\%, 99.125\%, 99.3\%, \nu_X = 9$ in (4.14)-(4.15) and compare the ratios $\rho_{\alpha_0}^{S2,VaR}(\nu_S)/\nu_S$ versus $\rho_{\alpha}^{as,VaR}(\lambda,\nu_X)/\nu_S$, resp. $\rho_{\alpha_0}^{S2,CVaR}(\nu_S)/\nu_S$ versus $\rho_{\alpha}^{as,CVaR}(\lambda,\nu_X)/\nu_S$. The different parameters satisfy the relationship $\lambda\nu_S^2 = 1+\nu_X^2$ introduced in (4.12). The made comparisons identify the expected number of claims as a main driving factor in solvency capital requirement, which has been neglected so far in the Solvency II standard approach.

Table 4.1: One-year asymptotic VaR (4.13) versus standard VaR (4.12)

confidence level				S2,VaR 0.995	as,VaR 0.995	reduction / penalty	as,VaR 0.9975	reduction / penalty	as,VaR 0.998	reduction / penalty
λ	vX	σX	vS							
50	9	2.099	1.281	5.300	3.458	34.8%	5.296	0.1%	6.023	-13.6%
100	9	2.099	0.906	4.904	3.193	34.9%	4.972	-1.4%	5.670	-15.6%
200	9	2.099	0.640	4.391	2.735	37.7%	4.439	-1.1%	5.105	-16.2%
300	9	2.099	0.523	4.100	2.359	42.5%	4.014	2.1%	4.658	-13.6%
400	9	2.099	0.453	3.911	2.035	48.0%	3.653	6.6%	4.281	-9.5%
500	9	2.099	0.405	3.776	1.747	53.7%	3.335	11.7%	3.950	-4.6%
600	9	2.099	0.370	3.674	1.486	59.6%	3.049	17.0%	3.654	0.5%
700	9	2.099	0.342	3.593	1.245	65.4%	2.787	22.4%	3.383	5.9%
800	9	2.099	0.320	3.528	1.021	71.1%	2.544	27.9%	3.132	11.2%
900	9	2.099	0.302	3.473	0.811	76.7%	2.317	33.3%	2.898	16.6%
1000	9	2.099	0.286	3.427	0.612	82.1%	2.103	38.6%	2.678	21.9%

Table 4.2: One-year asymptotic CVaR (4.15) versus standard CVaR (4.14)

conficence level				S2,CVaR 0.98675	as,CVaR 0.98675	reduction / penalty	as,CVaR 0.99125	reduction / penalty	as,CVaR 0.993	reduction / penalty
λ	vX	σX	vS							
50	9	2.099	1.281	5.625	4.287	23.8%	5.464	2.9%	6.194	-10.1%
100	9	2.099	0.906	5.083	3.963	22.0%	5.096	-0.3%	5.795	-14.0%
200	9	2.099	0.640	4.482	3.446	23.1%	4.525	-1.0%	5.188	-15.8%
300	9	2.099	0.523	4.159	3.035	27.0%	4.080	1.9%	4.721	-13.5%
400	9	2.099	0.453	3.954	2.687	32.0%	3.706	6.3%	4.330	-9.5%
500	9	2.099	0.405	3.810	2.380	37.5%	3.379	11.3%	3.990	-4.7%
600	9	2.099	0.370	3.701	2.104	43.2%	3.086	16.6%	3.686	0.4%
700	9	2.099	0.342	3.616	1.850	48.8%	2.818	22.1%	3.409	5.7%
800	9	2.099	0.320	3.547	1.615	54.5%	2.571	27.5%	3.154	11.1%
900	9	2.099	0.302	3.490	1.396	60.0%	2.340	33.0%	2.915	16.5%
1000	9	2.099	0.286	3.441	1.188	65.5%	2.122	38.3%	2.692	21.8%

The obtained results reveal that by increasing the confidence level, there is a better discrimination of these formulas with respect to insurers with small and large expected number of claims. In this respect, another open question is whether the accuracy of the asymptotic VaR and CVaR formulas increases by increasing the confidence level. In the examples, it seems that the choices $\alpha = 99.8\%$ for VaR and $\alpha = 99.3\%$ for CVaR provide a more balanced discrimination of the solvency requirements. However, in virtue of the unsolved accuracy question, it is not clear how to set in general the appropriate confidence levels of the asymptotic formulas in order to fulfill the Solvency II calibration test. According to the latter, the calculated SCR should be a fair, unbiased estimate of the risk as measured by the common SCR target criterion (e.g. Doff (2007), p.131).

5. Conclusions and outlook.

A summary of what has been obtained and a short outlook of possible further investigations might be helpful. Our starting point has been the Sparre Andersen risk model with geometric Lévy stochastic returns. Besides a classical modeling of the aggregate claims it allows for a flexible modeling of the investment returns (e.g. via the class of generalized hyperbolic distributions). We have decomposed the random accumulated surplus over any finite time horizon into insurance risk, market risk and future profit related components. By assuming that the initial available capital and the required solvency capital are invested at the risk-free rate, this decomposition has led to various natural multi-period economic capital amounts for both the VaR and CVaR measures. Besides the required total economic capital, we have justified notions of insurance risk and market risk economic capitals. We have observed that in a dual environment of random aggregate claims and random returns, the insurance risk and market risk components are dependent and lead to the measurement of the diversification effect between the associated economic capital measures.

We have focused our approach on the study of the multi-period insurance risk economic capital for claim size distributions with regularly varying tail (Section 3) and sub-exponential tail (Section 4). In particular, we have obtained some asymptotic economic capital formulas for the compound Poisson Pareto Lévy model (Theorem 3.2) and the compound Poisson model with log-normal claim size and constant force of interest (Theorem 4.2). Through numerical examples, we have documented interesting features for the compound Poisson Pareto Black-Scholes model (Examples 3.1). Moreover, a comparison with the standard Solvency II specification of the non-life insurance SCR has been undertaken (Examples 4.1).

Finally, the present approach suggests many open issues for further investigations:

(1) Error bounds for the asymptotic formulas and their speed of convergence are not known to the author. Moreover, no attempt to quantify the accuracy of them through Monte-Carlo simulation has been undertaken.

(2) The inflation of the claims has not been taken into account and will obviously reduce the premium loading defined in (2.3).

(3) Is it possible to use the available similar results for the Sparre Andersen risk model with geometric Lévy stochastic discounting in the form suggested in Remarks 3.1, point (iii)? For this purpose, it might be interesting to construct a Lévy (state price) deflator.

(4) The calculation of the total and market risk economic capital measures along the line of Section 2 has not been touched upon so far. This is of great interest and allows for a quantification of the diversification effect defined in (2.12).

(5) The possible stochastic dependence between claims and number of claims introduces further uncertainties in the evaluation of economic capital measures. Is it possible to make use of the many diverse asymptotic approximations in this area (e.g. Embrechts et al. (2009))?

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