

Economic Capital and the Aggregation of Risks using Copulas*

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Abstract

Insurance companies measure and manage capital across a broad range of diverse business products. Thus there is a need for the aggregation of the losses from the various business lines whose risk distributions vary. Risk dependencies between losses from different business lines have long been recognised in the insurance industry as integral factors driving the insurer's aggregate loss process. However, in the past, there has been limited attempt at adequately modelling the dependence structure to be factored in the aggregation process for capital determination purposes. The current industry standard is to solely use linear correlations to describe the dependence structure. While being computationally convenient and straightforward to understand, linear correlations fail to capture all the dependence structure that exist between losses from multiple business lines. Other more general dependence modelling techniques such as copulas have become popular recently. In this paper, we address the issue of the aggregation of risks using copula models. Copulas can be used to construct joint multivariate distributions of the losses and provide a rather flexible and realistic model of allowing for the dependence structure, while separating the effects of peculiar characteristics of the marginal distributions such as thickness of tails. This modelling structure allows us to explore the impact of dependencies of risks on the total required economic capital. Using numerical illustrations based on Australian general insurance data, the sensitivities of the capital requirement to the choice of the copula and other modelling as-

sumptions are investigated. The related issue of the diversification benefit from operating multiple business lines in the context of aggregation of risks by copulas is also explored. The key conclusion is that there is a large variation in the capital requirement as well as diversification benefit under different copula assumptions. The results of this paper serve as a reminder to actuaries and other industry practitioners of the significance of choosing an appropriate aggregation model for capital purposes.

1 Introduction

Using the method of copulas, this paper examines the impact of aggregating risks for purposes of computing economic capital for a multi-line insurance company. Insurance companies measure and manage capital across a broad range of diverse business products. This usually requires the companies to aggregate various business products whose risk distributions vary, that is, the loss distributions of the product lines are different. In recent years, we find that modelling dependencies using copulas have become popular in the actuarial, insurance, and finance literature. As we note in this paper, copulas may be used to construct joint multivariate distribution of losses and are rather flexible and realistic in terms of allowing a wide range of dependence structure. At the same time, they provide the flexibility of separating the effects of peculiar characteristics of the marginal distributions such as thickness of tails. This modelling structure allows the exploration of the impact of dependencies between risks on the total required economic capital.

The setting of this paper will be for a general insurance company writing multiple lines of business. In developing the analysis, we reasonably assume the Australian market to be representative of the industry in general due to its mature nature. Therefore, many aspects of the paper relate to specific Aus-

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tralian market conditions.

From the insurer’s perspective, the purpose of capital is to provide a financial cushion for adverse situations when its insurance losses exceed or asset returns fall below the levels expected. This cushion further enhances the insurer’s ability to continue paying claims and in most instances, to continue writing new business even under unfavourable financial circumstances. As well put by the International Actuarial Association (IAA) Insurer Solvency Assessment Working Party (2004) which has been primarily responsible for developing solvency standards suitable for global applications, capital is supposed to be a “rainy day fund, so when bad things happen, there is money to cover it.” There is no denying therefore that the management of capital forms an integral part of any insurance company’s risk management.

While there are many forms of capital such as capital required by regulatory authorities and capital required by rating agencies, this paper focuses on the form of capital, namely economic capital, that provides a measure of the amount that the insurance company should have at the minimum, to be able to withstand both expected and unexpected future losses. Economic capital is increasingly becoming a significant area of interest for the internal reporting and management of insurance companies. Mueller (2004) emphasises the distinction between economic and regulatory capital. Economic capital is the buffer set aside against potential losses that reflect risks specific to the insurer while regulatory capital often involve formulae based on industry averages and are designed for market wide application. Therefore, economic capital represents a far better measure of an insurer’s true capital requirements. Giese (2003) also gives a good overview of this concept and provides a discussion on recent developments of models for its calculation. Precise specification for the calculation of economic capital varies from company to company. However, it is generally accepted as the difference between the expected value of a risk portfolio and a worst tolerable value at a predetermined tolerable level. This paper focuses on a fundamentally equivalent definition which is the full amount of a worst tolerable value of a risk portfolio.

This paper addresses the issue of determining the aggregated economic capital of a multi-line insurance company when the losses from the several lines of business are dependent in some sense. We consider an insurance company with n different lines of business, each of which faces the risk of losing X_1, X_2, \dots, X_n at the end of a single period. The total company loss

is the random variable

$$Z = X_1 + X_2 + \dots + X_n$$

where the loss random vector $\mathbf{X}^T = (X_1, X_2, \dots, X_n)$ has a dependency structure characterized by its joint distribution using a copula function. It is well-known that for a given joint distribution function, say F , having marginal distributions F_1, \dots, F_n , there will always be a copula function C that links these marginals to their joint distribution as

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

This result is known as the Sklar’s Theorem (Sklar, 1959). For a proof, see Nelsen (1999).

For technical completeness, we shall assume that X_i is a random variable on a well-defined probability space (Ω, \mathcal{F}, P) . Suppose further that these random claims have a dependency structure characterised by the joint distribution of the vector \mathbf{X}^T . Clearly, the total capital required for the company, denoted by K , can be determined by the risk measure

$$\rho : Z \rightarrow R$$

which maps the risk Z to the set of real numbers R . In short, $K = \rho(Z) \in R$. Subsequently, we may also be interested in the contribution to the total capital of each line of business although this is beyond the scope of this paper. For a thorough discussion of risk measures such as the requirements of a coherent risk measure, see Artzner et al. (1999).

Risk measures are meant to provide a degree of magnitude of the severity of a potential loss in a portfolio and are therefore meaningful amounts to hold to cover for the risk exposure. Premium principles are clear examples of risk measures, and these have been extensively explored in Goovaerts et al. (1984). A further reference is the chapter on premium principles in Kaas et al. (2001). Risk measures should not only serve as a way to learn of the magnitude of risks, but could also be used to compare different risks.

Risk measures must be simple to apply and easily understood. It is for these purposes that we tended to focus on two widely known and used risk measures: the quantile risk measure, or fondly called the Value-at-Risk (VaR) in financial economics, and the tail conditional expectation (TCE) in this paper. Consider a loss random variable whose distribution function we shall denote by $F_X(\cdot)$ and survivorship function by $\bar{F}_X(\cdot)$. The random loss X may refer to the total claims for an insurance company or to the total loss in a portfolio of investment for an individual or institution.

For $0 \leq q \leq 1$, the q -th quantile risk measure is defined to be

$$VaR_q(X) = \inf(x | F_X(x) \geq q). \quad (1)$$

The TCE is defined to be

$$TCE_q(X) = E(X | X > x_q) \quad (2)$$

and is interpreted as the expected worst possible loss. Given the loss will exceed a particular value x_q , generally referred to as the q -th quantile with $\bar{F}_X(x_q) = 1 - q$, the TCE defined in (2) gives the expected loss that can potentially be experienced.

To make a meaningful comparison of numerical values of the risk measures for sums of dependent random variables, we draw on results of comonotonicity which provides an indication of the strongest possible positive dependence structure between random variables. See the papers of Dhaene et al. (2000a, 2000b) on the concept of comonotonicity and their relationships to various risk measures. We note the results that the VaR and TCE risk measures for sums of comonotonic random variables are in fact additive are essential to the analysis in this paper.

Consider a comonotonic random vector, say $\mathbf{X}^{C,T} = (X_1^c, X_2^c, \dots, X_n^c)$ and let the sum of the elements of this vector be $Z^c = X_1^c + X_2^c + \dots + X_n^c$. It has been demonstrated by Dhaene et al. (2002a, 2002b) that $VaR_q(Z^c) = \sum_{i=1}^n VaR_q(X_i^c)$ and that $TCE_q(Z^c) = \sum_{i=1}^n TCE_q(X_i^c)$.

As a matter of fact, one can therefore view this as that each business line is a stand alone business having to establish its own capital requirement. These results therefore imply that in terms of economic capital, if the losses from each business line were perfectly dependent with one another, then the aggregate economic capital required will simply be the sum of the economic capital required for each business line. However, there are diversification effects from writing several business lines that are less than perfectly dependent. In this case, the dependence structure is best specified using copulas.

If we then consider the VaR and TCE risk measures resulting from a particular copula structure, say $VaR_q(Z)$ and $TCE_q(Z)$ respectively, then the differences resulting from these measures

$$VaR_q(Z^c) - VaR_q(Z) \quad \text{and} \quad TCE_q(Z^c) - TCE_q(Z)$$

provide a measure of the degree of diversification benefit derived from constructing the entire portfolio. Note that we assume the same marginal distributions hold for the comonotonic copula as well as the other non-comonotonic copula under consideration. This difference will always be non-negative

for any risk measures that do indeed preserve stop-loss order and that are additive for comonotonic risks. Technically, the following result holds: any risk measure ρ that preserves stop-loss order and is additive for comonotonic random variables is considered sub-additive. That is,

$$\rho(Z) \leq \rho(Z^c) = \sum_{i=1}^n \rho(X_i^c) = \sum_{i=1}^n \rho(X_i).$$

For a proof, we suggest consultation of Vanduffel (2004).

In other words, from the insurer's perspective, the diversification benefit is the difference in the risk measure resulting from assuming comonotonic dependence to something less than comonotonic. This difference represents the savings in economic capital required due to the diversification of adding business lines to the portfolio.

It is well recognised in the insurance industry that there is some form of dependence between losses or claims occurring across the various business lines. Some intuitive explanation to these dependencies has been pointed out in Isaacs (2003). In the industry, catastrophic events such as major storms and earthquakes usually simultaneously affect more than a single line of business. Earthquake, for example, can clearly damage both building or housing structures and at the same time, other insured properties such as automobiles. It is also believed that general inflation can cause simultaneous repercussions on several lines of business, particularly long tail lines. Furthermore, in some sense, a single insurance company typically shares its company philosophy and strategy across the various business lines. There is, as a consequence, some common consistency in the implementation of underwriting rules and guidelines as well as in the establishment of reserves and capital. These consistencies may explain the possible dependencies that may exist across the business lines.

Dependencies have yet to be accurately factored into the capital calculations. The theoretical basis of incorporating the correlation structure is sometimes not well understood. Some argue that the calculations may be based on the multivariate Normal assumption, but even so, this assumption typically restricts dependencies of the business lines in the linear sense. This is because correlation explains only linear dependence, but in insurance as well as in some other financial products, other types of dependencies may exist. Embrechts et al. (1999) and Priest (2003) both provide such an argument that correlation can be a source of confusion in modelling dependencies. In an example illustrated by Embrechts et al. (1999),

it is possible to have two different probability models that can result from having equal marginals and equal correlation structure. It is for this reason that the authors are proponents for modelling a wide dependence structure by specifying the structure of the multivariate distribution function with copulas. As a matter of fact, the dependence of a multivariate random vector is entirely contained in its copula, and as noted by earlier researchers, copulas describe the “scale invariant” dependencies that exist between the elements of the random vector.

The rest of the paper has been structured as follows. Section 2 provides the necessary technical background on copulas. Section 3 discusses the methodology and assumptions used in developing the numerical simulation produced in Section 4 where the numerical results of the simulation and their discussion are presented. We conclude this paper in Section 5 with a few remarks on the findings, their limitations and potential direction for future research on the subject of aggregating risks for the purpose of setting capital requirements.

2 Aggregating Risks using Copulas

Today, there are several ongoing discussions about the implementation of copula models to account for possible dependencies between insurance risks. These are part of wider analyses into capital requirements that have been initiated by various professional bodies. First, there is the recent report, that is being circulated globally, by the International Actuarial Association (IAA), and assembled by IAA’s Insurer Solvency Assessment Working Party (IAA, 2004). Similarly, a British version of such an assessment titled “Risk and Capital Assessment and Supervision in Financial Firms” (Creedon et al., 2003) and a European Union version titled “Solvency II” is also being circulated. We note that these reports generally advocates the importance of recognising and modelling dependencies of multiple risks using copulas. In this section, we provide some introductory technical background on the subject of copulas.

As a mathematical tool to model dependencies, copulas are not a new invention but is a borrowed concept from statistics. Used as a tool for understanding relationships among multivariate outcomes, a copula is a function that links, or couples univariate marginals to their full multivariate distribution. Copulas were introduced by Sklar (1959) in the context of probabilistic metric spaces, a branch of mathematics

that deals with measures. Carriere (2003) provides a short history and the basic concepts of copulas. There is a rapidly developing literature on the statistical properties and applications of copulas. As pointed out in Frees and Valdez (1998), there is a variety of applications of this tool in actuarial science. See also Genest and MacKay (1986a, 1986b), Joe (1997), and Nelsen (1999) for further understanding of copulas.

Consider $\mathbf{u} = (u_1, \dots, u_n)$ belonging to the n -cube $[0, 1]^n$. A copula, $C(\mathbf{u})$, is a function, with support $[0, 1]^n$ and range $[0, 1]$, that is a multivariate cumulative distribution function whose univariate marginals are uniform $U(0, 1)$. As a consequence, we see that $C(u_1, \dots, u_{k-1}, 0, u_{k+1}, \dots, u_n) = 0$ and that $C(1, \dots, 1, u_k, 1, \dots, 1) = u_k$ for all $k = 1, 2, \dots, n$. Any copula function C is therefore the distribution of a multivariate uniform random vector.

The significance of copulas in examining the dependence structure of X_1, X_2, \dots, X_n comes from a result, mainly due to Sklar (1959). It relates the marginal distribution functions to copulas. Suppose $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$ is a random vector with joint distribution function F . According to Sklar (1959), there exists a copula function C such that

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n))$$

where F_k is the k th univariate marginal distribution function, for $k = 1, 2, \dots, n$. The function C need not be unique, but it is unique if the univariate marginals are absolutely continuous. For absolutely continuous univariate marginals, the unique copula function is clearly

$$C(u_1, \dots, u_n) = F(F_1^{-1}(x_1), \dots, F_n^{-1}(x_n)) \quad (3)$$

where $F_1^{-1}, \dots, F_n^{-1}$ denote the quantile functions of the univariate marginals F_1, \dots, F_n . From equation (3), it becomes apparent how the copula “links” or “couples” the joint distribution to its marginals.

As pointed out and proven by Embrechts et al. (1999), one interesting and attractive feature of the copula representation of dependence which is particularly useful for financial applications, is the invariance property of copulas. Suppose the random vector \mathbf{X} has copula representation C and the random vector $T(\mathbf{X})$ be a transformation of \mathbf{X} . That is, $T(\mathbf{X}) = (T_1(X_1), T_2(X_2), \dots, T_n(X_n))^T$ where T_i are non-decreasing and continuous functions, for $i = 1, 2, \dots, n$. Then $T(\mathbf{X})$ also has the same copula representation C as \mathbf{X} .

In the remainder of this section, we discuss, with examples, three classes of copulas: copulas of extreme dependence, Archimedean copulas and elliptical copulas.

2.1 Copulas of Extreme Dependence

To begin, an example of a copula is the independence copula which is given by

$$C(u_1, \dots, u_n) = u_1 \cdots u_n \quad (4)$$

and is the copula associated with the joint distribution of independent random variables X_1, X_2, \dots, X_n . This copula is often denoted simply by $\Pi(\mathbf{u})$.

The Frechet bounds for copulas are well-known results in mathematical statistics. The main results are given below and one is directed to consult Frechet (1951, 1957) for more details and discussions of these bounds. Define

$$M(\mathbf{u}) = \min(u_1, \dots, u_n)$$

and

$$W(\mathbf{u}) = \max(u_1 + \dots + u_n - n + 1, 0).$$

Then it is always true that for all \mathbf{u} in $[0, 1]^n$, we have

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}).$$

For all $n \geq 2$, the function $M(\mathbf{u})$ satisfies definition of a copula. For $n \geq 3$, the function $W(\mathbf{u})$ is not a copula. The copula $M(\mathbf{u})$ is a comonotonic copula and in fact, describes perfect positive dependence. For any random variable U that is uniform on $[0, 1]$, the random vector $(U, U, \dots, U)^T$ has distribution function described by the comonotonic copula. Furthermore, if the random vector \mathbf{X} has the comonotonic copula representation, then we say that its elements X_1, X_2, \dots, X_n are comonotonic random variables. Sometimes, it is convenient to place a superscript c on the random variables to denote they are comonotonic, i.e., $X_1^c, X_2^c, \dots, X_n^c$. The concept of comonotonicity has had tremendous applications in actuarial science, particularly in obtaining bounds for distribution functions of random variables. See Dhaene et al. (2002a, 2002b) for many interesting examples and illustrative applications of comonotonicity.

2.2 Archimedean Copulas

The use of Laplace transforms can lead us to construct a special type of copulas known as Archimedean copulas. This class of copulas is well discussed in Nelsen (1999). More formally, we say that a copula function C is Archimedean if it can be written in the form

$$C(u_1, u_2, \dots, u_n) = \psi^{-1}[\psi(u_1) + \dots + \psi(u_n)] \quad (5)$$

for all $0 \leq u_1, \dots, u_n \leq 1$ and for some continuous function ψ (often called the generator) satisfying:

- (i) $\psi(1) = 0$;
- (ii) ψ is strictly decreasing and convex. That is, for all $t \in (0, 1)$, $\psi'(t) < 0$ and $\psi''(t) \geq 0$; and
- (iii) ψ^{-1} is completely monotonic on $[0, \infty)$.

The ‘‘completely monotonic’’ requirement is a necessary and sufficient condition to extend Archimedean copulas into higher than two dimensions. See Nelsen (1999), for example, for an interesting proof of this proposition. A function $g(t)$ is said to be completely monotonic on a specific interval I if it is continuous on the interval and has derivatives of all orders that alternate in signs. This alternating signs requirement implies that we must have $(-1)^k \frac{d^k}{dt^k} g(t) \geq 0$, for $k = 1, 2, \dots, n$.

This class of copulas has also been extensively studied by Genest and Mackay (1986) who further demonstrate that this class of copulas possess several desirable and interesting properties that make them attractive for statistical inference and simulation. In addition, they are useful for extending copulas to higher dimensions. Since the copula is completely specified once the Archimedean generator is known, another advantage of this Archimedean representation is that when searching for a copula suitable to describe random variables, we reduce the task to searching for a single univariate function.

Example 1: Gumbel-Hougaard Copula

Using the generator defined by $\psi(t) = (-\log t)^\alpha$, this family has members with the following copula representation:

$$C(u_1, u_2, \dots, u_n) = \exp \left\{ - \left[\sum_{i=1}^n (-\log u_i)^\alpha \right]^{1/\alpha} \right\}. \quad (6)$$

It is easy to show that in this case, the inverse of the generator is $\psi^{-1}(s) = \exp(-s^{1/\alpha})$ and is completely monotonic for $\alpha \geq 1$ making the representation in (6) a valid multivariate copula function.

Example 2: Frank Copula

The Frank copula has the generator $\psi(t) = -\log \left(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1} \right)$ so that its multivariate copula representation is:

$$C(u_1, u_2, \dots, u_n) = -\frac{1}{\alpha} \log \left[1 + \frac{\prod_{i=1}^n (e^{-\alpha u_i} - 1)}{(e^{-\alpha} - 1)^{n-1}} \right]. \quad (7)$$

The inverse of the generator can be expressed as $\psi^{-1}(s) = -\frac{1}{\alpha} \log [1 + e^{\alpha s} (e^{-\alpha} - 1)]$ and is completely monotonic for $\alpha > 0$. See Frank (1979) and

Genest (1987) for details of the characteristics of this copula.

Example 3: Cook-Johnson Copula

Another important example of an Archimedean copula is the Cook-Johnson copula whose generator is defined by $\psi(t) = t^{-\alpha} - 1$ so that its multivariate copula representation is:

$$C(u_1, u_2, \dots, u_n) = \left[\sum_{i=1}^n u_i^{-\alpha} - n + 1 \right]^{-1/\alpha}. \quad (8)$$

The inverse of the generator can be expressed as $\psi^{-1}(s) = (t + 1)^{-1/\alpha}$ and is completely monotonic for $\alpha > 0$. These are sometimes called Clayton copulas and this family has been shown to be important in multivariate extreme value theory. See, for example, Juri and Wüthrich (2002) for some useful asymptotic results leading to Clayton copulas.

2.3 Elliptical Copulas

Another very important class of copulas that has been receiving attention in financial applications is the class of Elliptical copulas. Elliptical copulas are generally defined as copulas of elliptical distributions. There are a number of equivalent ways to define random vectors that belong to the class of elliptical distributions.

The n -dimensional vector \mathbf{X} is said to have a multivariate elliptical distribution, written as $\mathbf{X} \sim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \varphi)$, if its characteristic function has the form

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(it^T \boldsymbol{\mu}) \cdot \varphi\left(\frac{1}{2} \mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}\right)$$

for some column-vector $\boldsymbol{\mu}$, $n \times n$ positive-definite matrix $\boldsymbol{\Sigma}$, and some function $\varphi(t)$ called the characteristic generator. Members of the elliptical class has a special stochastic representation as follows. Assuming $\mathbf{X} \sim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \varphi)$ with $rank(\boldsymbol{\Sigma}) = r \leq n$, we can write the elliptical random vector as

$$\mathbf{X} = \boldsymbol{\mu} + \mathcal{R} \sqrt{\boldsymbol{\Sigma}} \mathbf{U}$$

where \mathbf{U} is a uniformly distributed random vector on $\{\mathbf{u} \in [-1, 1]^r \mid \|\mathbf{u}\| = 1\}$, the unit sphere, and \mathcal{R} is a non-negative random variable independent of \mathbf{U} . The following references, Fang et al. (1987) and Embrechts et al. (1999), provide comprehensive discussions on elliptical distributions. Two further references for elliptical distributions are Landsman and Valdez (2003) and Valdez and Chernih (2003). The reader is encouraged to consult these references for further study about the interesting properties of this class of distributions.

We now give some examples of copulas generated from this class of distributions.

Example 1: Gaussian (Normal) Copula

The copula generated by a multivariate Normal distribution with linear correlation matrix $\boldsymbol{\Sigma}$ is given by

$$C(u_1, \dots, u_n) = H(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n))$$

where H is the joint distribution function of a standard Normal random vector expressed as

$$H(x_1, \dots, x_n) = \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \dots \int_{-\infty}^{x_1} \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{\Sigma}|}} \times \exp\left(-\frac{1}{2} \mathbf{z}^T \boldsymbol{\Sigma}^{-1} \mathbf{z}\right) dz_1 \dots dz_n \quad (9)$$

and $\Phi^{-1}(\cdot)$ is the inverse of a standard Normal distribution and $\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \cdot e^{-w^2/2} dw$. It is critical to note that Normal copulas have zero tail dependence. See Embrechts et al. (2001) for a proof of this result.

Example 2: Student-t Copula

The copula generated by a multivariate Student-t distribution with linear correlation matrix $\boldsymbol{\Sigma}$ is given by

$$C(u_1, \dots, u_n) = T_v(t_v^{-1}(u_1), \dots, t_v^{-1}(u_n))$$

where T is the joint distribution function of a standard Student-t random vector expressed as

$$T_v(x_1, \dots, x_n) = \frac{\Gamma\left(\frac{v+n}{2}\right)}{\Gamma\left(\frac{v}{2}\right) (v\pi)^{n/2} \sqrt{|\boldsymbol{\Sigma}|}} \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \dots \int_{-\infty}^{x_1} \left(1 + \frac{1}{v} \mathbf{z}^T \boldsymbol{\Sigma}^{-1} \mathbf{z}\right)^{-(v+n)/2} dz_1 \dots dz_n \quad (10)$$

and $t_v^{-1}(\cdot)$ is the inverse of a standard Student-t with $t_v(z) = \int_{-\infty}^z \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right) (v\pi)^{1/2}} \cdot \frac{1}{\left[1 + \frac{z^2}{v}\right]^{\frac{v+1}{2}}} dz$. Unlike

Gaussian copulas, Student-t copulas have non-zero tail dependence. Again, a proof can be found in Embrechts et al. (2001) for this result.

Example 3: Cauchy Copula

The Cauchy copula is actually a special case of the Student-t copula where the degrees of freedom is $v = 1$. Thus, the copula generated by a multivariate Cauchy distribution with linear correlation matrix $\boldsymbol{\Sigma}$ is given by

$$C(u_1, \dots, u_n) = T_1(t_1^{-1}(u_1), \dots, t_1^{-1}(u_n))$$

where T_1 then is the joint distribution function of a standard Cauchy random vector expressed as

$$T_1(x_1, \dots, x_n) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)(\pi)^{n/2}\sqrt{|\Sigma|}} \int_{-\infty}^{x_n} \int_{-\infty}^{x_{n-1}} \dots \int_{-\infty}^{x_1} (1 + \mathbf{z}^T \Sigma^{-1} \mathbf{z})^{-(n+1)/2} dz_1 \dots dz_n \quad (11)$$

and $t_1^{-1}(\cdot)$ is the inverse of a standard Cauchy distribution with $t_1(z) = \int_{-\infty}^z \frac{1}{\pi} \cdot \left(\frac{1}{1+w}\right)^2 dw$. Due to its relationship with the Student-t copulas, we deduce that Cauchy copulas also have non-zero tail dependence.

Embrechts et al. (2001) provides overviews of all three examples of elliptical copulas and the reader is directed to that paper for further discussion.

3 Numerical Simulation

The primary goal of this paper is to assess the economic capital required for a multi-line insurer under various copula assumptions. Also, we try to quantify the diversification benefit for the same insurer from holding capital against the aggregate loss compared to holding the aggregate capital against losses from each business line under the different copulas. We discuss the existence of this diversification benefit in terms of capital in the Introduction section and have suggested a reference in Vanduffel (2004) for a proof.

A SAS program written using the Interactive Matrix Language (IML) procedure was developed to simulate and aggregate the prospective one year loss ratio distributions for each business line. For each line of business, 1,000 loss ratio simulations were generated for each copula to represent the sampling distribution. Appendix A provides the algorithm used for this simulation and the SAS program code is documented in Tang (2004).

The simulation performed in this paper have been motivated by historical data of losses for the aggregate Australian industry. First, we note that our simulation is based on loss ratios. The gross loss ratio (LR) defined as the ratio of the gross incurred claims and earned premium is a proxy for loss variables to make the measurement dimension invariant. The loss ratio for the period t derived from business unit i is defined as

$$LR_{i,t} = \frac{IC_{i,t}}{EP_{i,t}}$$

where $IC_{i,t}$ and $EP_{i,t}$ denote respectively the incurred claims and earned premium from line i during

period t . The loss ratio is in essence a standardised claims measure, in this case by a measure of the exposure to risk - gross earned premium. This standardisation allows valid comparison between losses from business lines with different levels of risk exposure.

For each copula, we calculate the distribution of the aggregate loss ratios at the company level taking the weighted average of each line's loss ratios according to pre-specified proportion of earned premium. The weighted averages are valid representations of the aggregate loss ratios due to the following argument. Suppose the following additional notation: LR_t denotes the aggregate loss ratio; IC_t denotes the aggregate incurred claims; and EP_t denotes the aggregate earned premium; at time t , and there are n lines of business in total.

$$\begin{aligned} LR_t &= \frac{IC_t}{EP_t} = \frac{\sum_{i=1}^n IC_{i,t}}{\sum_{i=1}^n EP_{i,t}} = \frac{\sum_{i=1}^n \frac{IC_{i,t}}{EP_{i,t}} * EP_{i,t}}{\sum_{i=1}^n EP_{i,t}} \\ &= \sum_{i=1}^n LR_{i,t} * \frac{EP_{i,t}}{\sum_{i=1}^n EP_{i,t}} = \sum_{i=1}^n LR_{i,t} * w_{i,t} \end{aligned}$$

where $w_{i,t} = \frac{EP_{i,t}}{\sum_{i=1}^n EP_{i,t}}$ represents the weight of line i in period t by earned premium. Figure 1 displays the time series of the observed historical loss ratios.

3.1 Data Supporting Assumptions Used for Simulation

Historical loss ratios for the aggregate Australian industry is used to derive the inter business line correlation and marginal distribution required as inputs for the parameterisation of the various copulas. In other words, the Australian industry is chosen as a proxy for all general insurance industries. This is a reasonable assumption due to three factors. First, the Australian industry is mature by global standards and offers a comprehensive, if not exhaustive range of products. Second, despite the country's relatively small population, the Australian general insurance market is disproportionately large on a per capita basis, accounting for several percentage points of the global market. This leads to Australian practices being representative of world standards and in fact, are often at the forefront of innovations in the industry. Finally, despite some concern over the degree of concentration in the current market due to rationalisation across the industry over the past decade, historically over the period from which the data was collected, there has been a reasonable amount of actively operating issuers, and hence competition, for a market of this size.

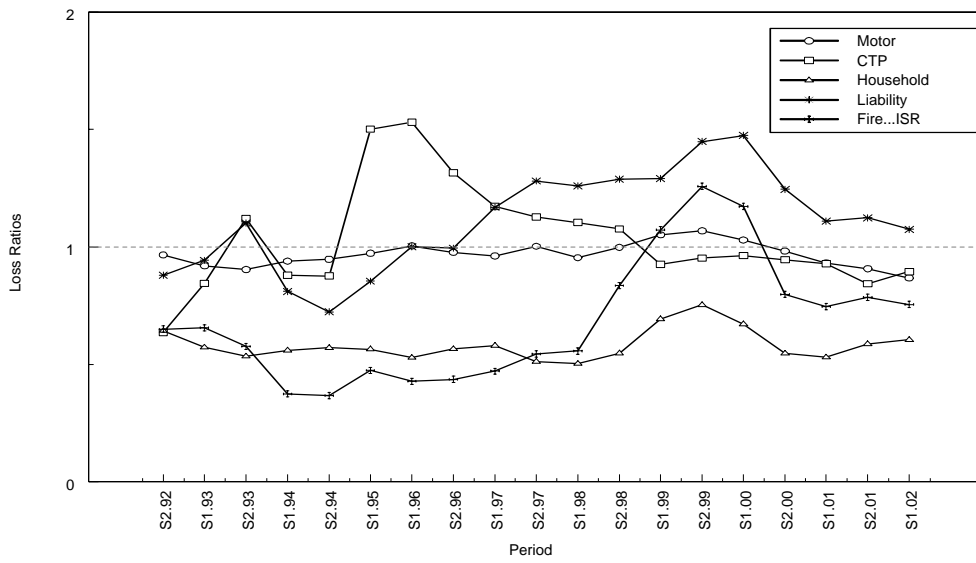


Figure 1: Historical Loss Ratios

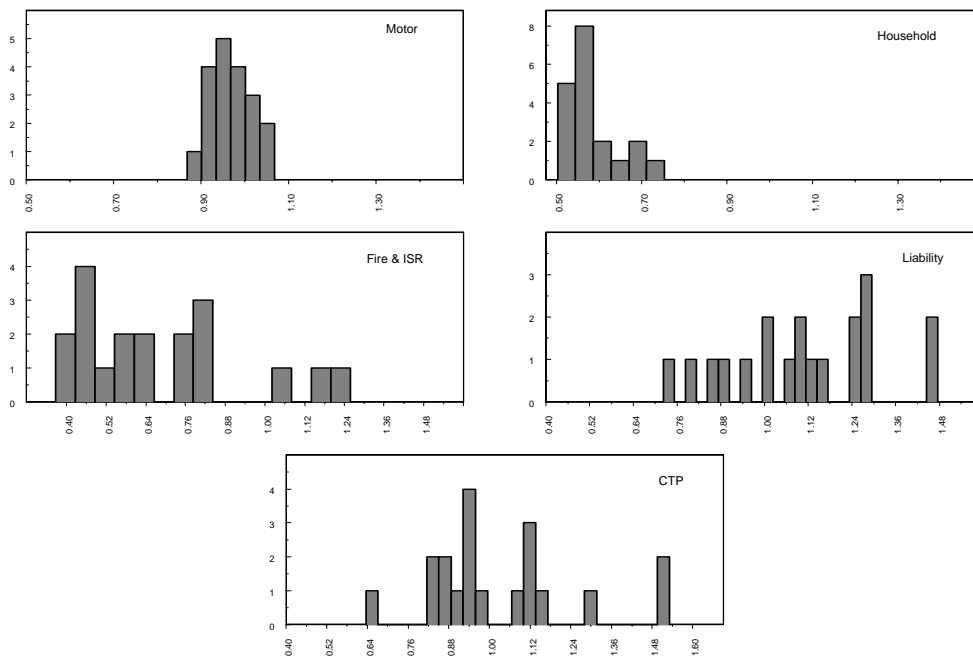


Figure 2: Distributions of Historical Loss Ratios

3.1.1 Data Source and Collection Period

All historical data used in this paper are presented in the various semi-annual issues of APRA and former Office of the Insurance Commissioner publications as listed below.

- APRA, Selected Statistics on the General Insurance Industry, December 1996 – June 2002
- Office of the Insurance Commissioner, Selected Statistics on the General Insurance Industry – Total Industry In and Outside of Australia, December 1992 – June 1996

Except where indicated, these publications report aggregate general insurance industry data that have been reported during the 12 month period prior to the publication dates of 30 June and 31 December of each year.

The Office of the Insurance Commissioner published similar industry statistics dating back to the 1970s. However, we find the data format prior to December 1992 is materially different to all subsequent periods and hence decided not to include these for use in this paper. The reason for this difference was due to a change in reporting procedures by the insurance companies to the regulatory authority and the details can be found in the June 1992 issue of the Selected Statistics (Office of the Insurance Commissioner, 1992).

Between December 1992 and June 2002, data for 19 periods, from June and December of each year were collected. At the time of writing, access to the December 1995 issue of the Selected Statistics was unavailable and hence there is a discontinuity of the data at this date. Also worthy of note are the data points from December 1992 and November 1997 (substituting December 1997). The data for December 1992 is for the 6 month period ending 31 December 1992 inclusive of 30 June 1992, and the data for November 1997 is for the 11 month period ending 30 November 1997. These are different from all the other data points in that they did not cover a full 12 month period. In particular, the difference is quantitatively significant for November 1997 due to its exclusion of the month of December during which numerous companies report. For this period, incurred claims and earned premium were respectively 42% and 40% lower than those of the subsequent period (June 1998). We do not omit either of these anomalies in the data set as the quantity of interest is ultimately the loss ratio, which as a ratio between incurred claims and earned

| Line of business | Category | Earned Premium * | Market Share |
|------------------|----------------|------------------|--------------|
| Motor | Short tail | 4,830,180 | 31.1% |
| Household | Short tail | 2,460,770 | 15.8% |
| Fire & ISR | Intermed. tail | 1,655,224 | 10.6% |
| Liability | Long tail | 2,429,945 | 15.6% |
| CTP | Long tail | 1,975,778 | 12.7% |

Source: APRA (2002); * (A\$,000)

Table 1: Business Line Assumptions and their Market Share

premium, is not materially impacted on by variations in the length of time coverage.

Since in general, annual data collected semi-annually has been used, there was some initial concern about the overlapping of the collection period and that this will distort the dependence structure. For example, the June 1998 data refers to the period from 1 July 1997 to 30 June 1998 whereas the December 1998 data refers to the period from 1 January 1998 to 31 December 1998, hence the period between 1 January 1998 and 30 June 1998 are included in both of the data points. On further consideration, we decided that this will not impact on the results as only the characteristics (correlation and distribution) between lines rather than serially through time are of concern in this paper.

3.1.2 Mapping to Business Lines

We decided that for the purpose of this paper the general insurance industry is to consist of a total of five business lines. This number of lines is large enough to allow for analysis in adequate depth without causing unnecessary complications in the analysis process. In order to create a realistic and representative reflection of the industry, the top five business lines by earned premium from the June 2002 Selected Statistics (APRA, 2002) are chosen for analysis. A broad mix of business lines resulted from this selection criterion which also fulfils the requirement for a good representation of the industry. Of the five business lines chosen, two are short tail, one intermediate tail and two are long tail lines. This selection criterion is broadly in line with the methodology of previous studies where similar empirical data by business lines were used. For an example, see Sherris and Sutherland-Wong (2004). Table 1 outlines the business lines assumed and their market share. This selection represents approximately 86% (excluding inwards reinsurance) of the aggregate market activities as measured by earned premium and hence agrees

| LoB | M | H | F&I | L | CTP |
|-----|-----|-----|-----|-----|-----|
| M | 100 | | | | |
| H | 50 | 100 | | | |
| F&I | 50 | 75 | 100 | | |
| L | 54 | 35 | 77 | 100 | |
| CTP | 21 | -38 | -38 | -2 | 100 |

Legend: M = motor; H=household;
F&I = Fire & ISR; L = liability

Table 2: Inferred Correlation Matrix

with the objective of creating a realistic reflection of the entire industry.

3.2 Correlation Matrix

We require a linear correlation matrix for specification of the copulas . We begin with an empirical correlation matrix inferred from the historical data discussed in the previous subsection, we call this the inferred matrix. The inferred matrix was deemed unsatisfactory and a second correlation matrix is considered necessary to give a subjectively more accurate representation of reality. We call this variation the alternative matrix and it is the correlation matrix we assume in the base scenario. Table 2 displays the linear correlations between the five business lines that are implied by the historical data.

The overall levels of correlation between business lines are generally higher than that can be intuitively expected. According to practising actuaries from Trowbridge Consulting, inter-line claims correlations in excess of 60% are highly unlikely (Collings and White, 2001). Of the ten pair-wise correlations inferred from the loss ratios data, two were above this 60% threshold while a further three were between 50% and 60%. While high correlations may be justified for certain pairs of lines whose businesses are similar in nature, in general, the high correlation values are in contrary to common perception.

Further, we note the three negative correlations between losses from the CTP and household, fire & ISR, and liability lines. Actuaries at Trowbridge Consulting hold the view that negative correlation between business lines are unlikely (Collings and White, 2001). It can be argued that the CTP line, providing cover for bodily injuries due to motor accidents, should in theory be uncorrelated with both the household and fire & ISR lines which covers damage to stationery property. In the pairing of CTP and liability, one would reasonably expect a positive relationship to exist as both of these lines essentially provide liability cover. We should take particular note of the ampli-

| Pairs | Inferred | Comment |
|---------|----------|------------------------|
| M/H | 50% | Excessive |
| M/F&I | 50% | Excessive |
| M/L | 54% | Excessive |
| M/CTP | 21% | Reasonable |
| H/F&I | 75% | Excessive |
| H/L | 35% | Should be uncorrelated |
| H/CTP | -38% | Should be uncorrelated |
| F&I/L | 77% | Excessive |
| F&I/CTP | -38% | Should be uncorrelated |
| L/CTP | -2% | Should be positive |

Table 3: Individual Pair-wise Inferred Correlation Commentary

| LoB | M | H | F&I | L | CTP |
|-----|-----|-----|-----|-----|-----|
| M | 100 | | | | |
| H | 20 | 100 | | | |
| F&I | 10 | 10 | 100 | | |
| L | 0 | 0 | 0 | 100 | |
| CTP | 25 | 0 | 0 | 25 | 100 |

Table 4: Tillinghast Correlation Matrix

fied diversification implications of the negative correlations. Table 3 provides the commentary for each individual pair-wise inferred correlation values.

The many counter-intuitive values in the inferred matrix appears to have been due to the lack of a reasonable amount of data. The inferred matrix is based on only 19 annual loss ratio data points and this leads to a higher volatility of each correlation estimate, and hence lower accuracy of the matrix as a whole in comparison to the desired situation. Given its inaccuracy, we decide to derive an alternative correlation matrix by including subjective views on what the true pair-wise correlations should be.

The alternative matrix is derived with reference to the outstanding claims correlation matrices from the Tillinghast (Bateup and Reed, 2001) and Trowbridge (Collings and White, 2001) reports. Collectively, we call these the industry reports. The original inferred matrix is also considered as a minor influence of the final choice of each pair-wise correlation value. Tables 4 and 5 are the Tillinghast and Trowbridge correlation matrices respectively.

Table 6 presents the resultant alternative matrix. We again emphasise that the pair-wise correlations are subjectively determined and consequently are all positive in sign. For a comprehensive discussion of the justification of this choice of correlation matrix, we suggest the reader to consult Tang (2004).

| LoB | M | H | F&I | L | CTP |
|-----|-----|-----|-----|-----|-----|
| M | 100 | | | | |
| H | 20 | 100 | | | |
| F&I | 20 | 40 | 100 | | |
| L | 0 | 0 | 0 | 100 | |
| CTP | 0 | 0 | 0 | 20 | 100 |

Table 5: Trowbridge Correlation Matrix

| LoB | M | H | F&I | L | CTP |
|-----|-----|-----|-----|-----|-----|
| M | 100 | | | | |
| H | 20 | 100 | | | |
| F&I | 20 | 50 | 100 | | |
| L | 10 | 0 | 20 | 100 | |
| CTP | 20 | 0 | 0 | 25 | 100 |

Table 6: Alternative Correlation Matrix

3.3 Marginal Distributions and their Parameterisation

The marginal distributions of each business line's loss are essential inputs in the simulation algorithm using copulas. In this subsection, we detail the choice of distributions, estimate the required parameters and discuss the issues that arose out of this process.

The variables of interest are the annual loss ratios from each business line and we assume they follow a predetermined distribution. As the loss ratios essentially represent the magnitude of the underwriting loss from each business line, their distributions are expected to behave similarly to claim severity distributions. The effect of the fluctuation in claim frequency on the distribution is removed by the standardisation process in the calculation of the loss ratios.

The risks that each line of business covers vary greatly in general insurance. Therefore, the distributional behaviour of each line's loss will also differ to one another. This requires us to select appropriate distributional assumptions on an individual line basis. In this paper, the main concern is the capital requirement as measured by two variations of the quantile risk measure, as outlined in Section 1, and is largely dependent on the right tail behaviour of the loss distribution. Therefore, in the selection of the marginals, it is imperative that the tail behaviour of the theoretical distributions are matched appropriately with that of each business line's losses. We consider three commonly used severity distributions: *Pareto* which has the heaviest tail weight, followed by *Log-normal* and finally *Gamma*.

Based on the histograms of each business line's loss ratio distributions from historical data, shown in Fig-

| Line of Business | Marginal Distribution |
|------------------|-----------------------|
| Motor | <i>Gamma</i> |
| Household | <i>Gamma</i> |
| Fire & ISR | <i>Log-normal</i> |
| Liability | <i>Pareto</i> |
| CTP | <i>Pareto</i> |

Table 7: Marginal Distribution Assumptions Initially Considered

ure 2, a ranking of the lines in order of increasing tail weight is determined. We map these to the three distributions under consideration to arrive at an initial set of distribution choices. These are displayed in Table 7. Clearly, both the motor and household lines display very thin tails so the *Gamma* distribution which has the lightest tail weight of the three candidates is suitable for these lines. Fire & ISR loss ratios displays a distinctly heavier tail than the first two lines and hence the heavier tailed *Log-normal* distribution is preferred over the *Gamma* in this case. The remaining lines of Liability and CTP displays extremely heavy tail behaviour and noting the comment that Hart et al. (1996) makes about the possible "inadequacy of fit" of the *Log-normal* distribution at extreme high values, the *Pareto* distribution is chosen for these classes.

Each of the chosen distributions are fully specified by two parameters and there are various statistical techniques that can be used for their estimation. The two most commonly used methods are the method of moments and method of maximum likelihood. The method of moments rely on matching the moments of the theoretical distribution with those of the sampling distribution represented by the historical loss ratios. On the other hand, the method of maximum likelihood entails a theoretical approach where the likelihood of observing the sampling distribution is maximised. Theoretically, the method of maximum likelihood produces estimates with desirable features such as being unbiased and efficient. However, the amount of data that constitutes the sampling distribution in our present case is very small (19 loss ratio observations for each business line), hence the difference between the two methods will not be pronounced. Due to the simpler numeric evaluation, we decide to use the method of moments for the purpose of this paper.

Table 8 summarises the key properties of the three parametric distributions considered as the marginals for the various business lines. In the table, the function $\Gamma(\cdot)$ refers to the gamma function defined by

$$\Gamma(a) = \int_0^{\infty} e^{-u} u^{a-1} du.$$

Table 8: Summary of Marginal Distributions Used

| Family | Density, constraints | Mean $E(X)$ | Variance $Var(X)$ |
|---------------------------|--|--------------------------------|--|
| $Gamma(\alpha, \beta)$ | $\frac{\beta^\alpha e^{-\beta x} x^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha, \beta > 0, x \geq 0$ | $\frac{\alpha}{\beta}$ | $\frac{\alpha}{\beta^2}$ |
| $Log-normal(\mu, \delta)$ | $\frac{e^{-\frac{1}{2}(\frac{\ln x - \mu}{\sigma})^2}}{x\sigma\sqrt{2\pi}}$, $\mu \in R, \sigma > 0, x \geq 0$ | $e^{\mu + \frac{\sigma^2}{2}}$ | $e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$ |
| $Pareto(a, b)$ | $\frac{ab^a}{(x+b)^{a+1}}$, $a, b > 0, x \geq 0$ | $\frac{b}{a-1}$ | $\frac{ab^2}{(a-1)^2 (a-2)}$ |

| LoB | Marginal | Estimated Parameters |
|-----|-------------------|---------------------------------------|
| M | <i>Gamma</i> | $\alpha = 354.4774, \beta = 366.2363$ |
| H | <i>Gamma</i> | $\alpha = 80.3886, \beta = 138.0149$ |
| F&I | <i>Log-normal</i> | $\mu = -0.4519, \sigma = 0.3712$ |
| L | <i>Log-normal</i> | $\mu = 0.0862, \sigma = 0.1882$ |
| CTP | <i>Log-normal</i> | $\mu = 0.0097, \sigma = 0.2169$ |

Table 9: Final Marginal Distribution and Parameter Assumptions

3.4 The Choice of Copulas

We propose to analyse the effect on capital requirements by aggregating losses using copulas and further investigate the sensitivity of results to the choice of copula. Therefore, the selection of copulas we use are of paramount importance to the results. We limit our choices for copula within the class of elliptical copulas. There are several merits to these choices. Elliptical copulas allow us to specify a variance-covariance structure which in some sense provides the linear dependence between the random variables. Except for the Normal copula which gives zero tail dependence, elliptical copulas allow for non-zero tail dependence. As capital requirements based on the VaR and TCE risk measures are concerned with the tails of the loss distribution, tail dependence plays a significant role in determining the correct capital. Elliptical copulas provide also the flexibility of simple simulation procedures. Isaacs (2003) explored using the Gumbel copula to model dependence between multiple business lines. While the Gumbel copula allows heavy tail dependence structures which is ideal for capital considerations, it is not flexible enough to capture dif-

ferences in pair-wise dependence structures. To simulate variables under a Gumbel copula, only two lines can be generated at any one time and dependencies between one of these lines and any other line outside the pair cannot be explicitly specified. This feature is in general true for copulas in the Archimedean class, simply because when specifying higher than two dimensional copulas, there is always only a single parameter describing the dependence. Since we are concerned with studying a more realistic multi-line operation with five business lines, each with a distinct dependence structure with one another, we prefer to use the family of more flexible copulas – elliptical copulas.

The elliptical copulas we consider in this paper are:

- *Gaussian (Normal) Copula*
- *Student-t Copula (with $n = 3$ and 10 degrees of freedom)*
- *Cauchy Copula*

All three elliptical copulas have been discussed in Section 2 where a discussion on copulas have been made including an outline of the forms of these copulas. Simulation procedures from these copulas are summarised in Appendix A. We choose the Normal copula as it is a very popular choice in practice due to the common assumption of normality in many financial modelling applications. It is also very well understood and tested in a diverse range of applications. However, it does not allow for tail dependence and as such becomes a less favourable candidate for capital applications. We are currently seeing a growing

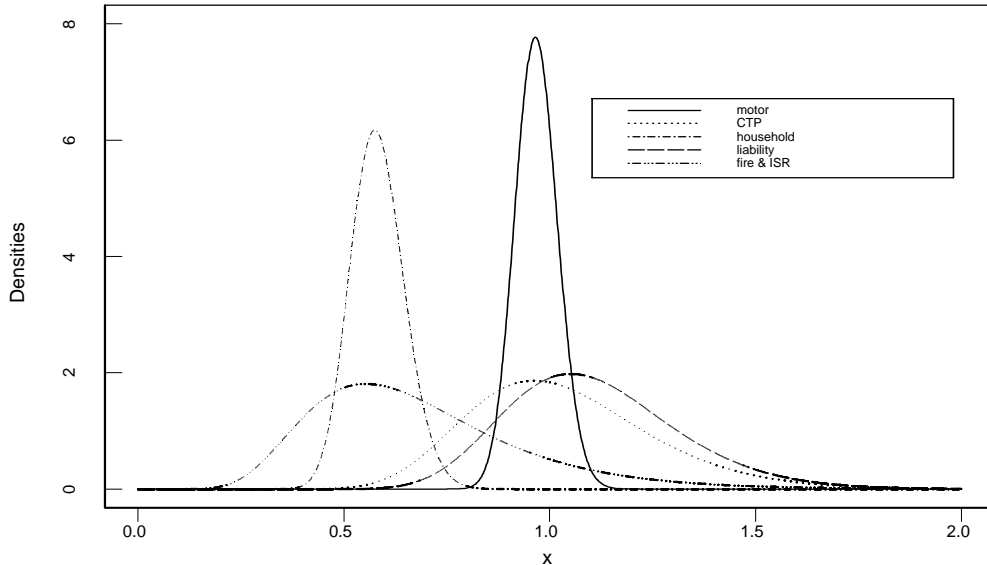


Figure 3: Marginal Densities

literature on the usefulness of the Student-t copula as an alternative to the Normal copula for modelling financial risks.

The main impetus for the Student-t copula's rise to notoriety is associated with its ability to incorporate tail dependence. When describing insurance losses, we are usually concerned with (upper) tail dependence and a common way to define this (upper) tail dependence is as follows. For any pair of random variables, say X and Y , the coefficient of (upper) tail dependence is defined as

$$\lambda = \lim_{\alpha \rightarrow 1^-} P(Y > F_Y^{-1}(\alpha) | X > F_X^{-1}(\alpha)) \quad (12)$$

provided this limit exists. See Embrechts et al. (1999) for details on the expression in (12). If X and Y has the Student-t copula with a correlation of ρ , then it can be shown that

$$\lambda = 2\bar{t}_n \left(\sqrt{n+1} \sqrt{1-\rho} / \sqrt{1+\rho} \right)$$

where \bar{t}_n denotes the tail of a univariate Student-t distribution with n degrees of freedom. It is to be noted also that the Normal copula is in fact, a limiting case (as $n \rightarrow \infty$) of the Student-t copula. We also noted in Chapter 3, that the Cauchy copula is in fact a special case of the Student-t copula with one degree of freedom. Therefore, in practice, we have chosen to aggregate the losses across business lines using four variations of the Student-t copula with different degrees of freedom.

| $n \setminus \rho$ | 0 | 0.5 | 0.9 | 1 |
|--------------------|------|------|------|---|
| 1 | 0.29 | 0.5 | 0.78 | 1 |
| 3 | 0.12 | 0.31 | 0.67 | 1 |
| 10 | 0.01 | 0.08 | 0.46 | 1 |
| ∞ | 0 | 0 | 0 | 1 |

Table 10: Coefficients of Tail Dependence for Student-t Copulas

Theoretically, the lower the degree of freedom, the heavier is the tail dependence for a Student-t copula. See again, Embrechts et al. (1999). Table 10 presents the coefficient of tail dependence, λ , for a selection of correlation values, ρ , and degrees of freedom parameters, n , that are relevant to our paper. Therefore, of the copulas chosen for this paper, we expect the rank in terms of increasing tail dependence to be the Normal, Student-t (3 df), Student-t (10 df) and Cauchy copula. We expect heavier tail dependence copulas to lead to heavier, or thicker, tails in the aggregate loss distribution. This can be explained by the fact that under heavy tail dependence, extreme losses occur together more frequently, leading to more extreme aggregate losses in general. Therefore, we expect the heavier the tail dependence of the copula, the higher will be the level of capital requirement.

The risk measures, or amounts therefore of capital required, are computed based on a specified level of probability. For our purposes, we have chosen either

$q = 97.5\%$ or $q = 99.5\%$. These arbitrary confidence levels are chosen to be consistent with the range implied by current industry practice for capital purposes and are confirmed with practising actuaries. In particular, $q = 99.5\%$ is chosen to facilitate the consistent comparison of our results with that of APRA's Prescribed Method (PM) in order to answer research question 6 of this paper.

Given the distribution of prospective loss ratios from the output of the SAS program, we can readily calculate the VaR and TCE at the chosen levels of q in Microsoft Excel. The VaR measure is calculated as the point on the ranked loss ratios distribution that corresponds to the particular level of q . That is, for each output distribution, of the 1000 simulated loss ratios, $VaR_{97.5\%}$ corresponds to the 975th value if the distribution is ranked in increasing magnitude. Similarly, $VaR_{99.5\%}$ corresponds to the 995th value in the same ranked distribution. To calculate the TCE, we simply take the arithmetic average, or expected value, of the values subsequent to the value corresponding to the VaR measure. Therefore, $TCE_{97.5\%}$ is calculated as the average of the 976th to 1000th value of the ranked loss ratios distribution and similarly, $TCE_{99.5\%}$ is calculated as the average of the 996th to 1000th value of the ranked loss ratios distribution. The respective VaR and TCE measures can be calculated for the distribution of loss ratios for each business line as well as for the insurer's portfolio in aggregate. Since the SAS program automatically aggregates the loss ratios from each business line, the procedure for calculating the VaR and TCE measure does not change for the aggregate portfolio case. That is, the procedure is simply applied to the distribution of loss ratios for the aggregate portfolio rather than the distribution for each business line.

4 Results of Simulation

Using the procedure as outlined in the previous section, for each of the five chosen copula models, we generated 1,000 observations of the loss ratios for each business line. These represent the loss, per unit of premium, distribution for each business line under the different copula assumptions.

4.1 The Simulated Loss Ratios

First, let us examine the resulting loss ratios, for each copula model, in the case where we assume the alternative correlation matrix as described in section 3.2. On their own, each line of business do not lead to any meaningful results in terms of the present inter-

est of capital requirements. However, as this paper is concerned with the dependence structure of the losses arising from the various lines of business, it is worthwhile to observe the implied dependence level in each case. As a visual guide, we present Figures 4 to 8 for which each provide a scatter plot of the loss ratios of the business lines for each of the copula models considered. These scatter plots provide a much better idea of the reasonableness of the simulated losses arising from the various dependence structures being considered. We outline the key observations from visually inspecting these plots for each copula below. Overall, the simulated loss ratios under all copulas conform to our expectations in terms of dependence structure, in particular, in respect of the level of tail dependence.

4.1.1 Normal Copula

Figure 4 clearly demonstrates that the dependence across losses in each business line is of a linear nature. For example, in pairs of lines that were assigned a high linear correlation value such as household/fire & ISR (50%), the scatter plot results in a positively sloping linear pattern. This is reasonable given that the Normal copula theoretically captures linear dependence. We further observe that for each pair, the observations tend to bunch up around the centre of the distribution, indicating strongest dependence around the mean and there is little evidence to suggest any form of tail dependence. This is also valid as in fact, the tail dependence measure for the Normal copula is zero if the pair-wise linear correlations are less than 1.

4.1.2 Student-t Copula (3 df)

Compared to the Normal copula case, Figure 5 lacks the overall linearity in the pair-wise associations because the Student-t copula also captures other forms of dependence as we have noted earlier. For the majority of the pairs, the observations are scattered randomly around the centre of the distribution. However in this case, we observe stronger evidence of tail dependence especially in the upper tail of the distribution where a large loss for one line results in a similarly large loss for the other line. This is expected as the Student-t copula allows for tail dependence. See for example, the pair-wise plots for the motor/CTP (20%) and CTP/liability (25%) business lines.

4.1.3 Student-t Copula (10 df)

Although also generated from a Student-t copula, the observations in Figure 6 appears more akin to the

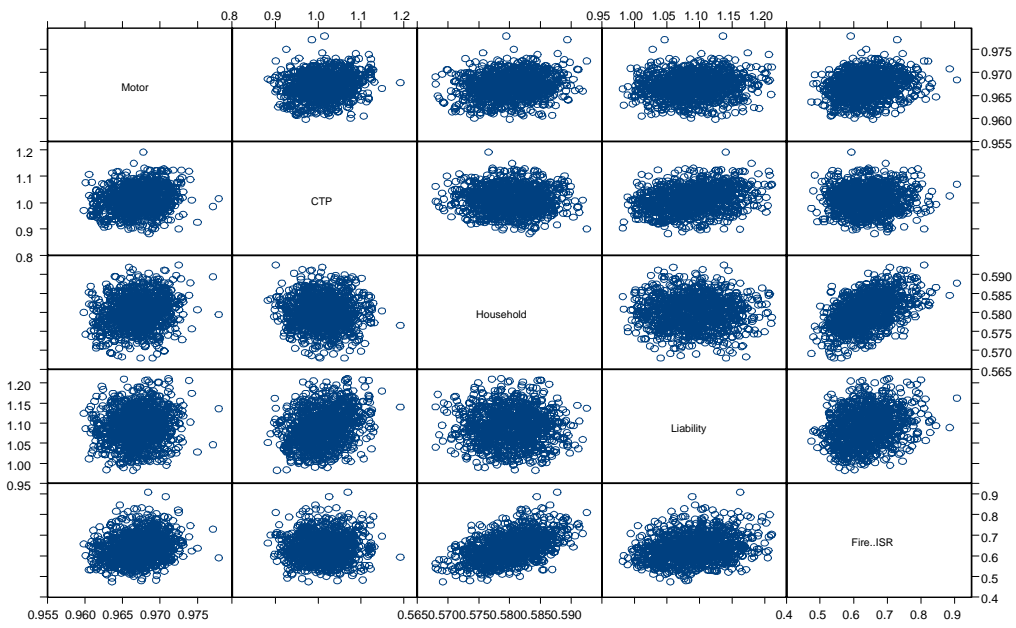


Figure 4: Scatter Plot of Simulated Loss Ratios - Normal Copula

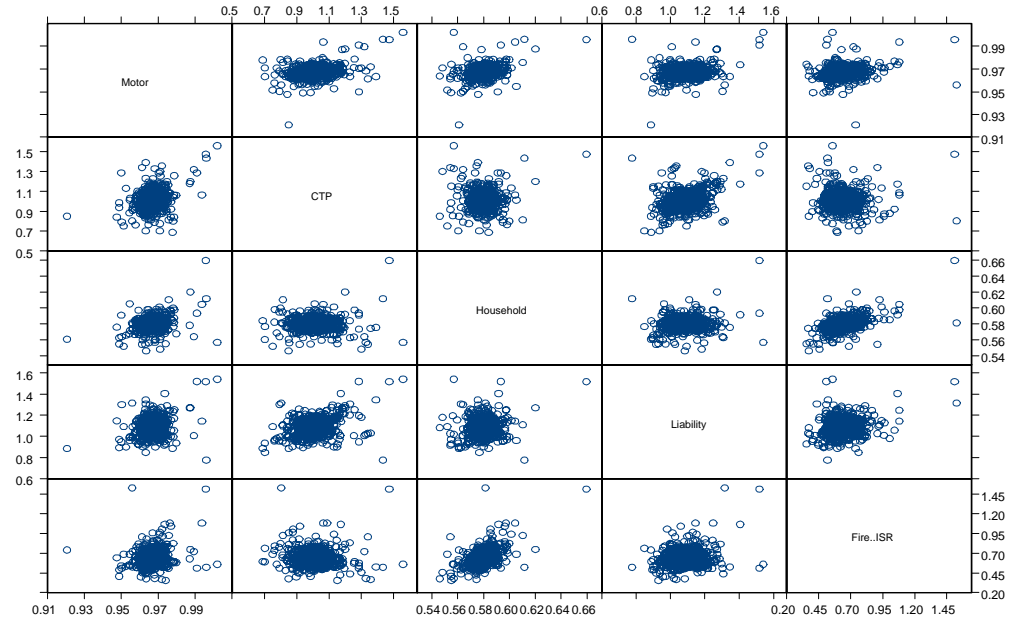


Figure 5: Scatter Plot of Simulated Loss Ratios - Student-t (3df) Copula

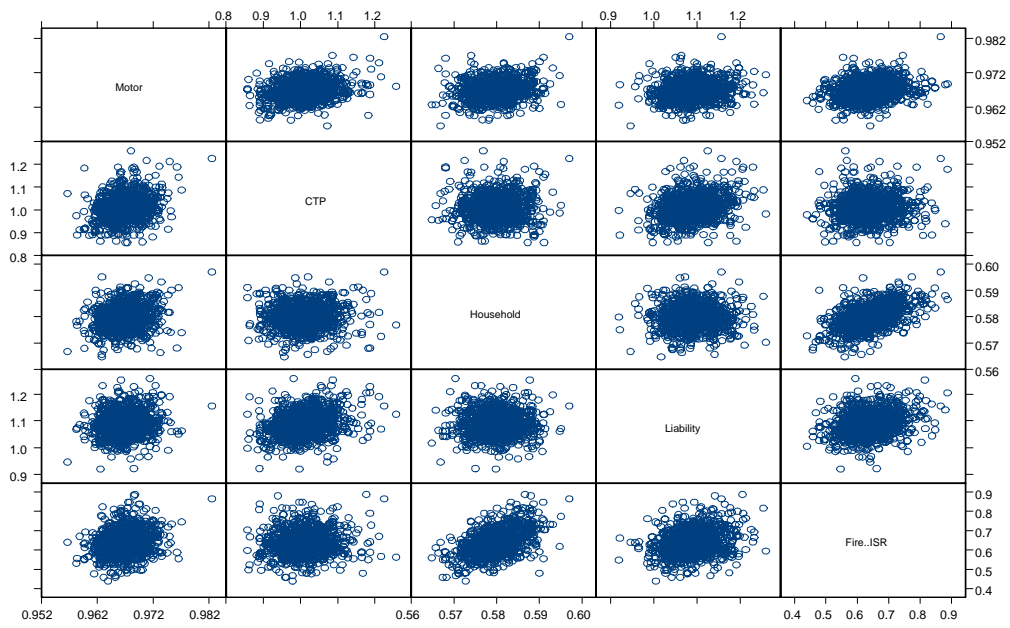


Figure 6: Scatter Plot of Simulated Loss Ratios - Student-t (10 df) Copula

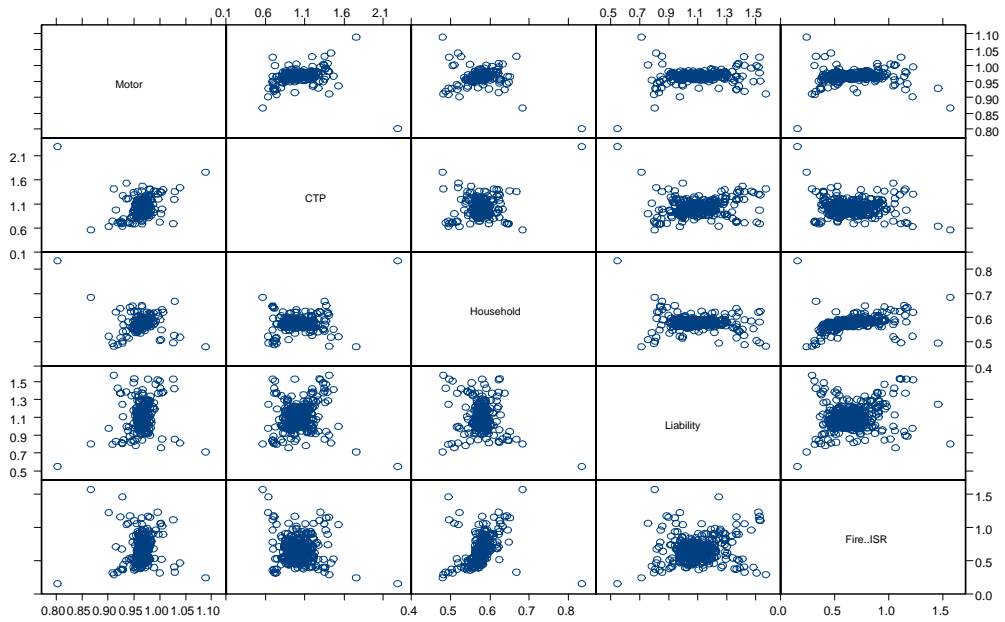


Figure 7: Scatter Plot of Simulated Loss Ratios - Cauchy Copula

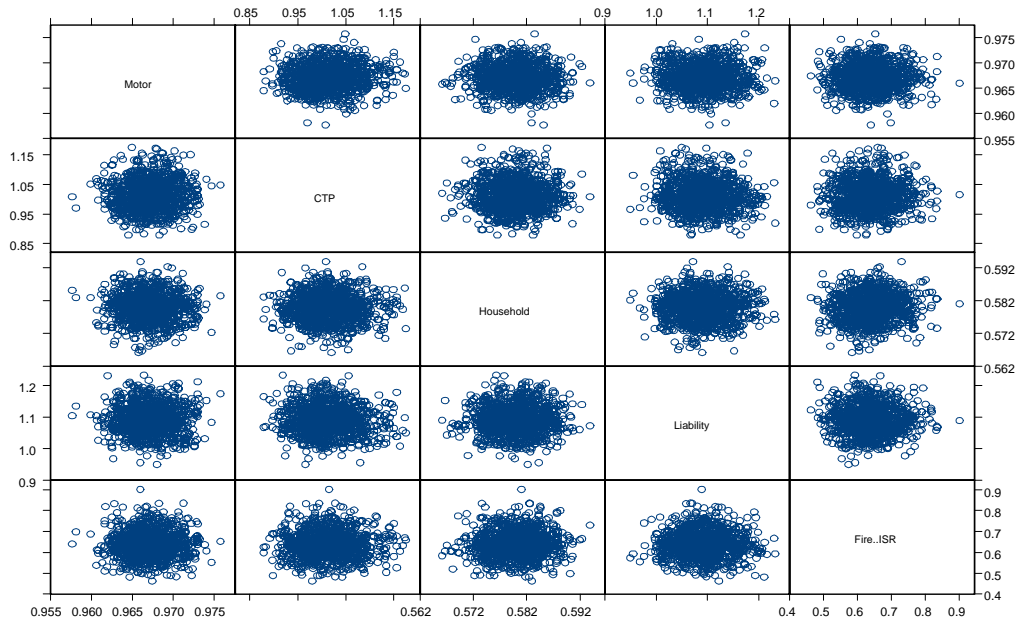


Figure 8: Scatter Plot of Simulated Loss Ratios - Independence Copula

Normal case than the previous Student-t case with more evidence of linear dependence for some pairs of business lines. This is attributable to the asymptotic behaviour of the Student-t copula, again as we noted earlier, that as the degree of freedom becomes large, the copula behaves more like that of a Normal copula. Therefore, although 10 degrees of freedom is not quite large enough for strict asymptotic behaviour, but it is larger than the 3 degrees of freedom, and we can still reasonably justify its similarity to the Normal case.

4.1.4 Cauchy Copula

Now inspecting Figure 7 for the Cauchy copula, it becomes more difficult to see the presence of the linear dependence between the losses from the different business lines. Other forms of dependencies are being captured in the Cauchy copula including possible strong dependence on the tails. For example, the household/fire & ISR losses appear to capture a quadratic dependence structure, one where it would not have been possible to capture using a Normal copula alone. Again as expected for this type of copula, there is stronger visual evidence of tail dependencies with many more pairs of simultaneously extreme value observations produced than those by the other forms of copulas.

4.1.5 Independence Copula

In the case of the independence copula, we would expect the pairs of observed loss ratios between the business lines to capture no discernible patterns because each line of business has been modelled independently of each other. These are evident from the random nature of the patterns in Figure 8 and therefore, this figure re-affirms our assumption.

4.2 The Aggregate Loss

Continuing with the procedure outlined earlier, the simulated loss ratios for each business line is then aggregated using the industry weights, by earned premium, to produce a distribution of the aggregate loss under each copula. The purpose of this aggregation is to enable us to examine the loss ratios for an insurer whose composition of business lines mimic that of the industry which represents the base scenario assumed in this paper. In Figure 9, we provide the histograms of the resulting aggregated loss ratio for this industry insurer for each copula model. As is observed from this figure, there are differences in the resulting distributions for the different copulas and this is therefore an initial indication that there will be differences in the level of capital requirements for different dependence structures. To further assess these differences,

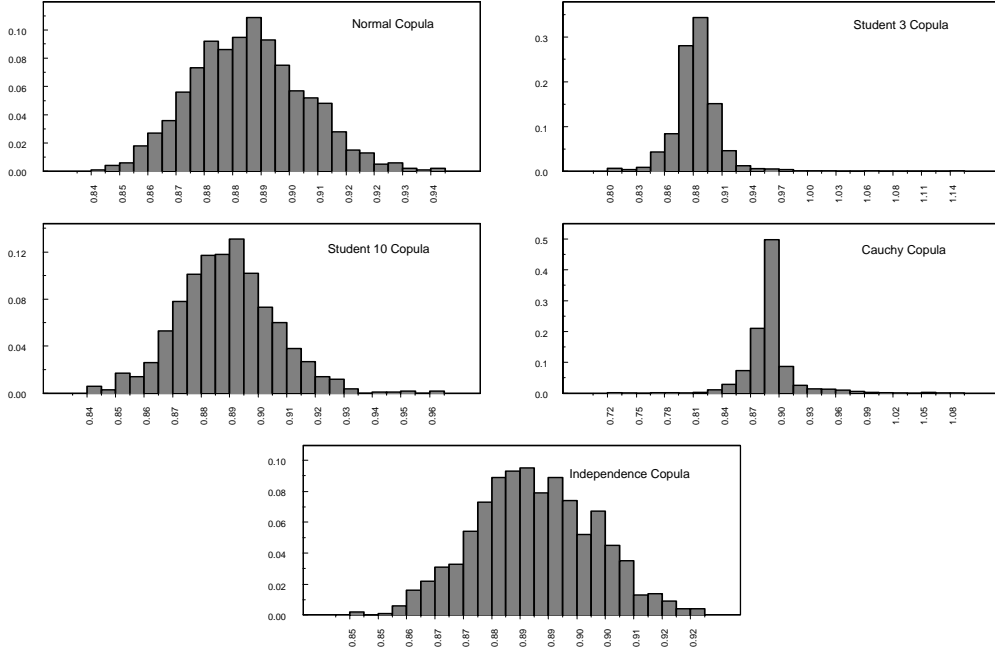


Figure 9: Aggregated Loss Ratio Distribution - Base Scenario

Table 11 provides some important summary statistics of these loss ratio distributions.

From Table 11 together with reference to Figure 9, we make the following key observations:

1. Measures of central tendency indicate a high degree of coherence between the aggregate loss under different copulas. In particular, the means are all within 0.1% of the mean of the independence case while the mode vary from -1.0% to +1.2% of that for the independence case. The near perfect match of the medians further affirms the similarity of each distribution in the centre portion of the range of aggregate losses.
2. As a measure of dispersion, the standard deviation indicates a large variety in the aggregate loss distributions. All of the copula models resulted in a more disperse distribution than the independence case. This result is expected as it conforms to risk pooling theory, which states that sums of positively correlated random variables will always be more variable than sums of independent random variables. There is also a vast difference in variability across the different copulas. The Normal and Student-t (10 df) copulas are less variable than the Student-t (3 df) and Cauchy copulas, being only 17.6% and 29.7% more volatile than the independence case

respectively. In contrast, the latter two copulas are much more widely dispersed with standard deviations 83.3% and 115.3% above that of the independence case. We also see these differences in dispersion from Figure 9 where the histograms for the Student-t (3 df) and Cauchy copulas both require a larger range of values on the x-axis. Similarly, observing the minimum and maximum values of each distribution in Tables 11 leads to the same conclusion regarding the comparative dispersion of the aggregate distributions.

3. Clearly from Figure 9, all of the aggregate loss distributions are asymmetric with a heavier tail extending in the positive direction. This is a key feature of insurance losses as earlier discussed. Skewness measures the degree of this asymmetry around the mean where positive skewness indicates a positive tail. The positive skewness values from Table 11 confirm the existence and direction of the asymmetry for all copulas assumptions. Depending on the copula used, we observe levels of skewness that varies dramatically. The independence case is the most symmetric while the Student-t (3 df) and Cauchy copulas are many times more skewed than the Normal and Student-t (10 df) copulas. This is reasonable as the copulas that have higher tail

Table 11: Summary Statistics of Aggregate Loss Distributions

| Statistic | Normal | t (3 df) | t (10 df) | Cauchy | Indep |
|--------------|-----------|-----------|-----------|-----------|-----------|
| Mean | 0.888269 | 0.888164 | 0.887526 | 0.889139 | 0.888527 |
| SD | 0.015541 | 0.024229 | 0.017150 | 0.028464 | 0.013218 |
| Mode | 0.868592 | 0.879274 | 0.868738 | 0.887653 | 0.877046 |
| Skewness | 0.222202 | 2.076989 | 0.288427 | 1.316450 | 0.057899 |
| Kurtosis | -0.054722 | 20.194530 | 0.942778 | 12.844188 | -0.259823 |
| Minimum | 0.844456 | 0.799064 | 0.837854 | 0.719465 | 0.847441 |
| 1st Quantile | 0.877388 | 0.877184 | 0.876068 | 0.881241 | 0.879448 |
| Median | 0.888053 | 0.887899 | 0.887781 | 0.887573 | 0.887905 |
| 3rd Quantile | 0.898136 | 0.897899 | 0.897883 | 0.894247 | 0.897487 |
| Maximum | 0.942946 | 1.156190 | 0.964726 | 1.094490 | 0.926259 |

dependence theoretically induces a heavier tail and hence a more skewed aggregate loss distribution.

4. Kurtosis measures the relative peakedness or flatness of a distribution compared with the Normal distribution. Positive kurtosis indicates a relatively peaked distribution whereas negative kurtosis indicates a relatively flat distribution. We also clearly observe that the rank of the copulas, in the decreasing order of the peakedness of their aggregate loss distributions is independence, Normal, Student-t (10 df), Student-t (3 df) and Cauchy. This observation is supportive of the conclusion drawn in point 2 above regarding the different dispersion of the distributions as in general, a more peaked distribution will lead to a lower dispersion.

In general, we conclude from examining the aggregate loss distributions that despite coherence around the central portion of the range of losses, different copulas lead to drastically different loss ratio distributions. These differences indicate different tail behaviour which ultimately leads to different capital requirements, as will be discussed in the following subsection.

4.3 Capital Requirements, Diversification Benefit, and the Dependence Structure

We previously observe that, for the base scenario, there are differences in the loss ratio distributions for the different copula models. We now translate these differences in terms of the capital requirements as well as the diversification benefit. In the following subsections, we discuss these capital requirements and

diversification benefit for the base scenario, and we also examine the resulting impact of assuming different copula forms.

4.3.1 Capital Requirements (CRs)

Capital requirements, hereafter denoted by CRs, resulting from the aggregate loss distributions in the previous section are calculated using both the Value-at-Risk (VaR) and Tail Conditional Expectation (TCE) risk measures at the 97.5% and 99.5% levels. For the rest of this chapter, these risk measures are empirically estimated from the simulated loss ratios in a fairly straightforward manner. For the VaR measure, we simply computed the sample quantiles and for the TCE, we computed the sample mean for the observed values above the corresponding quantile. Unless otherwise stated, we present the CRs hereafter in terms of loss ratios which represent an amount per unit of earned premium. To make the assessment, Table 12 presents the CRs for the base scenario for each copula model.

We recognise from Table 12 that the CRs under different risk measures and copulas vary widely between a range of 0.92 to 1.07 times earned premium. However, we leave discussion on these effects for the next two sections.

4.3.2 Diversification Benefits (DBs)

By themselves, the values in Table 12 merely indicate the absolute level of CR per unit of earned premium. However, one of our primary concern is to assess the diversification benefit derived from writing multiple lines of business. This benefit, hereafter denoted by DB, has been defined in Section 1 as the difference (savings) between the CR on the aggregate loss and the weighted sum, by each lines proportion of total

Table 12: Capital Requirements on the Aggregate Loss

| Risk Measure | Normal | t (3 df) | t (10 df) | Cauchy | Indep | Average |
|-----------------|----------|----------|-----------|----------|----------|----------|
| VaR 97.5 | 0.919556 | 0.937313 | 0.922053 | 0.963311 | 0.914818 | 0.931410 |
| VaR 99.5 | 0.931090 | 0.982005 | 0.943131 | 1.026140 | 0.921855 | 0.960844 |
| TCE 97.5 | 0.926698 | 0.974546 | 0.933319 | 1.000353 | 0.918941 | 0.950771 |
| TCE 99.5 | 0.936560 | 1.047614 | 0.954881 | 1.066644 | 0.923883 | 0.985916 |
| Average | 0.928476 | 0.985369 | 0.938346 | 1.014112 | 0.919874 | |

earned premiums, of the CRs for each business line as if each was a stand alone business. The weighted sum of the CRs for each line is presented in Table 13 and the DBs expressed as percentage of earned premium, being the difference between each element of Table 12 and Table 13, are presented in Table 14.

We immediately observe from Table 14 that for all risk measures and copulas, there exists a positive DB. This confirms the theoretical result, first introduced in Section 1, that there will always be a positive DB by aggregating business lines under a multi-line business set up rather than running individual stand alone businesses. It is also immediately recognisable that the choice of risk measure and copula drastically affect the level of DB which ranges from 1.78% to 13.16% of earned premium. There appears to be a positive relationship between the absolute level of CR and the resulting DB as copulas that give rise to a high CR tends to be associated with a high DB. From Tables 13 and 14, we see that the Cauchy copula case is a good example of this phenomenon. To put these results into perspective, assuming the case of the “average” insurer¹, the capital savings in Table 14 are converted to monetary terms in Table 15.

The range of capital savings for the “average” insurer is therefore from \$23.8 million to \$175.6 million. This range is large and the amount of savings concerned is significant by any measure. Therefore, it is imperative that we conduct further analyses to understand the key drivers of these DBs in the following sections. In conclusion, the results presented so far demonstrate the existence of a DB in operating multiple lines of business in terms capital savings and illustrates the importance to choose the appropriate risk measure as well as copula structure to model the aggregate loss. There also appears to be a positive relation between the levels of CR and DB across the copulas.

¹We assume the “average” insurer has a 10% share of the Australian market as at June 2002, with total earned premium of \$13,351,897, similar to the 6th largest insurer.

4.3.3 The Effect of the Copula Assumption

As we noted previously, Tables 12 and 14 provide the necessary results to gauge the effect of the various copula assumptions on both the level of CR and the implied DB. These effects can be visualised graphically in Figures 10 and 11² respectively, for the different copula models.

Clearly from Figure 10, there is a consistent effect due to the choice of copula on the total capital required regardless of the risk measure used. The Cauchy copula results in the highest capital requirement, followed by the Student-t (3 df) in all cases. The Normal and Student-t (10 df) copulas produce relatively similar result for all risk measures with the latter always resulting in a marginally higher CR. This observation is expected due to the asymptotic Normal behaviour of the Student-t copula as the degree of freedom increases. As we expected, due to its lack of account of the inherent positive dependence of losses across business lines, the independence copula results in the lowest capital requirement. On average across all risk measures, the range of CRs were from 92% to 101% of earned premium for the different copulas. The observed discrepancy of the CR between copulas highlights the importance of correctly modelling the dependence structure, in particular tail dependence, between losses across business lines. In conclusion, we find that copulas allowing for high tail dependence, i.e., the Cauchy and Student-t (3 df) results in drastically higher CRs compared to those that do not allow for it. This difference is significant for all risk measures with the extreme being the TCE (99.5%) case where the discrepancy between the CRs under the Cauchy and Normal copulas reaches 14.3% of total earned premium, or equivalently, a difference of \$191million for the “average” insurer.

The choice of copula also have a profound effect on the resulting diversification benefit. We make the following key observations:

²For all figures in this chapter, risk measures 1 to 4 denote the VaR(97.5%), VaR(99.5%), TCE(97.5%) and TCE(99.5%) measures respectively.

Table 13: Aggregated Capital Requirements of Individual Business Line Losses

| Risk Measure | Normal | t (3 df) | t (10 df) | Cauchy | Indep | Average |
|-----------------|----------|----------|-----------|----------|----------|----------|
| VaR 97.5 | 0.937405 | 0.962058 | 0.942277 | 1.004449 | 0.941166 | 0.957471 |
| VaR 99.5 | 0.951284 | 1.035403 | 0.966556 | 1.110557 | 0.955346 | 1.003829 |
| TCE 97.5 | 0.946906 | 1.013448 | 0.957959 | 1.082981 | 0.950686 | 0.990396 |
| TCE 99.5 | 0.959170 | 1.107498 | 0.978780 | 1.198195 | 0.962960 | 1.041320 |
| Average | 0.948691 | 1.029602 | 0.961393 | 1.099045 | 0.952540 | |

Table 14: Diversification Benefits

| Risk Measure | Normal | t (3 df) | t (10 df) | Cauchy | Indep | Average |
|-----------------|--------|----------|-----------|--------|-------|---------|
| VaR 97.5 | 1.78% | 2.47% | 2.02% | 4.11% | 2.63% | 2.61% |
| VaR 99.5 | 2.02% | 5.34% | 2.34% | 8.44% | 3.35% | 4.30% |
| TCE 97.5 | 2.02% | 3.89% | 2.46% | 8.26% | 3.17% | 3.96% |
| TCE 99.5 | 2.26% | 5.99% | 2.39% | 13.16% | 3.91% | 5.54% |
| Average | 2.02% | 4.42% | 2.30% | 8.49% | 3.27% | |

Table 15: Diversification Benefits for the "Average" Insurer (1,000)

| Risk Measure | Normal | t (3 df) | t (10 df) | Cauchy | Indep | Average |
|-----------------|--------|----------|-----------|---------|--------|---------|
| VaR 97.5 | 23,831 | 33,039 | 27,003 | 54,927 | 35,180 | 34,796 |
| VaR 99.5 | 26,962 | 71,296 | 31,277 | 112,712 | 44,717 | 57,393 |
| TCE 97.5 | 26,982 | 51,943 | 32,898 | 110,324 | 42,386 | 52,907 |
| TCE 99.5 | 30,189 | 79,956 | 31,909 | 175,645 | 52,175 | 73,975 |
| Average | 26,991 | 59,059 | 30,772 | 113,402 | 43,615 | |

Table 16: Comparative Diversification Benefit across Copulas

| Risk Measure | Normal | t (3 df) | t (10 df) | Cauchy | Indep |
|-----------------|--------|----------|-----------|--------|-------|
| VaR 97.5 | 68% | 94% | 77% | 156% | 100% |
| VaR 99.5 | 60% | 159% | 70% | 252% | 100% |
| TCE 97.5 | 64% | 123% | 78% | 260% | 100% |
| TCE 99.5 | 58% | 153% | 61% | 337% | 100% |
| Average | 62% | 132% | 71% | 251% | 100% |

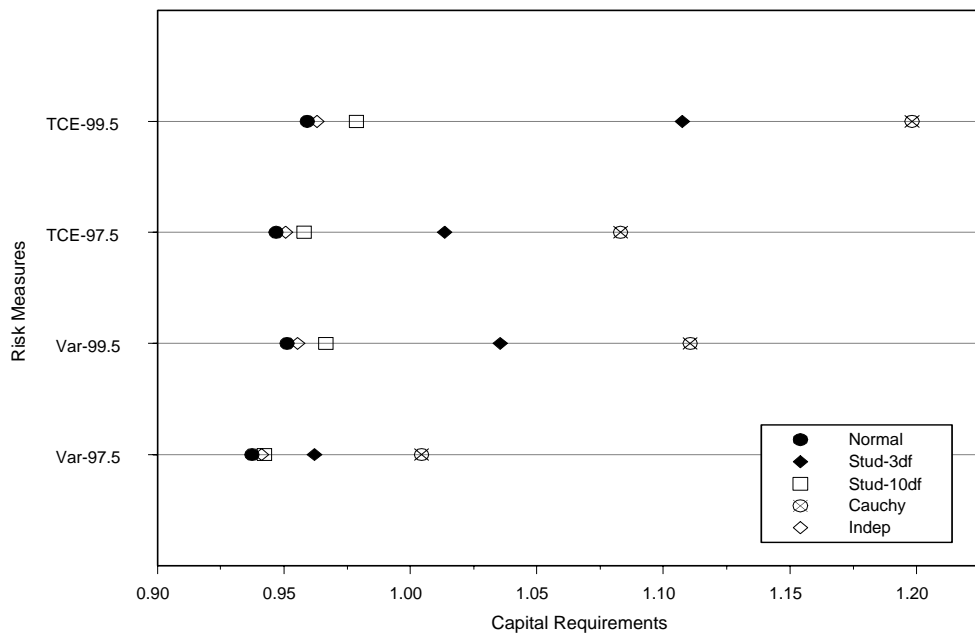


Figure 10: The Effect of Copulas on Capital Requirements

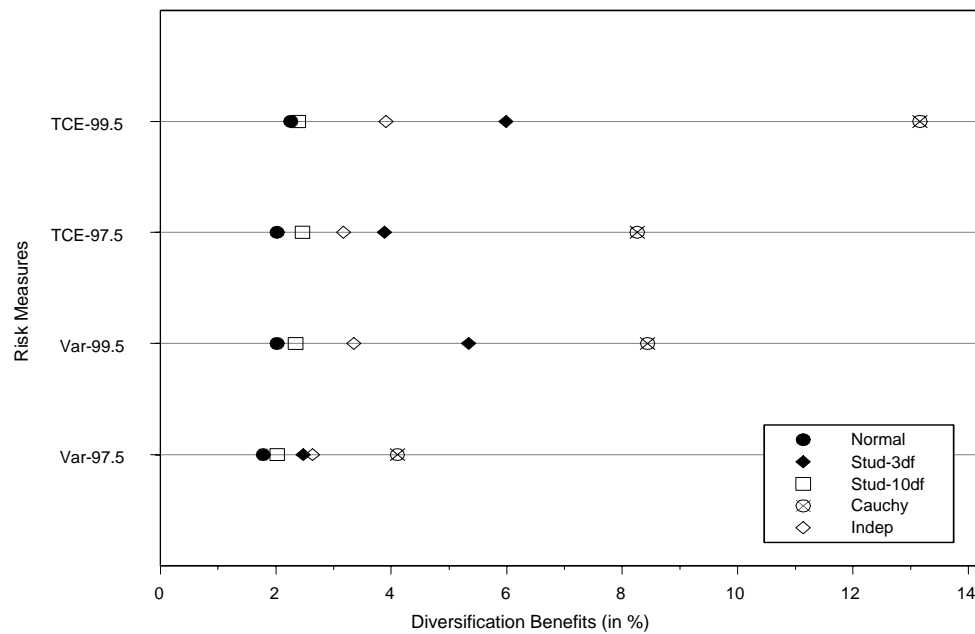


Figure 11: The Effect of Copulas on the Diversification Benefit

1. In Table 14, the average value of the DB according under each copula is calculated and this provides an initial crude measure of the magnitude of the DB resulting from each copula. In decreasing order, the ranking of the copulas is Cauchy (8.5%), Student-t (3df) (4.4%), independence (3.3%), Student-t (10df) (2.3%) and Normal (2.0%). For the “average” insurer, this implies capital savings of between \$27million and \$113million, depending on the copula used³. Therefore, modelling dependencies between losses from different business lines with the Cauchy copula will give by far the highest level of DB while using the Normal copula will result in the lowest level of DB.
2. Similar to the analysis on the CRs, we find that copulas that allow high tail dependence, i.e., the Cauchy and Student-t (3 df) copula, result in the highest DB. The range of diversification benefits for the Cauchy copula is from 4.1% to 13.2% depending on the capital risk measure while the range is between 2.5% to 6.0% for the Student t-copula (3 df). On the other hand, the copulas that allows little or no tail dependence, i.e., the Normal copula and the Student t-copula (10 df), result in relatively low DBs of approximately 2% regardless of the choice of risk measure. For the Normal copula, the range of DBs is 1.8% to 2.3%. The Student-t copula (10 df) approximates the Normal copula, and results in only slightly higher range of DBs of 2.0% to 2.4%. Despite resulting in the lowest CRs, the independence copula induces a DB in excess of those induced by the Normal and Student-t (10 df) copulas. This is true regardless of the risk measure and in the case of VaR (97.5%), it is even slightly higher than the Student-t (3 df) case. The range of the independence copula’s DBs is 2.6% to 3.9%.
3. For the independence, Normal and Student-t (10 df) copulas, the degree of variation of the DBs across risk measures is relatively small. This is consistent with the fact that these copula structures does not induce much tail dependence and hence the tail of the aggregate loss distributions are not affected by them. This contrasts with the cases of the Cauchy and Student-t (3 df) copulas where the fluctuations under different risk measures are very apparent.
4. With reference to Table 16, which illustrates the comparative level of the DBs standardised by the

³Refer to Table 15 for details on the capital savings for the "average" insurer.

independence copula values, the above observations can be put into a different perspective. We immediately see that the Normal copula provides consistently low diversification benefits with a range of DBs between 58% and 68% of the independence case and an average of 62%. Similarly, the Student-t (10 df) copula also displays a consistently low range of 61% to 78% with an average of 71%. The Student-t (3 df) copula gains on average 32.3% more DB compared to the independence copula but the DB is in fact comparatively less than the independence case if VaR (97.5%) is used as the risk measure. The Cauchy copula results in a average DB that is about 2.5 times of the independence case, however, the advantage is much more profound for the TCE measure and for the lower level of ruin tolerance (97.5%).

Overall, we conclude that the choice of copula has a paramount effect on the CR as well as DB for a multi-line insurer. This effect is driven mainly by the amount of tail dependence that the copula allows for losses between business lines. The more tail dependence allowed by a copula, the higher the CR as well as the DB if losses are aggregated under that copula. In terms of the DB, the Normal and Student-t (10 df) copulas perform worse than the independence copula while the Student-t (3 df) and Cauchy copulas both perform better.

4.4 Comparison between the Copula Approach and a Current Approach

Currently, APRA requires all Australian insurers that does not use an internal model⁴ to determine their appropriate CR to use the so-called Prescribed Method (PM). See Tang (2004). In this section, we compare the CR resulting from the PM to those assuming different copula models. In doing so, we particularly pay attention to the CRs based on the VaR at the 99.5% level under each copula as this corresponds to APRA’s requirements under the PM. Table 17 summarises the comparison between the PM CR and the copula model CRs for the base scenario of the industry portfolio.

For all copula assumptions except the Cauchy copula, we see that the PM overestimates the true CR for the industry portfolio of liabilities. That is to say, if multi-line insurers are to use a copula based

⁴At the time of writing, there has not been any internal models approved by APRA for capital purposes.

Table 17: Comparison of PM and Copula Model CRs - Industry Portfolio

| | Normal | t (3 df) | t (10 df) | Cauchy | Indep |
|--------------------------|---------------|-----------------|------------------|---------------|--------------|
| PM CR | 1.010291 | 1.010233 | 1.008857 | 1.002536 | 0.999034 |
| VaR 99.5% CR | 0.931090 | 0.982005 | 0.943131 | 1.026140 | 0.921855 |
| Excess Capital | 0.079201 | 0.028228 | 0.065726 | -0.023604 | 0.077179 |
| % Savings | 7.84% | 2.79% | 6.51% | -2.35% | 7.73% |
| “Average” Insurer | | | | | |
| Savings (\$,000) | 105,748 | 37,689 | 87,757 | -31,515 | 103,048 |

model rather than the PM to aggregate their losses across business lines for CR purposes, savings of between 2.8% and 7.8% of the PM capital can be made. For the “average” insurer, this amounts to savings of between \$37.7million and \$105.7million. However, we must note that the level of savings diminishes as copulas with a heavier tail dependence structure are used. For example, the savings on the Normal copula model which allows zero tail dependence is 7.8% while it is only 2.8% in the case of the Student-t (3 df) copula. If an extremely tail dependent copula such as the Cauchy copula is used, the PM CR in fact represents a shortfall of 2.4% from its intended level. Therefore, in making the assertion that copula models lead to capital savings compared to the PM in general, we must be aware that this result is extremely sensitive to the choice of copula and is by no means definitive. Further, it is worthwhile to compare the PM and copula model CRs for a short tail and a long tail portfolios. These results are presented in Tables 18 and 19 respectively.

We see that support for the argument of capital savings under copula models is strong for the short tail portfolio as all copula assumptions lead to savings within the range of 1.5% to 19.5%. However, the results are less clear cut for the long tail portfolio where shortfalls of up to 17.2% exist under the independence copula if the PM is used. This is a significant result in monetary terms, representing a capital shortfall of \$197.2 million for the “average” insurer if the independence copula model is used. Although it is unimaginable for any insurer to assume independence between their business lines and use the independence copula, other copula structures also lead to similar results. In particular, the Cauchy copula leads to a shortfall of 12.8% which represents \$184.4million for the “average” insurer. This inconsistency in the results between the short tail and long tail portfolios highlights the deficiency of the PM as a comprehensive, “one size fits all” solution for calculating capital for the diverse range of insurers in the industry. The

significant capital savings (shortfalls in some cases) from using the copula models compared to the PM discussed in this sections indicates two issues. First, it highlights the deficiency of the formula driven PM as the current primary method of capital calculation for different types of insurers in the industry. Consequently, this leads to the urgent need for more accurate and flexible internal models to be developed for capital determination purposes where copulas are incorporated for the aggregation of losses from different business lines. However, as the actual CRs are found to be extremely sensitive to the choice of copulas, and further this sensitivity varies depending on the portfolio composition of the particular insurer, it is imperative that in constructing the internal models, an appropriate copula assumption is made. The most important aspect of the choice of copulas is the tail dependence behaviour and the copula that best represents the tail behaviour of the particular portfolio’s situation should be chosen.

5 Concluding Remarks

Today, there is a significant number of general insurance companies that write insurance contracts in multiple business lines. To ensure solvency, insurers are required both for regulatory purposes and as a going business concern to hold capital to back their insurance liabilities. In aggregating losses from different business lines for the purpose of capital determination, insurers have traditionally either ignored the dependence structure between business lines or used simple linear correlations to model such dependence. In this paper, we aggregated each business line’s losses using the independence and four variants of the Student-t copula, and assessed the capital requirements in each case using the value-at-risk and tail conditional expectation risk measures. Further, we proposed the existence of a diversification benefit from holding capital in aggregate for multi-line insurers rather than on an individual business line basis.

Table 18: Comparison of PM and Copula Model CRs - Short Tail Portfolio

| | Normal | t (3 df) | t (10 df) | Cauchy | Indep |
|-------------------------|----------|----------|-----------|----------|----------|
| PM CR | 0.951609 | 0.952025 | 0.951191 | 0.948628 | 1.093202 |
| VaR 99.5% CR | 0.876892 | 0.911036 | 0.885701 | 0.934066 | 0.880529 |
| Excess Capital | 0.074717 | 0.040989 | 0.065490 | 0.014562 | 0.212673 |
| % Savings | 7.85% | 4.31% | 6.89% | 1.54% | 19.45% |
| Savings (\$,000) | 99,761 | 54,729 | 87,442 | 19,443 | 283,959 |

Table 19: Comparison of PM and Copula Model CRs - Long Tail Portfolio

| | Normal | t (3 df) | t (10 df) | Cauchy | Indep |
|---|----------|-----------|-----------|-----------|-----------|
| PM CR | 1.098314 | 1.097543 | 1.095357 | 1.083399 | 0.857781 |
| VaR 99.5% CR | 1.021380 | 1.135560 | 1.026240 | 1.221500 | 1.005440 |
| Excess Capital | 0.076934 | -0.038017 | 0.069117 | -0.138101 | -0.147659 |
| % Savings | 7.00% | -3.46% | 6.31% | -12.75% | -17.21% |
| “Average” Insurer Savings (\$,000) | 102,721 | -50,760 | 92,284 | -184,391 | -197,152 |

We analysed this diversification benefit for the different copula assumptions. The following are the key findings of this paper.

First, the choice of copula has a dramatic effect on both the capital requirement and diversification benefit for a multi-line insurer. In particular, the more tail dependence a copula allows, the higher is the required capital. The same relationship between the choice of copula and the diversification benefit also exists. In the extreme case, this diversification benefit amounts to \$113 million in capital saving for the “average” insurer writing an industry weight portfolio. Therefore, because of the potential for massive modelling errors, it is imperative for insurers to select a dependence structure that is most reflective of their own unique situation to avoid the risk of mis-calculating their capital requirement.

Second, there is a positive relationship between the capital requirement and diversification benefit under all copulas. In terms of regulatory capital requirements, this means that the more stringent a capital regime is, the higher the incentive is for insurers to write multiple lines of business. This finding coincides with the presence of many multi-line insurers in Australia, where the recently enacted capital regime involved a tightening from the previous one.

Lastly, the adequacy of the current APRA Prescribed Method is assessed against the capital requirements implied by the copula models. While the Prescribed Method appears to overestimate the cap-

ital requirement for an insurer with a short tail or industry portfolio, it underestimates the capital requirement for a long tail insurer under certain copulas. Again, the higher the copula’s allowance for tail dependence, the higher the resulting capital requirement will be under that model, and hence the larger the shortfall (or smaller the savings) of holding the Prescribed Method capital will be. The inconsistency between the short and long tail comparisons highlights the inadequacy of the Prescribed Method as being a “one size fits all” solution for determining capital for all types of insurers in the industry. Consequently, this points to the urgent need for insurers to develop internal models which can account for their business’ specific dependency structure. Other findings, please see Tang (2004).

The reader of this paper should be aware that there are simplifying assumptions inherent in our analysis. In calculating the capital requirements and diversification benefits, we have limited our focus on purely the insurance or underwriting risk of an insurer while ignoring other sources of risk such as investment and operational risk. In reality, different sources of risks interact and they all affect the capital requirement for an insurer. Further, reinsurance is a common risk management tool for insurers and hence forms an essential part of their liability portfolios. However, due to data limitations, amongst other reasons, we have excluded the effect of reinsurance from our analysis. Investment of assets was also ignored although this

would have only had a minor effect due to the short term nature of general insurers' investments.

Due to these assumptions, the single most significant limitation to the results of this paper is that they merely serve to compare the effect of the dependence structure on the capital requirements and does not quantify the required capital that a particular insurer should hold. In particular, the results cannot be directly compared with other studies that consider the total capital requirement for an insurer. The capital requirements calculated in this paper is not representative beyond the amount of assets required to ensure a specific level of confidence in meeting its insurance losses within a one year period. The numerical results in monetary terms presented in this paper must also be taken with caution as this is only representative of the "average" insurer as defined in this paper and again, we emphasise that it only accounts for the insurance risk component of the total capital requirements. However, the modelling procedure of aggregating risks using copulas as demonstrated in this paper can be readily adapted.

Given that our findings in this paper indicate a pronounced effect of the copula structure on the capital requirements for insurers, we suggest further research into using other copulas to model the dependence structure of multi-line insurers' portfolios. In this paper, we have explored using elliptical copulas to model the dependence structure but these are only a few of the vast pool of copula structures that one can draw from. Isaacs (2003) explored the same aggregation process using the Gumbel copula, a representative of heavy tail dependence copulas which appear to be very suitable for capital determination purposes. However, the specification of a Gumbel copula only involves pair-wise associations and hence is not appropriate for application to multi-line insurers with more than two business lines.

We can also apply copula approaches to model other risk dependencies in the general insurers' business. For example, rather than modelling the dependency at the business line level, one may investigate dependencies between risk sources such as those between investment and operational risks.

Catastrophes that affect multiple lines of business are often modelled as a separate class in a dynamic model. To this end, we may find interesting results if the analysis of this paper was performed on a portfolio with a separate catastrophe business line.

Furthermore, we can relax some of our assumptions in our modelling to alleviate some of the limitations previously mentioned. For example, we can perform the analysis on a net rather than gross basis so that reinsurance effects on the capital requirements are al-

lowed for. Another suggestion is to factor in expenses and possibly the effects of investments as these also influence capital requirements. Sensitivities of the capital requirement under the copula models to the size of the insurer and the correlation matrix assumption can also be explored.

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A Generating Multivariate Random Vectors

The purpose of this appendix is to show how one can go about generating random vectors. One of the useful applications of copulas is the resulting ease of generating random vectors. One procedure is described in Nelsen (1999), by considering the conditional distribution

$$\begin{aligned} c(v|u) &= \text{Prob}(V \leq v | U = u) \\ &= \lim_{\Delta u \rightarrow 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial}{\partial u} C(u, v) \end{aligned}$$

then the following generates (u, v) pairs:

1. generate two independent $U(0, 1)$, denote them by u and t ;
2. set $v = c^{(-1)}(t|u)$, the quasi-inverse of $c(t|u)$;
3. (u, v) is the desired pair with uniform marginals and copula C .

Recall that the copula defined by

$$C(u_1, \dots, u_n) = H(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n))$$

is called the Normal copula. The following algorithm generates (x_1, x_2, \dots, x_n) from the normal copula:

1. Construct the lower triangular matrix B so that the covariance matrix $V = BB^T$ using Choleski's decomposition;
2. Generate a column vector of independent standard normal random variables $Z = (Z_1, Z_2, \dots, Z_n)^T$;

3. Take the matrix product of B and Z , i.e. $Y = BZ$;
4. Set $U_i = \Phi(Y_i)$ for $i = 1, 2, \dots, n$;
5. Set $X_i = F_{X_i}^{-1}(U_i)$ for $i = 1, 2, \dots, n$.
6. (x_1, x_2, \dots, x_n) is the desired vector with marginals F_{X_1}, \dots, F_{X_n} and normal copula C .

Recall that the copula defined by

$$T_\nu(z_1, \dots, z_n) = \int_{-\infty}^{z_n} \int_{-\infty}^{z_{n-1}} \dots \int_{-\infty}^{z_1} f_\nu(x_1, \dots, x_n) dx_1 \dots dx_n$$

is called the t copula. The following algorithm generates (x_1, x_2, \dots, x_n) from the t copula:

1. Construct the lower triangular matrix B so that the covariance matrix $V = BB^T$ using Choleski's decomposition;
2. Generate a column vector of independent standard normal random variables $Z = (Z_1, Z_2, \dots, Z_n)^T$;
3. Take the matrix product of B and Z , i.e. $Y = BZ$;
4. Generate a chi-squared random variable $S \sim \chi^2(\nu)$ with ν degrees of freedom, independent of Z ;
5. Set $T_i = \sqrt{\frac{\nu}{S}} Y_i$ for $i = 1, 2, \dots, n$;
6. Set $U_i = t_\nu(T_i)$ for $i = 1, 2, \dots, n$;
7. Set $X_i = F_{X_i}^{-1}(U_i)$ for $i = 1, 2, \dots, n$.
8. (x_1, x_2, \dots, x_n) is the desired vector with marginals F_{X_1}, \dots, F_{X_n} and t copula C .

To generate from a Cauchy copula, we simply set the degrees of freedom to be $\nu = 1$ and followed the above procedure for simulating random vectors from a Student-t copula. We will need the above simulation procedures to simulate claim values with different copulas. Programs written in SAS and Splus.