Annuity Decisions with Systematic Longevity Risk*

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ABSTRACT

In this paper we investigate the effect of systematic longevity risk, i.e., the risk arising from uncertain future survival probabilities, on the attractiveness of different types of annuities. We consider a life-cycle framework with expected utility where an individual faces both investment and longevity risk. In contrast to existing literature we allow not only for idiosyncratic, but also for systematic longevity risk. When comparing the expected lifetime utility, conditional on the type of annuity which is purchased, we find for a 65-year old male that (i) systematic longevity risk reduces the attractiveness of annuities, (ii) when an immediate annuity is purchased, the expected lifetime utility is decreasing in the postponement period, (iii) when in the future purchasing an immediate annuities, the effect of the evolution of the survival probabilities on the optimal fraction of annuitized wealth is large, and (iv) the optimal annuity to purchase at retirement is a deferred annuity which starts to pay after only a short deferral period. However, when the purchase of an annuity with the optimal deferral period is compared to the purchase of an immediate annuity at retirement date, the utility gain is negligibly small.

Keywords: Optimal life-cycle portfolio choice, Life annuities, Asset allocation, Longevity risk, Market price of longevity risk, Optimal annuitization age.

JEL Classifications: C61, D14, D91, G11, G22, G23, J11.

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1 Introduction

Our goal in this paper is to investigate the optimal annuity decision in a life-cycle model when there is systematic longevity risk. Life expectancy has increased substantially over the past decades, and is expected to increase further in the future. However, there is considerable uncertainty regarding the exact development in future life expectancy. This uncertainty is called systematic longevity risk.[1] Systematic longevity risk can be modeled by allowing future survival probabilities to be stochastic. In studies investigating optimal annuity decisions systematic longevity risk is typically ignored. However, its presence affects the optimal life-cycle decisions in a number of ways. First, systematic longevity risk is a non-diversifiable risk and therefore it will have a nonzero price of risk, complicating the pricing of annuities. Second, stochastic future survival probabilities imply stochastic future annuity prices, further complicating the optimal life-cycle decisions of the individual. We allow for a nonzero price of risk in the annuity prices using risk-neutral survival probabilities, following Cairns, Blake, and Dowd (2006). Third, we allow for stochastic future survival probabilities and optimal decisions which depend on the evolution of these survival probabilities. To solve the optimization problem with stochastic future survival probabilities we follow Brandt, Goyal, Santa-Clara, and Stroud (2005) and Carroll (2005), and using extensions proposed by Koijen, Nijman, and Werker (2009). In addition to allowing for systematic longevity risk,[2] we extend the current literature on optimal annuity decisions by not only investigating the possibility of investing in an immediate annuity but also in a deferred annuity.

The existing literature on optimal annuitization in the context of immediate annuities but without systematic longevity risk is extensive. The literature was initiated by the seminal paper by Yaari (1965). Yaari (1965) and others (see, for example, Merton, 1983; and Davidoff, Brown, and Diamond, 2005) show that an individual's optimal investment choice is to invest all his wealth in annuities. This is shown in a standard Modigliani life-cycle model of savings and consumption without a bequest motive, with

[1]Naturally, we also allow for idiosyncratic longevity risk, which is due to a random individual remaining lifetime, conditional on given survival probabilities. This is also referred to as non-systematic longevity risk.

[2]Cocco and Gomes (2009) also allow for systematic longevity risk in a life-cycle model, but in their paper the individual maximizes the expected lifetime utility in a setting with only a risk-free asset and a longevity bond, without annuities or equities as investment opportunities.

as only investment opportunity a risk-free asset and actuarially fair annuities. The rationale behind this result is that the returns from annuities dominate the risk-free return, since the capital invested in annuities is allocated only to the survivors. Although these results suggest that retirees will voluntarily purchase annuities, in most countries very few actually do so (see, among others, Friedman and Warshawsky, 1990; Poterba and Wise, 1998; Moore and Mitchell, 2000; Bütler and Teppa, 2007; and Dushi and Webb, 2004b). This "annuity puzzle" has generated a lot of literature aimed at solving this puzzle.[3] We show that systematic longevity risk reduces the attractiveness of an immediate annuity, thereby reducing the optimal level of annuitized wealth when purchasing an annuity at retirement date. Thus, systematic longevity risk seems to be an important ingredient in understanding the annuity puzzle.

This paper also contributes to the literature on the optimal timing of the purchase of annuities. Much research has been devoted to finding the optimal fraction of wealth invested in annuities and the best timing for purchasing annuities. Due to actuarial unfairness of annuities postponing the purchase of an annuity purchase may be rational, because the mortality credit is too low just after retirement age.[4] Milevsky (1998) proposed postponing the annuity purchase until the mortality credit is larger than or equal to the equity risk premium. However, this annuitization strategy would only be optimal for risk-neutral individuals. Blake, Cairns, and Dowd (2003) found an optimal annuitization age in the range of 65 to 80, depending on individual characteristics such as risk aversion and bequest motive. Milevsky and Young (2002) estimated that the real option to annuitize remains valuable until the age range 75-85, also depending on individual characteristics. Different assumptions of the utility function have been made to find the optimal time to purchase annuities. These include the HARA utility (see Kingston and Thorpe, 2005) and the power utility (see Stabile, 2006). Others

[3]For example, the evidence for the size of an individual's bequest motive and the corresponding effect in a life-cycle model is mixed (see, for example, Yaari, 1965; Friedman and Warshawsky, 1990; Bernheim, 1991; Brown and Poterba, 2000; Hurd and Smith, 2001; and Vidal-Meliá and Lejárraga-García, 2006). Other authors have examined the strategic bequest motive (see, for example, Bernheim, Shleifer, and Summers, 1985). Another possible explanation for the annuity puzzle is the default risk of the annuity issuer (see Babbel and Merrill, 2007) or the illiquidity or irrevocability of annuities (see Sinclair and Smetters, 2004; and Peijnenburg, Nijman, and Werker, 2009). In addition, behavioural effects (see, for example, Hu and Scott, 2007; Brown, 2007; Brown, Kling, Mullainathan, and Wrobel, 2008; and Gazzale and Walker, 2009) may influence the decision to forgo voluntary annuitization.

[4]The mortality credit is defined as the (yearly) excess return of an annuity relative to the return on the risk-free investment. The mortality credit is formally defined in Section 4.

have investigated the optimal gradual annuity purchase pattern during retirement. For example, Kapur and Orszag (1999) and Horneff, Maurer, and Stamos (2008) found that gradual annuitization is optimal until the mortality credit is larger than the equity return. In our setting we find that for a 65-year-old individual postponing the annuity purchase is utility-decreasing due to systematic longevity risk. This difference with the existing literature illustrates the importance of systematic longevity risk in the life-cycle optimization problem.

This paper also contributes to the literature on the attractiveness of deferred annuities. The literature on the optimal deferral period of a deferred annuity which is purchased at retirement date is not very extensive. Milevsky (2005) provides a description of (inflation-linked) deferred annuities, referred to as Advanced-Life Deferred Annuities (ALDAs). This paper states that deferred annuities, purchased at the retirement date, starting to pay after a deferral period of around 15 to 25 years, are optimal. A deferred annuities is optimal because such an annuity provides longevity insurance at a low price. Hu and Scott (2007) mention that deferred annuities may be more desirable for individuals than immediate annuities, because the former overweight small probabilities. Dus, Maurer, and Mitchell (2005) show that deferred annuities can enhance the expected payout and cut the expected shortfall risk. In a setting with only a risk-free asset and given rules of thumb for the consumption level, Gong and Webb (2009) show that deferred annuities provide longevity insurance at a low cost. Horneff and Maurer (2008) find that a deferred annuity which starts income payments at the age of 65 might become more appealing than an immediate one when the loading factor is high enough. Bayraktar and Young (2009) find that it is always optimal to purchase an immediate annuity instead of a deferred one, when an individual's objective is to minimize the probability of financial ruin. We extend this literature in two ways. First, we compare annuities with different deferral periods. Second, we take into account that the actuarial unfairness in annuity prices may be due to a risk premium for systematic longevity risk. We use a risk-neutral pricing approach to model the risk premium in annuities due to systematic longevity risk. This results in annuities with a risk premium for systematic longevity risk that is dependent on the deferral period. We find that the optimal deferral period is short and that the utility gain from purchasing an annuity with the optimal deferral period instead of an immediate annuity is very small. Moreover, we find that when an individual purchases an annuity with a moderate deferral period (around 10 years) he can hold a substantial amount of liquid wealth with a low reduction in the expected lifetime utility.

The paper is organized as follows. In Section 2 we present the preferences of the representative individual and describe the stochastic forecast models we use to forecast the probability distribution of future survival probabilities. Section 3 presents the parameter calibration of the distributions of the equity returns and the distribution of the future survival probabilities. The attractiveness of an annuity is affected by the price and the payment stream. Therefore, in Section 4 we first illustrate the effect of systematic longevity risk on both the price of a deferred annuity and an immediate annuity. In addition, we illustrate the effect of systematic longevity risk on the mortality credit. In Section 5 we determine the optimal choices of an individual in the expected lifetime utility model. We show the effect of different annuity choices and the effect of systematic longevity risk on the optimal decisions of the individual. Robustness checks are subsequently performed in Section 6. Section 7 presents the conclusions.

2 Preferences, survival probabilities, and annuities

This paper investigates an individual's optimal fraction of wealth invested in either a deferred annuity or an immediate annuity, and the effect of systematic longevity risk on this decision. The optimal annuity decision is determined in a setting with three sources of risk:

i) *investment risk*, caused by a random return in the equity market;

ii) *idiosyncratic longevity risk*, due to a random individual remaining lifetime (conditional on given survival probabilities);

iii) *systematic longevity risk*, due to random future survival probabilities.

Section 2.1 defines an individual's expected lifetime utility function and describes the constraints the individual faces. The optimal choices of an individual depend on the probability distribution of future survival probabilities which is described in Section 2.2, and on the pricing of annuities, which is described in Section 2.3.

2.1 The individual's optimization problem

In this section we describe the optimization problem including the constraints faced by an individual who maximizes an expected utility of lifetime consumption. The investment choice consists of the fractions of wealth invested in a risky asset, a risk-free asset, and in an annuity. We consider two types of annuities:

i) An *immediate annuity* which yields a nominal yearly payment of 1, with a final payment in the year the insured dies;

ii) a *deferred annuity* which yields a nominal yearly payment of 1, after a deferral period of d years when the insured is still alive, with a final payment in the year the insured dies.

Let $A^{(d)}_{x,t}$ denote an annuity with a deferral period of d years bought by an individual aged x at time t. Note that an immediate annuity represents a special type of a deferred annuity, namely one with a deferral period equal to one (i.e., $d = 1$).[5] We determine an individual's lifetime expected utility and optimal choices for two cases, namely currently purchasing a deferred annuity with a fixed deferral period d and postponing the purchase of an immediate annuity until a fixed time s.[6]

We assume that the individual has an intertemporally separable, expected lifetime constant relative risk aversion (CRRA) utility function, without a bequest motive. To avoid overloaded notation, the time at which we calculate the expected lifetime utility, i.e., the base year, is set equal to zero, unless otherwise mentioned. We assume that the individual invests in an annuity only once, at a fixed time $s \geq 0$, and invests in only one type of annuity, i.e., an annuity with deferral period d (with $d = 1$ for an immediate

[5]An annuity can either be an *ordinary annuity* or an *annuity-due*. The difference between the two types of annuities is that an annuity income payments can either be at the beginning of a specified period (i.e., an annuity-due) or at the end of the specified period (i.e., an ordinary annuity). An annuity with $d = 1$ is an ordinary immediate annuity and an annuity with $d = 0$ is an immediate annuity-due. Note that using only a deferred annuity with a deferral period of zero years, one can obtain the same payment stream as with a deferred annuity with a deferral period of one year. This occurs because the difference between the two annuities is only a certain immediate payment. Under arbitrage-free pricing the price of an annuity with an initial payment in the following year and an annuity with an immediate initial payment equals the level of the current payment. In this paper, when we refer to immediate annuities, we mean an ordinary annuity with an initial payment in the following year, i.e., $d = 1$.

[6]Note that one can also investigate the effect of postponing the purchase of deferred annuities. However, as we will argue in Section 7, this will probably not be optimal due to the systematic longevity risk.

annuity). To investigate the effect of the different types of annuities and the effect of postponing the annuity purchase we calculate the optimal consumption, investment, and annuity choices conditional on the annuity type (i.e., conditional on d), and the time when an annuity is purchased (i.e., conditional on s). We obtain the optimal annuity choice by comparing the corresponding expected lifetime utilities.

Let ${}_{\tau}p_{x,t}$ be the probability that an x-year old at time t will survive τ years; let γ denote the coefficient of relative risk aversion; let β be the time preference parameter (also referred to as the subjective discount factor); let W_t be the liquid wealth level in period t; let C_t be the consumption level in period t; and let A_t be the annuity income in year t. An individual is characterized by his time-t age x, wealth level before annuity income and consumption, W_t, and the time-t state variables corresponding to the annuity income, A_t, and B_t.

Now we consider a given time s at which annuities with a deferral period of d years are bought, and determine the optimal investment and consumption choices. At time t the endogenous state variables are W_t, A_t, and B_t and the exogenous state variables at time t are denoted by the vector X_t.[7] Let (C_t, w_t) be the set of control variables in year t, i.e., the time-t level of consumption and the fraction of wealth after annuity income and consumption invested in equity, respectively, and let $a_s(d)$ be an additional control variable at time s, i.e., the fraction of after-consumption wealth which in year s is invested in an annuity with a deferral period of d years. The time-t expected lifetime utility J_t of an individual is defined by:

$$\begin{aligned} J_t\left(x, W_t, A_t, B_t, X_t\right) &= \max_{a_s(d), \{w_\tau, C_\tau\}_{\tau \geq t}} \left\{ \mathbb{E}_t \left[\sum\nolimits_{\tau \geq 0} {}_{\tau}p_{x,t} \cdot \beta^\tau \cdot \tfrac{(C_{t+\tau})^{1-\gamma}}{1-\gamma} \right] \right\}, \quad \text{if } t \leq s \\ &= \max_{\{w_\tau, C_\tau\}_{\tau \geq t}} \left\{ \mathbb{E}_t \left[\sum\nolimits_{\tau \geq 0} {}_{\tau}p_{x,t} \cdot \beta^\tau \cdot \tfrac{(C_{t+\tau})^{1-\gamma}}{1-\gamma} \right] \right\}, \quad \text{if } t > s. \end{aligned} \tag{1}$$

At or before time s, the individual maximizes his expected lifetime utility with the fraction of liquid wealth invested in equity, the consumption, and the fraction of wealth invested in annuities at time s as control variables. After time s the individual does not purchase new annuities, and hence the control variables are only the sequence of current and future fractions of liquid wealth invested in equities, and the yearly consumption

[7]The exogenous state variables depend on the evolution of the future survival probabilities up to time t. The evolution of the survival probabilities is described in Section 2.2. The evolution of an individual's information about the distribution of the future survival probabilities is described in Appendix A.

levels.

The wealth dynamics of the individual, for all $\tau \geq 0$, are given by:

$$W_{\tau+1} = \begin{cases} (W_\tau - C_\tau) \cdot (1 - a_s(d)) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau = s, \\ (W_\tau + A_\tau - C_\tau) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau \neq s, \end{cases} \tag{2}$$

where r^{rf} is the time-independent risk-free return, and r_τ is the (risky) return on equity between year τ and $\tau+1$. The first equation corresponds to the wealth dynamics in the year in which the individual purchases an annuity and the second corresponds to the wealth dynamics in the years in which the individual does not purchases an annuity. The individual faces a sequence of short-selling constraints and liquidity constraints. These constraints imply that an individual cannot borrow against future income. Hence, the objective function for the individual, as represented in (1), is maximized subject to the wealth dynamics in (2) and the following constraints:

$$0 \leq a_s(d) \leq 1, \tag{3}$$

$$0 \leq w_\tau \leq 1, \qquad \text{for } \tau \geq 0, \tag{4}$$

$$C_\tau \leq W_\tau + A_\tau, \qquad \text{for } \tau \geq 0. \tag{5}$$

Equations (3)–(4) correspond to the no short-selling constraints, and equation (5) implies that the individual cannot borrow against future income. Given that the individual purchases an annuity at time s with a deferral period of d years, the annuity income level is, by definition, given by:

$$A_{t+1} = \begin{cases} A_t, & \text{if } t \neq s + d - 1, \\ B_t, & \text{if } t = s + d - 1, d > 1, \\ \frac{a_s(1) \cdot (W_s - C_s)}{V_s\left(A^{(1)}_{x+s,s}\right)}, & \text{if } t = s, d = 1, \end{cases} \tag{6}$$

with $A_0 = 0$, $V_s\left(A^{(1)}_{x+s,s}\right)$ the time-s price of an immediate annuity, and

$$B_{t+1} = \begin{cases} B_t, & \text{if } t \neq s, \\ \frac{a_s(d) \cdot (W_s - C_s)}{V_s\left(A^{(d)}_{x+s,s}\right)}, & \text{if } t = s, \end{cases}$$

with $B_0 = 0$, and $V_s(A^{(d)}_{x+s,s})$ the time-s price of an annuity with a deferral period of d years. The state variable B_t does not play a role when $d = 1$.

To obtain the optimal consumption and investment choices we use a simulation-based method which can deal with many exogenous state variables, proposed by Brandt, Goyal, Santa-Clara, and Stroud (2005) and by Carroll (2006). In addition, we include several extensions which were proposed by Koijen, Nijman, and Werker (2009). In Appendix A the method used to obtain the optimal consumption and investment choices is described.

2.2 Survival probabilities

As can be observed from equation (1) the optimal life-cycle choices of an individual depend on future survival probabilities of the individual. In this paper future survival probabilities are stochastic. In this section we describe the modeling of the probability distribution of future survival probabilities. We use the model proposed in Cairns, Blake, and Dowd (2006). This CBD model is attractive because it uses only a few parameters to obtain a good fit of the mortality probabilities and it has already been extended to include a market price of systematic longevity risk in an empirically justified method. In the CBD model the mortality curve is a special case of the Perks models (see, for example, Perks, 1932, and Benjamin and Pollard, 1993). Let $q_{x,t}$ be the time-t one-year mortality probability for the cohort aged x at time t. In the CBD model the logit of the one-year mortality probability is modeled as:

$$\log\left(\frac{q_{x,t}}{1-q_{x,t}}\right) = k^{(1)}_t + x \cdot k^{(2)}_t + \epsilon_{x,t}, \tag{7}$$

where $k^{(i)} = \left[k^{(i)}_{\underline{t}}, k^{(i)}_{\underline{t}+1}, \dots\right]'$ for $i \in \{1, 2\}$ are stochastic processes with $\underline{t}$ being the first year of mortality data, and $\epsilon_{x,t}$ an the age- and time-specific idiosyncratic residual assumed to be independent and identically distributed (i.i.d.) normally distributed with zero mean and age-specific variance.

We estimate the process for mortality probabilities using mortality data. Let $\overline{t}$ be the last year of the mortality data. Then, in order to project the future mortality probabilities the individual needs the future values of the stochastic processes $k_t =$

$\left[k_t^{(1)}\ k_t^{(2)}\right]'$, for $t > \bar{t}$. Following Cairns, Blake, and Dowd (2006) we assume that the individual forecasts these stochastic processes by a two-dimensional random walk with drift:

$$k_{t+1} = k_t + \mu + C \cdot N_t, \tag{8}$$

where μ is a constant 2×1 vector, C is a constant 2×2 upper triangular matrix, and N_t is a two-dimensional standard Gaussian process. For longevity risk it is common to include not only process risk, i.e., the risk arising from the random process N_t, but also parameter risk (see, for example, Cairns, Blake, and Dowd, 2006). There is a consensus in the literature that the exclusion of parameter uncertainty, given a specification like (7)–(8), would lead to a significant underestimation of longevity uncertainty.[8] Let $D_t = k_t - k_{t-1}$, and $D = [D_{\underline{t}+1}, \ldots, D_{\underline{t}+n}]'$ with $n = \bar{t} - \underline{t}$. To incorporate parameter risk in the parameters μ and $V = C \cdot C'$ we use the Jeffrey's prior as in the CBD model, a non-informative prior distribution, which is a common prior for the multivariate Gaussian distribution in which both μ and V are unknown:

$$p(\mu, V) \propto |V|^{-3/2},$$

where $|V|$ is the determinant of the covariance matrix V. The posterior distribution for $(\mu, V|D)$ at time τ satisfies:

$$V^{-1}|D \sim \text{Wishart}\left(\tau - 1, \tau^{-1}\widehat{V}_\tau^{-1}\right), \tag{9}$$

$$\mu|V, D \sim MVN\left(\widehat{\mu}_\tau, \tau^{-1}V\right), \tag{10}$$

where $\widehat{\mu}_\tau$ and $\widehat{V}_\tau$ are the maximum likelihood estimates of the parameters of the stochastic process based on the revealed information up to time τ. At any time $\tau > \underline{t}$ these

[8] Likewise, the investor might allow for parameter uncertainty in the equity process. Since our focus in this paper is on the effect of systematic longevity risk, we assume that the investor only accounts for process risk in the equity process.

parameters are estimated by:

$$\widehat{\mu}_\tau = \frac{1}{\tau - \underline{t}} \cdot \sum_{t=\underline{t}+1}^{\tau} D_t, \tag{11}$$

$$\widehat{V}_\tau = \frac{1}{\tau - \underline{t}} \cdot \sum_{t=\underline{t}+1}^{\tau} (D_t - \widehat{\mu}_\tau) \cdot (D_t - \widehat{\mu}_\tau)'. \tag{12}$$

The one-year survival probabilities are given by:

$$p_{x,t} = 1 - q_{x,t},$$

where $p_{x,t}$ denotes the probability that an x-year-old at time-t will survive at least another year. Let ${}_\tau p_{x,t} \equiv p_{x,t} \cdot p_{x+1,t+1} \cdots p_{x+\tau-1,t+\tau-1}$ denote the time-t probability that an x-year-old at date-t will survive at least another τ years.

2.3 Annuity prices

The price of an annuity depends on the probability distribution of an individual's remaining lifetime. The actuarially fair price (i.e., the expected discounted cash flows) of an annuity can be determined using the probability distribution of the stochastic future survival probabilities which is described in Section 2.2. The actuarially fair value of an annuity does not incorporate the effect of systematic longevity risk on the price the annuity. Idiosyncratic longevity risk is diversifiable (i.e., the risk becomes negligible when the the portfolio size is sufficiently large) and thus will not be priced in an efficient market without arbitrage opportunities. In contrast, systematic longevity risk does not decrease with the portfolio size, and may thus lead to a risk premium.

There is strong empirical evidence that the market price of annuities exceeds the actuarially fair one (see Mitchell, Poterba, Warshawsky, and Brown (1999) for the US market and Frinkelstein and Poterba (2002) for the UK market). To account for this, the existing life-cycle model literature commonly uses a loading factor. Commonly, the loading factor is seen as a transaction cost, (see, among others, Mitchell et al., 1999). However, the actuarial unfairness might be due to the price of systematic longevity risk rather than transaction costs, which is also mentioned in Milevsky and Young (2007). The premium for systematic longevity risk in an annuity might be significant which implies that the actuarial fair value might be a significant underestimation of the market price.[9] More importantly, a loading factor independent of the deferral period of an annuity does not necessarily properly reflect the risk premium for systematic longevity risk.

No liquid market exists yet for systematic longevity risk (see Blake, Cairns, and Dowd, 2008). Therefore, it is difficult to calibrate the market price of systematic longevity risk. The existing literature proposes different approaches to obtain a fair value for annuities when systematic longevity risk exists. These approaches include the utility maximization principle (see Malamund, Trubowitz, and Wüthrich, 2008); the Sharpe ratio approach (see, for example, Milevsky, Promislow, and Young, 2006, 2008; Bayraktar, Milevsky, Promislow, and Young, 2009; and Bauer, Börger, and Ruß, 2009); the Wang transform (see Lin and Cox, 2005; Cox and Lin, 2007; Denuit, Devolder, and Goderniaux, 2007; and Lin and Cox, 2008), and risk-neutral pricing (see Cairns, Blake, and Dowd, 2006). An excellent overview of different pricing methods is given in Bauer, Börger, and Ruß (2009).

In this paper we use the risk-neutral approach to calculate the risk premium for systematic longevity risk. The risk-neutral approach is based on long-established financial economic theory and states that, if the overall market is arbitrage-free, there exists a risk-neutral measure such that the price of an annuity equals the expected discounted payments under the risk-neutral measure. Due to market incompleteness many risk-neutral risk measures might exist. Therefore, we shall assume that an individual is acting in an equilibrium setting, and that this equilibrium selects a market consistent (unique) risk neutral measure. Following the existing literature on life-cycle models we also use, as alternative to the risk-neutral approach, a loading factor for pricing annuities. As loading factor we take 7.3%, which is in line with Mitchell et al. (1999), and commonly used in the life-cycle literature (see, for example, Horneff, Maurer, and Stamos, 2008). Hence, this paper considers annuity market prices that exceed the actuarially fair ones which are modeled using:

[9] For example, the Solvency II project (Group Consultatif Actuariel Europeen, 2008) requires the valuation of annuities using a market-to-model approach. The approach proposed in the Solvency II guidelines leads to a valuation of an annuity which is approximately 6% to 9% higher than the real-world expected discounted cash flows of an annuity for an annuitant aged 65, due to systematic longevity risk (see Olivieri and Pitacco, 2008).

i) *risk-neutral survival probabilities*;

ii) a *loading factor*, which is independent of the deferral period of an annuity.

Especially for deferred annuities the method used to price annuities is important. Using our risk-neutral survival probabilities the actuarial unfairness (measured by the fraction of the price of an annuity due to the risk margin for systematic longevity risk) in deferred annuities is an increasing function of the deferral period. In contrast, using a loading factor the actuarial unfairness is independent of the deferral period. To illustrate that the results are robust for these different types of pricing annuities, we will use both methods (i.e., the risk-neutral pricing method and the loading factor) to price annuities.

Let us now describe the risk-neutral method to obtain the price of an annuity. Recall that ${}_\tau p_{x,t}$ is the probability that an x-year old at time t will survive at least τ years, and r^{rf} is the risk-free return. Let $\mathbb{E}_t^{\mathcal{Q}}[\cdot]$ denote the time-t expectation under the risk-neutral measure. Then the time-t market price of an annuity $V_t(A_{x,t}^{(d)})$ is given by:[10]

$$V_t(A_{x,t}^{(d)}) = \mathbb{E}_t^{\mathcal{Q}}\left[\sum_{\tau\geq d} 1_\tau^{(x,t)} \cdot \left(\frac{1}{1+r^{rf}}\right)^\tau\right] = \sum_{\tau\geq d} \mathbb{E}_t^{\mathcal{Q}}\left[{}_\tau p_{x,t}\right] \cdot \left(\frac{1}{1+r^{rf}}\right)^\tau, \tag{13}$$

where $1_\tau^{(x,t)}$ is an indicator which equals one if the individual with age x at time t is alive at time $t+\tau$, and zero otherwise.

As can be observed from equation (13), to calculate the price of an annuity using the risk-neutral approach we need the risk-adjusted expectation of future survival probabilities. The probability distribution of future survival probabilities is described in Section 2.2. To include the market price of systematic longevity risk we follow the method proposed in Cairns, Blake, and Dowd (2006). In this method the risk-adjusted pricing measure $\mathcal{Q}(\lambda)$ is modeled using an adjustment in the dynamics of the stochastic process k_t. Let $\lambda = [\lambda_1\ \lambda_2]'$ be the vector representing the market price of systematic longevity risk, which is assumed to be time-independent. The dynamics of the process k_t under the real-world measure is described in equation (8). The process $\widetilde{k}_t$ under the

[10] In case of uncertainty in the instantaneous forward rate curve it has been shown that, under the assumption of independence of the instantaneous forward rate and survival probabilities, equation (13) still holds, replacing $\left(\frac{1}{1+r^{rf}}\right)^\tau$ on the right hand by the time-t price of a zero-coupon bond with face value 1 maturing in year $t+\tau$ (see, for example, Cairns, Blake, and Dowd, 2006).

risk-adjusted measure $Q(\lambda)$ is given by:

$$\begin{aligned}\widetilde{k}_{t+1} =& \widetilde{k}_t + \mu + C \cdot (N_t - \lambda), \\ =& \widetilde{k}_t + \widetilde{\mu} + C \cdot N_t, \end{aligned} \tag{14}$$

where $\widetilde{\mu} = \mu - C \cdot \lambda$. Whereas the individual updates the parameters μ and V continuously, we assume that the parameter λ is not updated over time.

3 Parameter calibration

3.1 Financial market

In this section we describe the financial market return processes. We assume that, besides different types of annuities, the financial market consists of a risk-free asset, and a risky asset. The yearly return on the risk-free asset is set at 4% and assumed to be time-independent. Hence, we have that $r^{rf} = 0.04$. Define S_t as the time-t stock price, assuming that there are no dividends.[11] Then $r_t \equiv \frac{S_{t+1}}{S_t} - 1$ is the yearly equity return between year t and $t+1$. The stock price is modeled as a Brownian motion with drift:

$$dS_t = \mu^S \cdot S_t \cdot dt + \sigma^S \cdot S_t \cdot dZ_t,$$

where $\mu^S \equiv r^{rf} + \lambda^S \cdot \sigma^S$ and σ^S are model parameters, with λ^S the parameter for the market price of equity risk, and Z_t is a standard Brownian motion. Following the life-cycle literature we set λ^S equal to 0.155 and σ^S equal to 0.158, resulting in an expected yearly excess equity return equal to 4% and a standard deviation of the yearly equity return equal to 17% (see, for example, Gomes and Michaelides, 2005; Yao and Zang, 2005; and Cocco, Gomes, and Meanhout, 2005). The equity risk premium of 4% is lower than the historical one, which is very common in this literature. The lower-than-historical return is an adjustment in order to take transaction costs into account, most of which are in the form of mutual fund fees. Due to the high dimensionality of modeling the transaction costs explicitly (as is, for example, done in Heaton and Lucas, 1996) it

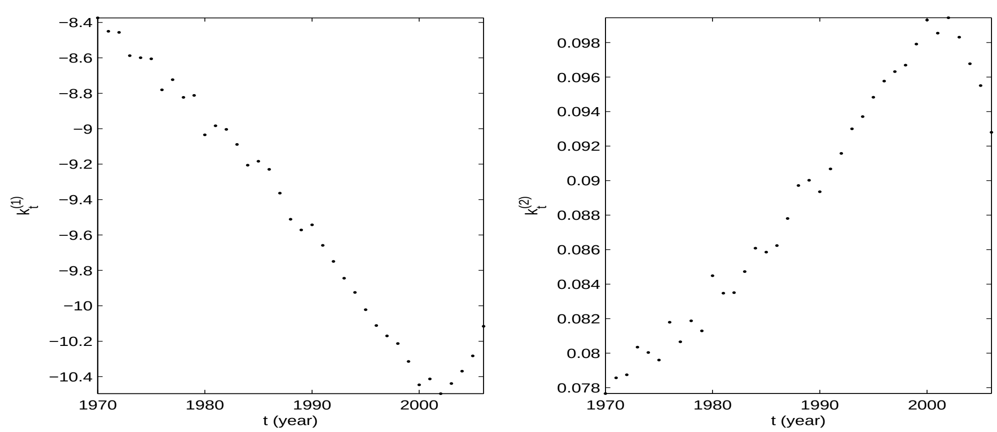
[11] Note that when there are dividends, the return dynamics are not affected when the dividends are reinvested in equities.

is common to use this shortcut representation of a lower expected return to take into account transaction costs. In Section 6 as a robustness check we set λ^S equal to 0.343 resulting in an expected yearly equity excess return equal to 7%. This leads to an equity risk premium which is in line with historical results (see, for example, Brennan and Xia, 2002).

3.2 Systematic longevity risk

To estimate the parameter values of the stochastic processes $k^{(1)}$ and $k^{(2)}$ we use age-, gender-, and time-specific mortality probabilities for the United States, obtained from the Human Mortality Database.[12] Figure 1 displays the parameter values of the stochastic process for males in the US from $\underline{t} = 1970$ to $\bar{t} = 2006$, by fitting (7). The value of $k_t^{(1)}$ displays the time-t general level of mortality. The general decrease over time in the level of the stochastic process $k_t^{(1)}$ implies that there is generally a decrease in the level of the mortality probabilities over time. The general increase over time in the value of $k_t^{(2)}$ implies that the mortality reduction is generally lower at higher ages.

Figure 1: Estimated parameter values of the stochastic processes k.



This figure displays the estimated parameter values of the stochastic processes $k^{(1)}$ and $k^{(2)}$. The left panel displays the estimated parameter values of the stochastic processes $k^{(1)}$, the right panel displays the estimated parameter values of the stochastic processes $k^{(2)}$. The stochastic processes are estimated using US male mortality data from 1970 to 2006.

Using US male mortality probabilities from 1970 to 2006 we obtain the estimates

[12]Available from the Human Mortality Database: www.mortality.org.

of the parameters in equations (11) and (12):

$$\widehat{\mu}_{\overline{t}} = \begin{bmatrix} -0.048383 \\ 0.00042065 \end{bmatrix}$$

$$\widehat{V}_{\overline{t}} = \widehat{C}_{\overline{t}} \cdot \widehat{C}'_{\overline{t}} = \begin{bmatrix} 0.0069237 & -0.00010012 \\ -0.00010012 & 1.4765 \cdot 10^{-6} \end{bmatrix},$$

where $\widehat{C}_{\overline{t}}$ can be recovered from a Choleski decomposition of $\widehat{V}_{\overline{t}}$.

Recall that the base year is set equal to 0. To forecast the distribution of the future mortality probabilities we take as the starting value of k_0 the estimate $\widehat{k}_{\overline{t}}$ corresponding to $\overline{t} = 2006$, i.e., $k_0 = \widehat{k}_{\overline{t}} = [-10.1157\, 0.092799]'$. The mortality probabilities are forecasted by simulating the parameters μ and V for each path. Let MA be the maximum attainable age, which is set at 110 years. Then, given the simulated parameters μ and V, we simulate the paths of k_t for each future time period, i.e., for $t = 1, \ldots, MA - x$, given the assumed values of k_0. The distribution of future mortality probabilities is obtained by the simulated values of the stochastic processes k_t and the simulated residuals $\{\epsilon_{x,t} | x \in \{x, \ldots, MA\}, t \in \{1, \ldots, MA - x\}\}$. Each such path gives us a path of the future mortality probabilities.

3.3 Pricing systematic longevity risk

In Section 2.3 we mentioned that the market price of an annuity exceeds the actuarially fair one, which might be due to systematic longevity risk. Since there is no liquid market for systematic longevity risk it is difficult to calibrate the risk-neutral survival probabilities using empirical data. Although little information is available, it is reasonable to assume that systematic longevity risk leads to a positive risk premium.[13] This implies that the physical expectation of the discounted cash flows is lower than the risk-adjusted expectation.

The risk adjusted process for survival probabilities, as defined in equation (14),

[13]There exists also natural counterparties of systematic longevity risk in annuities, for example death benefit providers. However, it is still likely that there is a positive risk premium, because short (for example annuities) exposure is 30-40 larger than long (for example death benefits) exposure (source: American Council of Life Insurers, "U.S. Life Insurance Moodys Statistical Handbook", August 2006; "Pension Markets in Focus", OECD, October 2006; and Moody's U.K. Life Insurance Industry Outlook, January 2007).

depends on the parameter λ. Cairns, Blake, and Dowd (2006) calibrated as parameter value $\lambda = [0.175\ 0.175]'$, using the EIB/BNP longevity bond, which was announced in November 2004.[14] We will use this calibrated value of λ for pricing annuities.[15] Using the calibrated parameter $\lambda = [0.175\ 0.175]'$ in the risk-neutral pricing we obtain a market price for an immediate annuity which is 7.35% higher than the actuarially fair one. Using the empirical prices of immediate annuities Mitchell et al. (1999) found a loading factor of 7.3%.[16] This might imply that using only the risk-adjusted process $Q(\lambda)$ with $\lambda = [0.175\ 0.175]'$, one could explain the price observed in the real-world by only using the risk-neutral pricing method for systematic longevity risk.

One might argue that the price of systematic longevity risk using the calibrated parameter from Cairns, Blake, and Dowd (2006) (i.e., $\lambda = [0.175\ 0.175]'$) might be an overestimation of the real one, since the longevity bond was withdrawn prior to issue. As an alternative, we assume that the individual postulates a uniform distribution on $\lambda = [\lambda_1\ \lambda_2]'$, i.e., $\lambda_1 = \lambda_2 \sim U(0, 0.175)$. Note that this stochastic λ leads to a lower risk premium for systematic longevity risk than the calibrated $\lambda = [0.175\ 0.175]'$.

In summary, for the pricing of annuities we distinguish three cases:

i) the calibrated parameter on the EIB/BNP longevity bond, i.e., $\lambda = [0.175\ 0.175]'$;

ii) a stochastic λ, with $\lambda_1 = \lambda_2 \sim U(0, 0.175)$;

iii) a loading factor, which is set equal to 7.3%, irrespective of the deferral period.

The last case is included in order to compare our results with the existing literature on life-cycle models. As we will show in the following section, for deferred annuities the risk-neutral pricing method implies that the risk premium as fraction of he actuarially

[14] The EIB/BNP longevity bond was withdrawn prior to issue. One of the reasons why this issue was unsuccessful might be the price of the longevity bond indicating that this calibrated λ overestimates the price of systematic longevity risk. However, there are several design issues (see Blake, Cairns, and Dowd (2008) for an extensive investigation of the failure of this longevity bond) which might explain why the bond was withdrawn prior issue.

[15] The longevity bond was based on publicly available Office for National Statistics (ONS) data on English and Welsh mortality for a cohort of males aged 65 in 2003. Tuljapurkar, Nan, and Boe (2000), among others, have shown that the mortality development in western countries has similar patterns. This indicates that that the driving forces for the decline in mortality may be the same in western countries, which implies that the price of longevity risk would be similar for western countries. Hence, it might indicate that the market price of risk (λ) is approximately the same in western countries.

[16] This holds for immediate annuities. Since we do not have information on the loading factor for deferred annuities we do not know whether this also holds for deferred annuities.

fair price of an annuity premium is increasing in the deferral period. By determining the optimal choices in the life-cycle model in a setting where annuities are priced using a constant loading factor we show that our results also hold when the actuarial unfairness in annuities is independent of the deferral period of a deferred annuity.

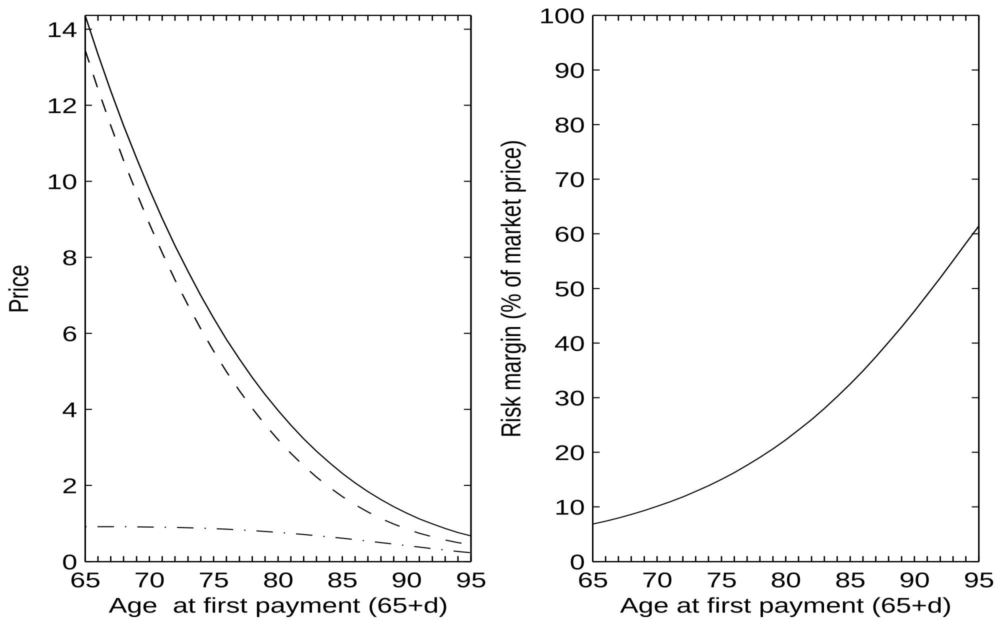
4 The effect of longevity risk on annuity prices

This paper extends the existing literature on life-cycle models in two ways, namely by including systematic longevity risk and by allowing an individual to purchase a deferred annuity. Clearly, the optimal annuity decision also depends on the current and future prices of the different annuities. Therefore, before investigating the optimal annuity decision in a utility framework setting in Section 5, we investigate in this section the effect of systematic longevity risk on the price of immediate and deferred annuities.

Systematic longevity risk affects the price of annuities in two ways. First, systematic longevity risk leads to a risk premium in the price of annuities. Second, it leads to uncertain future survival probabilities and thus to stochastic future annuity prices. We determine the current price of deferred annuities and the probability distribution of the price of immediate annuities purchased at time $s > 0$, using the model to forecast the distribution of future survival probabilities. For illustrative purposes we consider a 65-year-old male at time $t = 0$ and set $\lambda = [0.175\,0.175]'$ (see Section 3.3).

Figure 2 displays the effect of systematic longevity risk on the current price of deferred annuities, as a function of the deferral period. The left panel displays the price as a function of the deferral period (solid curve, the expectation under the $\mathcal{Q}$-measure), the actuarially fair price (dashed curve, the expectation under the $\mathcal{P}$-measure), and the risk premium for the systematic longevity risk of the annuity (dashed-dotted curve, the difference in the expectation under the $\mathcal{Q}$- and the $\mathcal{P}$-measure). The right panel displays the fraction of the price of the annuity which is due to the risk premium, i.e., the risk premium as fraction of the price, as a function of the deferral period.

Figure 2: Price of deferred annuities.



The left panel of this figure displays components of the price of a deferred annuity as a function of the deferral period. The solid curve corresponds to the market price of a deferred annuity; the dashed curve to the actuarially fair price, i.e., the expected discounted cash flows; and the dashed-dotted curve to the risk premium for systematic longevity risk. The right panel of this figure displays the risk premium for systematic longevity risk as a percentage of the price of a deferred annuity.

From Figure 2 we observe that, as expected, both the price of an annuity and the risk premium are decreasing functions of the deferral period. This occurs because a longer deferral period reduces the (expected) number of payments to be made. The decline in the risk premium for systematic longevity risk is small for short deferral periods, and large for long deferral periods.[17] This occurs because systematic longevity risk for payments to be made for short durations is much smaller than it is for long durations. Finally, we observe that the risk premium as fraction of the price of an annuity is an increasing function of the deferral period. The uncertainty in the survival

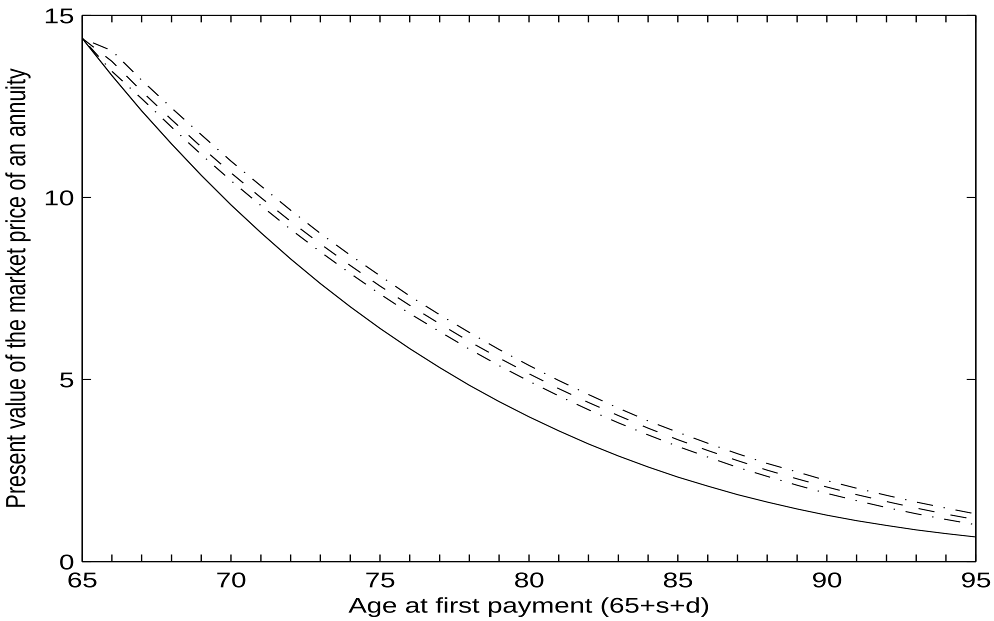
[17]The risk premium for a deferred annuity with a long deferral period is high. This occurs because there is a large amount of uncertainty in the probability of surviving until advantaged ages conditional on being alive at the age of 65. In addition, the risk premium is high due to the skewness of the distribution of the probability of surviving.

probabilities is greater for survival probabilities farther in the future, which leads to (relatively) higher risk premiums for payments which have a longer maturity.

Instead of purchasing a deferred annuity the individual can also postpone the purchase of an immediate annuity. The income stream of a deferred annuity with deferral period d can be mimicked by the following strategy: when the individual is alive at time $d-1$ he purchases an immediate annuity (with $d=1$), and when the individual is not alive at time $d-1$ he does not purchase any annuities. Although the income stream of a deferred annuity can be mimicked using an immediate annuity, when currently purchasing a deferred annuity the price is known, whereas the future price for an immediate annuity is currently stochastic. As new mortality information becomes available, it can be consulted before setting the annuity prices. Therefore, when the annuity purchase is postponed there is generally less uncertainty about the development of future survival probabilities. This reduces the risk premium for longevity risk for an annuity, thus making it more attractive. However, a disadvantage of postponing the purchase of the annuity is that there is currently uncertainty about what the future price of an annuity will be.

Figure 3 displays the median (dashed curve) and 95% confidence intervals (dashed-dotted curves) of the discounted price of an immediate annuity at date s, i.e., $V_s(A^{(1)}_{s,65+s}) \cdot \left(\frac{1}{1+r^{rf}}\right)^s$, as a function of the postponement period s. For comparison, the current price of a deferred annuity with a nominal yearly payment of one, as a function of the deferral period (solid curve) is also displayed in Figure 3.

Figure 3: Present value of annuity prices.



This figure displays selected quantiles of the present value of the date-s price of an immediate annuity and the price of a deferred annuity as a function of the age at which the initial payment is made. The solid curve corresponds to the current market price of a deferred annuity; the dashed curve corresponds to the present value of the median market price of an immediate annuity purchased at time s; and the dashed-dotted curves correspond to the present value of the 95% confidence bounds of the market price of an immediate annuity purchased at time s.

From Figure 3 we observe that it is generally cheaper to currently purchase a deferred annuity than to postpone the purchase of an immediate annuity. Currently purchasing a deferred annuity instead of postponing the purchase of an immediate annuity has the advantage that some of the buyers will not survive until the payoff phase. This can be observed from the following relation between the deferred annuity price and immediate annuity price:

$$V_t\left(A_{x,t}^{(d)}\right) = \mathbb{E}_t^{\mathcal{Q}}\left[\left(\frac{1}{1+r^{rf}}\right)^{d-1} \cdot \left({}_{d-1}p_{x,t} \cdot V_{t+d-1}\left(A_{x+d-1,t+d-1}^{(1)}\right) + (1 - {}_{d-1}p_{x,t}) \cdot 0\right)\right].$$

The effect that part of the buyers of a deferred annuity are not alive at time $d-1$

dominates the effect of a risk premium for systematic longevity risk which is generally lower when the moment of purchase is postponed. Note that, even for an individual without any bequest motive, it might still improve utility to postpone the purchase of an annuity instead of currently purchasing a deferred annuity, since the money invested in deferred annuities cannot be invested in equities, which reduces the capital gains from the equity risk premium.

Let us finally discuss the effect of systematic longevity risk on the attractiveness of annuities. The attractiveness of annuities as an investment opportunity is due to pooling: i.e., the individuals who live longer than expected are subsidized by those who do not. This reallocation of the contributions of those who die to those who survive, is referred to as the mortality credit advantage, see, for example, Milevsky (1998), Milevsky and Young (2002), and Horneff, Maurer, and Stamos (2008). The mortality credit (MC) is defined as the return from currently purchasing an annuity and selling it (at market price) the following year in excess of the risk-free return. This is equivalent to the excess return of purchasing an annuity instead of postponing its purchase to the succeeding year. In a setting without systematic longevity risk the mortality credit is defined as:

$$MC\left(t,d,x\right) = \frac{V_{t+1}\left(A^{(\max\{0,d-1\})}_{t+1,x+1}\right) + 1_{d=0}}{V_t(A^{(d)}_{t,x})} - \left(1+r^{rf}\right) \tag{15}$$

$$= \frac{1+r^{rf}}{{}_1p_{x,t}} - \left(1+r^{rf}\right), \tag{16}$$

where $1_{d=0}$ is an indicator function which equals one if $d = 0$, and zero otherwise, and ${}_1p_{x,t}$ is the deterministic one-year survival probability. Because ${}_1p_{x,t}$ is between zero and one, the mortality credit is always positive. This occurs due to the risk-sharing principle, i.e., in the following year the annuitized wealth is re-allocated to the survivors.

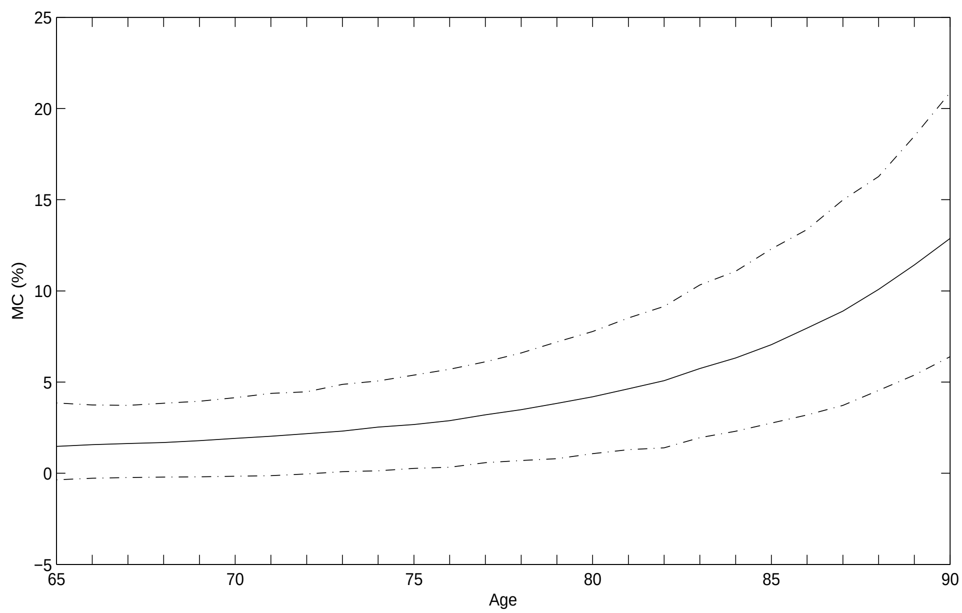
In a setting with systematic longevity risk the mortality credit from (15) equals:

$$MC\left(t,d,x\right) = \frac{\sum_{s\geq\max\{d-1,0\}} \mathbb{E}^{\mathbb{Q}}_{t+1}\left[{}_sp_{x+1,t+1}\right]\cdot\left(1+r^{rf}\right)^{-s} + 1_{d=0}}{\sum_{s\geq d}\mathbb{E}^{\mathbb{Q}}_{t}\left[{}_sp_{x,t}\right]\cdot\left(1+r^{rf}\right)^{-s}} - \left(1+r^{rf}\right). \tag{17}$$

Compared to the mortality credit in a setting with deterministic mortality probabilities, the mortality credit in a setting with stochastic mortality probabilities differs in two ways. First, from equation (17) we observe that the mortality credit is stochastic instead of deterministic, because $V_{t+1}\left(A_{t+1,x+1}^{(\max\{0,d-1\})}\right)$ depends on the evolution of the mortality probabilities until time $t+1$. Second, the mortality credit is dependent on the deferral period d in a setting with systematic longevity risk, whereas it is independent of the deferral period in a setting without systematic longevity risk. This is due to the fact that the price of an annuity in the following year depends on the change in the distribution of future survival probabilities due to revealed mortality information between time t and time $t+1$. This change may be different for different ages, resulting in various changes to the risk-adjusted expected discounted cash flows of the different annuity payments.

Figure 4 displays selected quantiles of the distribution of the mortality credit as a function of the age of the individual, for immediate annuities (i.e., $d = 1$).

Figure 4: Mortality credit for immediate annuities.



This figure displays selected quantiles of the distribution of the mortality credit for immediate annuities under the real-world measure, as a function of the age of the individual. The solid curve corresponds to the median mortality credit for immediate annuities under the real-world measure and the dashed curves correspond to the 95% confidence bounds of the mortality credit for immediate annuities under the real-world measure.

From Figure 4 we observe that in a setting with systematic longevity risk, in contrast to a setting without systematic longevity risk, the mortality credit can be negative. A negative mortality credit implies that it is cheaper for the individual to currently invest in the risk-free asset and purchase an immediate annuity in the following year than to currently purchase an immediate annuity. This can occur due to systematic longevity risk, which might lead to a change in the distribution of future survival probabilities when new mortality information is revealed. Note that from age 72, the effect of the positive mortality probability is generally larger than the effect of the new information on mortality probabilities, leading to a positive mortality credit with a probability of more than 97.5%. Moreover, recall that the expected excess return of equity is set at 4%, which is lower than the median of the mortality credit for the ages above 80. As discussed in Milevsky and Young (2002) in a setting without systematic longevity risk, when the mortality credit is higher than the equity risk premium it is optimal for an individual to annuitize all his wealth, because annuities yields a higher expected return with less uncertainty.

5 Optimal life-cycle choices

In this section we quantify the effect of the choice of the deferral period and the time at which an annuity is purchased on an individual's expected lifetime utility. Both the price and the payoff stream of an annuity influence the expected lifetime utility. Recall from Section 2.1 that the individual is a rational expected lifetime utility maximizer with a CRRA utility function. The individual's preference parameters of the CRRA utility are set equal to values used in the life-cycle literature (see, for example, Gomes and Michaelides, 2005): $\gamma = 5$ and $\beta = 0.96$. The individual is a male currently aged 65, who faces longevity risk and, when he invests in equities he faces investment risk. We assume that the individual invests only once in an annuity and only in one type, i.e., either an immediate annuity ($d = 1$, with $s \geq 0$), or a deferred annuity with a fixed deferral period d (with $s = 0$).

We quantify the attractiveness of the different types of annuities by the certainty equivalent consumption. The certainty equivalent consumption is the yearly consumption level CEC for which the utility of this consumption stream equals the utility given the optimal choices conditional on the type of annuity, $J_0\left(x, W_0, 0, 0, X_0\right)$. Hence, the certainty equivalent consumption is determined by the following equation:

$$\mathbb{E}_0\left[\sum_{\tau\geq 0} {}_\tau p_{x,0}\cdot\beta^\tau\cdot\frac{(CEC)^{1-\gamma}}{1-\gamma}\right]=J_0\left(x,W_0,0,0,X_0\right), \tag{18}$$

with $J_0\left(x,W_0,0,0\right)$ as defined in (1). In our results we compare the utility (quantified by the certainty equivalent consumption) obtained by the optimal consumption and investment choices to the utility of the constant consumption level that arises from currently investing all after-consumption wealth in annuities. We refer to this investment strategy as the *fully annuitized (fa) strategy*. The corresponding certainty equivalent consumption in this strategy equals $CEC_{fa}=W_0/\left(1+V_0\left(A_0^{(1)}\right)\right)$, where the denominator equals the "price" of a yearly consumption of one, i.e., the current consumption plus the price of an immediate annuity.

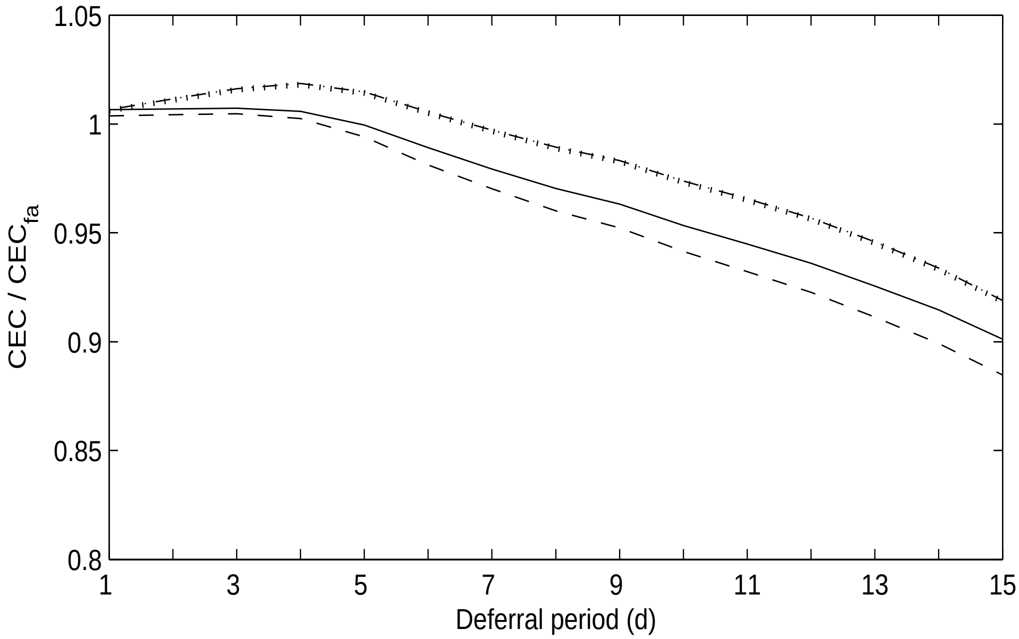
In Section 5.1 we investigate the optimal fraction of wealth invested in a deferred annuity and the corresponding certainty equivalent consumption conditional on the immediate purchase of a deferred annuity, i.e., $s=0$, for different deferral periods. In Section 5.2 we investigate the optimal fraction of wealth invested in an immediate annuity and the corresponding certainty equivalent consumption conditional on postponing the purchase of an immediate annuity, i.e., $d=1$. We assume that the individual purchases an immediate annuity only once.

5.1 Purchasing a deferred annuity at retirement date

In this section we investigate the effect of the deferral period on the expected lifetime utility of an individual, conditional on currently ($s=0$) purchasing a deferred annuity. We maximize the individual's expected lifetime utility as given in (1) given constraints (2)–(5) for annuities with $d=1,\ldots,15$ (i.e., different deferral periods, including an immediate annuity), respectively and $s=0$ (i.e., immediately purchasing an annuity).

First, let us investigate the effect of the deferral period on an individual's expected lifetime utility. Figure 5 displays the certainty equivalent consumption relative to the certainty equivalent consumption in the fully annuitized strategy, as a function of the deferral period. The figure also illustrates the effect of the pricing method of annuities (i.e., using a risk-neutral approach or using a constant loading factor of 7.3%) on the optimal decision.

Figure 5: Certainty equivalent consumption conditional on purchasing a deferred annuity at retirement date.



This figure displays the certainty equivalent consumption relative to the fully annuitized strategy as a function of the deferral period of the annuities. The solid curve corresponds to the certainty equivalent consumption when annuities are priced using risk-neutral pricing with $\lambda = [0.175\ 0.175]'$. The dashed curve corresponds to the certainty equivalent consumption when annuities are priced using the stochastic λ. The dashed-dotted curve corresponds to the certainty equivalent consumption when annuities are priced using a loading factor of 7.3%, irrespective of the deferral period. The dotted curve corresponds to a setting without systematic longevity risk with a loading factor of 7.3%.

In Figure 5 we observe the following:

i) the optimal annuity is a deferred annuity with a short deferral period;

ii) the effect of systematic longevity risk on the utility gain (or loss) of a longer deferral period is negligibly small.

As expected, we observe that when currently purchasing an immediate annuity (i.e., $d = 1$) the certainty equivalent consumption relative to the fully annuitized strategy is greater than one. This indicates that the investment and consumption choices in the fully annuitized strategy are not the optimal ones. The utility gain, relative to the fully annuitized strategy, is similar in a setting with and without systematic longevity risk, using a constant loading factor for annuity prices. When the annuities are priced using the risk-neutral survival probabilities, the utility gain obtained by purchasing an annuity with the optimal deferral period instead of purchasing an immediate annuity is negligibly small. We also observe that the optimal deferral period is short: only three years when risk-neutral pricing is used (for both $\lambda = [0.175\ 0.175]'$ and the stochastic λ) and four years when a loading factor of 7.3% is used. An increase in the deferral period has two effects, namely:

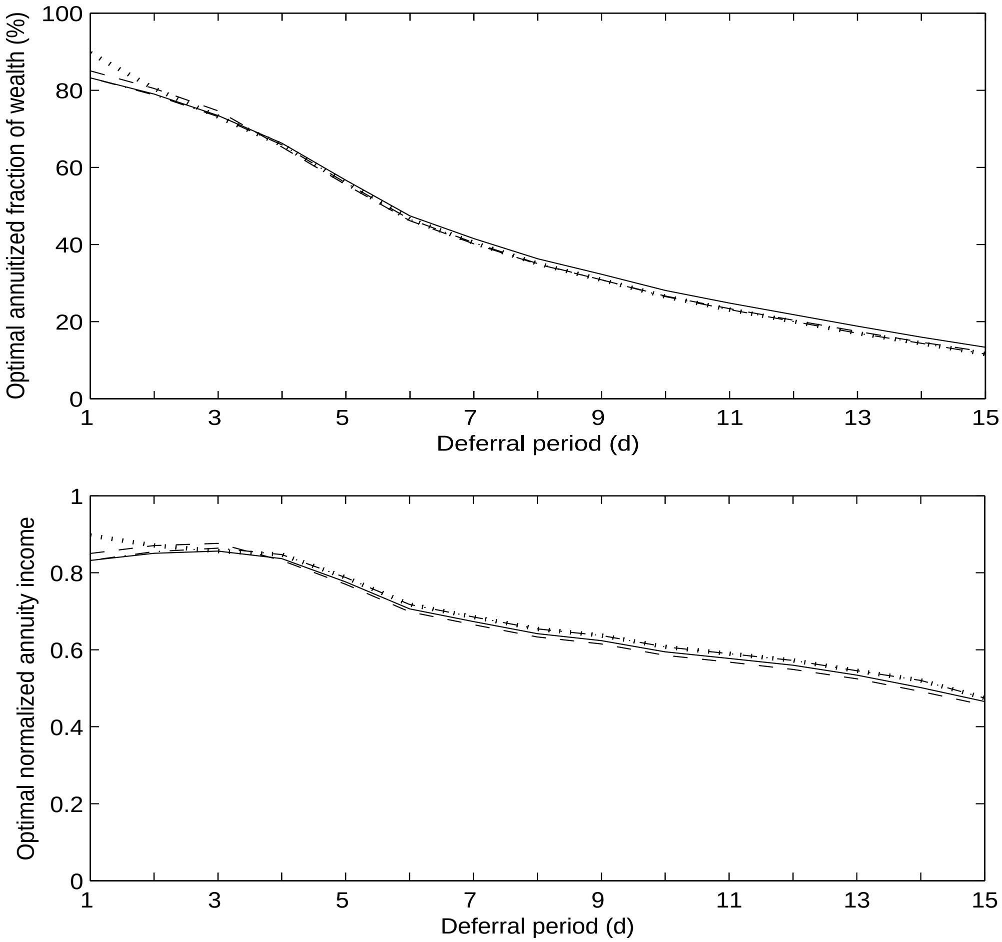
i) A longer deferral period leads to cheaper annuities. Because annuities with a longer deferral period are cheaper the individual can invest more in other assets, i.e., in the risk-free asset or in equities, and/or purchase an annuity with a higher income stream. When the individual chooses to invest a lower proportion of initial wealth in an annuity, a higher fraction of initial wealth may be invested in equities, leading to a higher equity risk premium. When the individual does not choose to invest less in a deferred annuity, a higher deferral period leads to a higher income level in the payoff phase.

ii) A longer deferral period leads to fewer periods with an income guarantee. Hence, there are more periods with greater uncertainty in the consumption level. This uncertainty reduces an individual's expected lifetime utility.

We observe that for deferred annuities with a short deferral period, the first effect dominates. However, for longer deferral periods, the second effect dominates, resulting in a short optimal deferral period.

The attractiveness of an annuity with a longer deferral period is that it is cheaper. As mentioned previously, the reduction in the price of annuities can be used to either increase the income level after the deferral period and/or to increase the fraction of wealth invested in equities or in the risk-free asset. To illustrate this, the upper panel of Figure 6 displays the optimal fraction of initial wealth which is invested in a deferred annuity as a function of the deferral period. The lower panel of Figure 6 displays the optimal level of normalized (i.e., relative to the fully annuitized strategy) yearly annuity income after the deferral period as a function of the fixed deferral period.

Figure 6: Optimal annuity decision.



This figure displays the optimal annuity decision as a function of the deferral period of the currently purchased annuity. The upper panel displays the optimal fraction of after-consumption wealth which is currently used for the purchase of deferred annuities, i.e., $a_0(d)$. The lower panel displays the optimal annuity income relative to the optimal annuity income in the fully annuitized strategy, i.e., $a_0(d) \cdot V_0\left(A^{(1)}_{65,0}\right)/V_0\left(A^{(d)}_{65,0}\right)$. The solid curve corresponds to annuities priced using the risk-neutral pricing for systematic longevity risk with $\lambda = [0.175\ 0.175]'$. The dashed curve corresponds to annuities priced using the stochastic λ. The dashed-dotted curve corresponds to annuities priced using a loading factor of 7.3%, irrespective of the deferral period. The dotted curve corresponds to a setting without systematic longevity risk and annuities priced using a loading factor of 7.3%, irrespective of the deferral period.

In Figure 6 we observe the following:

i) the optimal fraction of annuitized wealth is decreasing in the deferral period of the annuity;

ii) after a short deferral period the optimal annuity payment in the payout phase is decreasing in the deferral period;

iii) systematic longevity risk reduces the attractiveness of annuities.

In Figure 6 we observe that the fraction of wealth which is invested in a deferred annuity is a decreasing function of the deferral period. However, the annuity income level is an increasing function of the deferral period up to a deferral period of three years. This implies that an increase in the deferral period (for $d \leq 3$) initially leads to less annuitized wealth, but in the payoff phase to more wealth invested in annuities. This leads initially to fewer, but in the payoff phase to more capital gains from the mortality credit. This occurs because initially the optimal annuity income increases in the deferral period. When the deferral period is longer than three years both the optimal fraction of annuitized wealth and the optimal annuity payment level are decreasing functions in the deferral period. This occurs because annuities are illiquid. The illiquidity of an annuity restricts the individual's consumption smoothing. Therefore, when the individual is faced with much lower than expected returns in the financial market, he cannot adjust his annuitized wealth level. This may lead to a substantial reduction in the consumption level until the payoff phase of the deferred annuity; this is not optimal.

Let us finally investigate the effect of systematic longevity risk on the optimal fraction of annuitized wealth. Systematic longevity risk may have two effects when the individual currently purchases a deferred annuity. First, it leads to a risk premium which is an increasing function of the deferral period. Compared to a loading factor which is independent of the deferral period, an increasing loading factor leads to a higher fraction of annuitized wealth. In addition, it makes a longer deferral period less attractive. Second, systematic longevity risk leads to stochastic values of the survival probabilities. The uncertainty in the future survival probabilities makes annuities less attractive. In Figure 6 we observe that excluding systematic longevity risk leads to an increase the optimal the fraction of annuitized wealth when purchasing an immediate annuity from 83.2% to 89.7%.[18] This occurs because systematic longevity risk leads to uncertainty in the utility gain of an annuity; i.e., when future survival probabilities are lower than expected they are less attractive. In the setting with systematic longevity risk the individual adjusts his consumption level to the newly revealed mortality information: he consumes more when the survival probabilities are lower than expected and he saves more when the survival probabilities are higher than expected. In order to be able to adjust his consumption, the individual needs liquid wealth. Although there is a small effect of systematic longevity risk on the fraction of annuitized wealth, the effect of systematic longevity risk on the utility is negligible small. This implies that the utility gain from setting the choices optimal conditional on the newly revealed mortality information is negligible small when the individual purchases deferred annuities. This occurs due to the annuity income, when an individual lives much longer than expected he still has a substantial annuity income at high ages and this reduces the effect of an adjustment in the consumption when new mortality data reveals. Interestingly, the effect of systematic longevity risk on the fraction of annuitized wealth is negligible small after a short deferral period. This occurs because the adjustments of the choices on a change in the distribution of the future survival probabilities are small. In addition, the individual has more liquid wealth and thus is more able to adjust his consumption level as survival probabilities are realized. When the deferral period is larger than three years the difference in the optimal fraction of annuitized wealth with and without systematic longevity risk is negligibly small, due to the substantial amount of liquid wealth. Hence, the exclusion of systematic longevity risk leads to a higher utility for an immediate annuity, but not for a deferred annuity (with a moderate or long deferral period).

Let us now discuss how our findings relate to the existing literature on deferred annuities. For our representative agent it would be optimal to purchase a deferred annuity which starts with an initial payout at the age of 68. Our finding deviates from the literature that argues that deferred annuities with a long deferral period are preferable (see Milevsky, 2005; Dus, Maurer, and Mitchell, 2005; Horneff and Maurer, 2008; and Gong and Webb, 2009). Our findings primarily differ from existing literature since we determine the optimal choices dynamically using the objective function to

[18] For the sake of comparison, we kept the same prices of deferred annuities, using a loading factor of 7.3%.

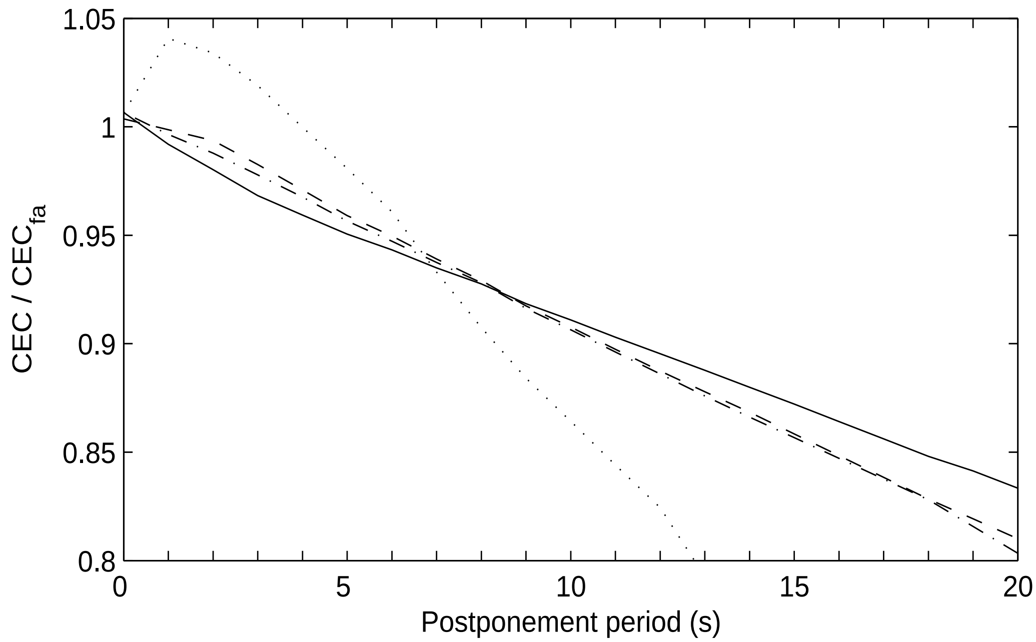
maximize an individual's expected lifetime utility. This differs from existing literature which uses either expected shortfall risk using rules of thumbs for the choices or using an individual's utility where an individual's choices are determined using rules of thumbs. The existing literature argues that a deferred annuity allows for mortality credits when they are high, and allows for liquid wealth to earn the equity risk premium when they are low. We find that the low mortality credit just after the annuity transaction does not offset the utility loss because of the illiquidity of the annuity. The illiquidity of annuities restricts an individual's opportunities to smooth consumption. This affects the expected lifetime utility, especially when the individual is faced with adverse shocks in the equity market before the payoff phase. Therefore, conditional on the purchase of a deferred annuity with a long deferral period, it is not optimal to invest a large proportion of the liquid wealth in an annuity; i.e., the individual needs much more precautionary saving when the deferral period is long. This is in line with the results of Bayraktar and Young (2009). They found that immediate annuities are more preferable than deferred annuities, in a setting where an individual minimizes lifetime ruin probability instead of maximize expected lifetime utility.

5.2 Postponing the purchase of an immediate annuity

An alternative to purchasing a deferred annuity at retirement date is to postpone the purchase of an immediate annuity. The advantage of postponing the annuity purchase is that the individual maintains only liquid assets until the moment the annuity is purchased. This allows the individual to adjust his consumption and annuity income level to the realizations of the financial market, at the moment the annuity is purchased. The disadvantage of postponing the annuity purchase is that the price of the annuity is currently stochastic and that the individual does not receive the mortality credit until the moment of the purchase of an annuity. In this section we maximize the individual's expected lifetime utility as given in (1) given constraints (2)–(5) for an immediate annuity (i.e., $d = 1$) with $s = 0, \ldots, 20$ (i.e., different fixed postponement periods, with $s = 0$ immediately purchasing an annuity).

Figure 7 displays the certainty equivalent consumption conditional on purchasing an immediate annuity at time s, relative to the currently fully annuitized strategy, and as a function of the postponement periods. The individual maximizes his utility by investing in the risk-free asset and in equities, and, at time s, in an annuity.

Figure 7: Certainty equivalent consumption conditional on the postponement period of an immediate annuity.



This figure displays the certainty equivalent consumption relative to the fully annuitized strategy as a function of the postponement period of the purchase of immediate annuities. The solid curve corresponds to the certainty equivalent consumption when annuities are priced using risk-neutral pricing with $\lambda = [0.175\, 0.175]'$. The dashed curve corresponds to the certainty equivalent consumption when annuities are priced using the stochastic λ. The dashed-dotted curve corresponds to the certainty equivalent consumption when annuities are priced using a loading factor of 7.3%, irrespective of the deferral period. The dotted curve corresponds to the certainty equivalent consumption in a setting without systematic longevity risk and annuities are priced using a loading factor of 7.3%, irrespective of the deferral period.

Let us first investigate the effect of postponing the purchase of an immediate annuity on an individual's expected lifetime utility. In Figure 7 we observe that with systematic longevity risk the certainty equivalent consumption is decreasing in the postponement period. The three main effects of postponing the annuity purchase on the expected lifetime utility of the individual are:

i) *Mortality credit.* By postponing the annuity decision the individual does not earn the mortality credit of the annuity until the moment of purchasing the annuity.

ii) *Equity risk premium.* By postponing the annuity decision the individual has until time s only liquid wealth. Therefore, he is less restricted in the fraction of total wealth he invests in equities. This might lead to a higher capital gain from the equity risk premium.

iii) *Conversion rate risk.* Due to stochastic survival probabilities the price of an annuity in the future is stochastic, which results in conversion rate risk.

We observe that the positive effect of postponing the annuity purchase, i.e., the equity risk premium, is smaller than the negative effects, i.e., the missed mortality credit and the conversion rate risk. This occurs because just after retirement, the missed mortality credit is small, but the conversion rate risk is substantial. This conversion rate risk is affected by changes in survival probabilities in two ways. The annuity prices in the future are affected by the difference in realized and expected survival probabilities on the one hand and by a change in the expected trend in the evolution of future survival probabilities on the other hand.

Next, let us investigate the effect of systematic longevity risk on the effect of an individual's expected lifetime utility conditional on postponing the purchase of an immediate annuity. In Figure 7 we observe that the inclusion of systematic longevity risk has two effects, namely:

i) whereas it is optimal to postpone the purchase of an immediate annuity in a setting without systematic longevity risk, this is not the case in a setting with systematic longevity risk;

ii) after a short postponement period the utility loss of increasing the postponement period with an additional year is much larger in a setting without longevity risk than in a setting with systematic longevity risk.

There are two opposite effects of systematic longevity risk on the expected lifetime utility when the purchase of an annuity is postponed. On the one hand systematic longevity risk reduces the attractiveness of an immediate annuity, as discussed in Section 5.1. On the other hand, systematic longevity risk leads to uncertainty in the future prices of annuities which leads to a lower utility when the annuity purchase is postponed in a setting with systematic longevity risk than in one without it. When the postponement period is short, excluding systematic longevity risk and thus a deterministic price of annuities purchased in the future leads to a higher utility when the annuity purchase is postponed. When the postponement period is longer, excluding systematic longevity risk leads to a larger decrease in the certainty equivalent consumption. This is due to the fact that immediate annuities are more attractive in a setting without systematic longevity risk than in a setting with systematic longevity risk.

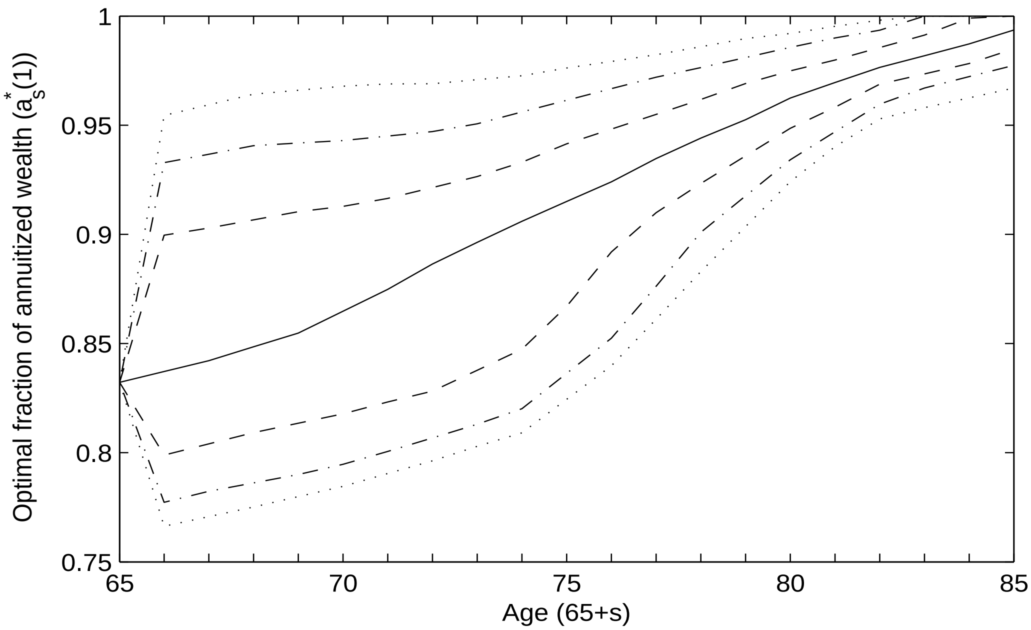
Figure 8 displays selected quantiles of the optimal fraction of annuitized wealth as a function of the postponement period (i.e., as function of s). Note that the individual has CRRA preferences, which implies that the optimal fraction of annuitized wealth is independent of the past equity returns and the wealth level of the individual.[19] This implies that the optimal fraction of wealth annuitized is only affected by the uncertainty in the survival probabilities. First, in Figure 8 we observe that, as expected, the fraction of wealth invested in an immediate annuity generally increases with the length of the postponement period, or equivalently, with the age of the individual. This occurs because the mortality probabilities increase with age which generally leads to a higher mortality credit (see Figure 4).

Second, in Figure 8 we also observe that the uncertainty in the distribution of the optimal fraction of annuitized wealth is large. Therefore, we investigate how the evolution of future survival probabilities affects the fraction of annuitized wealth. To do so, we decompose the total effect in an effect due to changes in the risk premium for systematic longevity risk (which is primarily affected by changes in the uncertainty in the future survival probabilities) and an effect due to changes in the actuarial fair value (which is primarily affected by changes in expected survival probabilities). We calculate the correlation between the optimal fraction of wealth invested in an annuity and the actuarially fair value. We find that they are positively correlated.[20] This

[19] Note that this only holds when the return dynamics of the asset portfolio are modeled in such a way that past returns do not have an influence on current and future returns. In our setting the return process is stationary, since we use a random walk with drift process to model the price of equities.

[20] The correlation between the actuarially fair value and the fraction of annuitized wealth is approximately 0.8 depending on the time of the purchase of an annuity.

Figure 8: The optimal fraction of wealth invested in an annuity conditional on the postponement period.



This figure displays the selected quantiles of the optimal fraction of after-consumption wealth which is invested in annuities as a function of the age of the individual at the moment of purchase of the immediate annuity. The solid curve corresponds to the median; the dashed curves correspond to the 50% confidence bounds; the dashed-dotted curves correspond to the 80% confidence bounds; and the dotted curves correspond to the 90% confidence bounds.

occurs because a change in the expected survival probabilities has a small effect on the attractiveness of an annuity, but a much larger effect on the price of an annuity. We also calculate the correlation between the optimal fraction of wealth invested in an annuity and the risk premium for systematic longevity risk in the annuity, and we find a negative correlation.[21] This occurs because an increase in the risk premium in an annuity leads to a higher cost without additional expected payments, making an annuity less favorable. Interestingly, the effect of a change in the actuarially fair value on the optimal fraction of annuitized wealth is larger than the effect of a change in the risk premium. This occurs because a change in the actuarially fair value is due to a change in the expected survival

[21]The correlation between the risk margin and the fraction of annuitized wealth is between 0 and -0.3, and a decreasing function in the time of the purchase of an annuity.

probabilities. The consequence of higher survival probabilities is a higher actuarially fair value of an annuity on the one hand, and a larger consumption effect at higher ages on the expected lifetime utility on the other hand. Therefore, when the individual expects to live longer due to newly revealed survival information, the optimal fraction of annuitized wealth increases. This is due to the higher price of an annuity and due to a higher weight in the expected lifetime utility function of consumption at advanced ages.

6 Alternative individual characteristics and financial market parameters

The results shown in the previous sections suggest that systematic longevity risk may significantly affect an individual's annuity decision. In this section we show that these results are robust to alternative assumptions in individual characteristics and financial market parameters. In the existing literature (see, for example, Blake, Cairns, and Dowd, 2003; Horneff, Maurer, and Samos, 2008; and Babel and Merrill, 2007) different assumptions on the risk aversion coefficient and the equity risk premium are used. In this section we investigate the robustness of the results for changes in the risk aversion coefficient and in the equity risk premium. In particular, we compute the optimal annuity decisions for a less risk averse individual, i.e., an individual with a risk aversion coefficient of 2 instead of 5. Moreover, we compute the optimal annuity decisions when the expected excess return on equities is 7% (i.e., $\lambda^s = 0.343$) instead of 4% (i.e., $\lambda^s = 0.155$). As discussed in Section 3.1, in the standard life-cycle model literature the excess return is set lower than the empirical one in order to cope with transaction costs.

First, we investigate the robustness of the utility-loss when postponing the annuity purchase. We find that for the different alternative individual characteristics and financial market parameters, i.e., for both a lower value of the risk aversion parameter ($\gamma = 2$, with $\lambda^s = 0.155$) and for a higher equity risk premium ($\lambda^s = 0.343$, with $\gamma = 5$) it is optimal not to postpone the annuity purchase. Either a lower risk aversion ($\gamma = 2$) or a higher equity risk premium ($\lambda^s = 0.343$) does lead to a smaller utility loss when the annuity purchase is postponed than in the case where $\gamma = 5$ and $\lambda^s = 0.155$. In a setting with systematic longevity risk, for the investigated values of the risk aversion and the equity risk premium, we find that it is only optimal to postpone the annuity purchase when both the risk aversion is low and the equity risk premium is high ($\gamma = 2$ and $\lambda^s = 0.343$). Assuming a higher equity risk premium ($\lambda^s = 0.343$), we find that the optimal fraction of wealth currently invested in an immediate annuity is approximately 60%. This is lower than the average observed fraction of annuitized wealth of the current retiring US cohort, taking into account the pre-existing annuitized wealth, such as social security benefits and defined benefit pension plans (see Dushi and Webb, 2004a, using data of the Health and Retirement Survey). Although a high equity risk premium of 7% explains the empirical level of annuitization this equity risk premium seems empirically to be too high (see the discussion in Section 3.1). One possible explanation of the annuity puzzle might be that the individual does not take transaction costs into account. Hence, when the individual would not be rational and expects a too high equity risk premium, this might explain the individual's choice to forgo annuitization.

Next, we investigate the robustness of the optimal deferral period to alternative individual characteristics and financial market parameters. Table 1 displays the optimal deferral period under the different assumptions.

Table 1: Optimal deferral period

γ	λ^s	pricing annuities	$d^\star$	gain in CEC
5	0.155	$\lambda = [0.175\ 0.175]'$	3	0.06%
5	0.155	stochastic λ	3	0.10%
5	0.155	loading factor (7.3%)	4	1.19%
5	0.343	$\lambda = [0.175\ 0.175]'$	3	0.07%
2	0.155	$\lambda = [0.175\ 0.175]'$	5	0.30%

This table displays the optimal deferral period of annuities currently purchased for several alternatives of the parameters in the model. The first column corresponds to the risk aversion parameter; the second column corresponds to the parameter for the expected excess return on equity (i.e., $\lambda^s = 0.155$ if the expected excess return is 4% and $\lambda^s = 0.343$ if the expected excess return is 7%); the third column corresponds to the pricing of annuities (using a loading factor of 7.3% or using risk-neutral survival probabilities); the fourth column corresponds to the optimal deferral period; and the last column corresponds to the gain in certainty equivalent consumption when purchasing an annuity with the optimal deferral period instead of purchasing an immediate annuity.

In this table we observe that the optimal deferral time is short, also in cases where risk aversion is lower or the equity risk premium is higher. As expected, when risk aversion is low enough or the return on an alternative to an annuity payment (i.e., the expected return on equity) is higher, it becomes more favorable to defer the first annuity payment. When deferring the first annuity payment, the individual optimally invests a lower fraction of wealth in a deferred annuity at retirement date, which is partly invested in equities, depending on risk aversion. The utility gain is very small when purchasing an annuity with the optimal deferral period instead of an immediate one and this result is robust for all five examined alternatives of the parameters of the individual characteristics and the financial market.

7 Conclusions

This paper investigates the effect of systematic longevity risk on an individual's optimal annuitization decision in a life-cycle model. In addition, we investigate the optimal annuity product an individual should purchase, i.e., a deferred annuity at retirement date or an immediate annuity purchased either at retirement date or at a fixed time in the future. We argue that systematic longevity risk affects the optimal annuity decision in three ways. First, due to systematic longevity risk the price of an annuity purchased in the future depends on the distribution of future survival probabilities, and therefore it is currently stochastic. Second, systematic longevity risk affects the current market price of a deferred annuity. The impact of systematic longevity risk on the market price of an annuity depends on the type of the annuity. For payments with greater longevity uncertainty (payments at advanced ages), the risk premium as fraction of the expected discounted cash flow of the payment, is much higher than for payments with smaller longevity uncertainty (payments in the first years following the purchase of an annuity). Compared to a loading factor which is independent of the deferral period, this makes deferred annuities less attractive. Third, systematic longevity risk leads to annuities which are less attractive. This occurs because the systematic longevity risk leads to uncertainty in the value of annuities, thereby making annuities a more risky investment.

In the context of our life-cycle model we show that systematic longevity risk affects an individual's annuity decision in two ways. First, due to the uncertainty in the future prices of annuities, it is utility-increasing for an individual aged 65 to purchase an annuity currently instead of postponing the annuity purchase. This differs from the existing literature, which ignores systematic longevity risk. Second, we show that systematic longevity risk makes deferred annuities less attractive. We find that it is optimal to currently purchase an annuity with a short deferral period. However, we find that the utility loss of currently purchasing an immediate annuity, instead of currently purchasing an annuity with the optimal deferral period, is negligibly small.

This paper can be extended in several ways. For example, many modifications and extensions can be made with respect to the CRRA utility function, such as adding a bequest motive. Obviously, this will affect the quantitative results. However, since the mechanisms as described in this paper will probably remain in place, the qualitative results, i.e., the effect of systematic longevity risk and the effect of an increase in the deferral period, will probably also remain in place. This paper could also be extended by including other types of annuities, such as variable annuities, or different options in the annuity (for example, a period-certain, or a lump-sum option), allowing for the purchase of a portfolio of different types of annuities, or allowing the individual to gradually purchase annuities. For example, the model could be extended by allowing for the postponement of the purchase of deferred annuities. However, this might not be optimal, because, as observed in the case of immediate annuities, it is utility-reducing to postpone the annuity purchase due to the currently stochastic price of these annuities. It would be more interesting to extend the literature on optimal gradual annuitization by allowing for a combination of annuity products purchased at different moments in the life-cycle.

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A Method to calculate optimal annuity decision

In this appendix we describe the technique to obtain optimal choices in a life-cycle model which was proposed by Brandt, Goyal, Santa-Clara, and Stroud (2005) and by Carroll (2006), with several extensions proposed by Koijen, Nijman, and Werker (2009).

Let us first summarize the individual's life-cycle problem. Denote

$$\widehat{\mu}_t = [\widehat{\mu}_t^{(1)} \ \widehat{\mu}_t^{(2)}]', \ \ \mu_t = [\mu_t^{(1)} \ \mu_t^{(2)}]',$$

$$\widehat{V}_t = \begin{pmatrix} \widehat{V}_t^{(1,1)} & \widehat{V}_t^{(1,2)} \\ \widehat{V}_t^{(1,2)} & \widehat{V}_t^{(2,2)} \end{pmatrix}, \widehat{C}_t = \begin{pmatrix} \widehat{C}_t^{(1,1)} & \widehat{C}_t^{(1,2)} \\ 0 & \widehat{C}_t^{(2,2)} \end{pmatrix}, \text{ and } C_t = \begin{pmatrix} C_t^{(1,1)} & C_t^{(1,2)} \\ 0 & C_t^{(2,2)} \end{pmatrix}.$$

Let $\widehat{\theta}_t = [\widehat{\mu}_t^{(1)} \ \widehat{\mu}_t^{(2)} \ \widehat{V}_t^{(1,1)} \ \widehat{V}_t^{(1,2)} \ \widehat{V}_t^{(2,2)}]'$ be the vector of the maximum likelihood estimates of the parameters of the stochastic survival process based on the information revealed up to time t and let $\mathbb{E}_t[\cdot]$ be the expectation conditional on the exogenous and endogenous state variables at time t.

Formally, the individual solves:

$$J_0(x, W_0, A_0, B_0, X_0) = \max_{a_s(d), \{w_\tau, C_\tau\}_{\tau \geq 0}} \left\{ \mathbb{E}_0 \left[\sum_{\tau \geq 0} {}_\tau p_{x,0} \cdot \beta^\tau \cdot \frac{(C_\tau)^{1-\gamma}}{1-\gamma} \right] \right\}.$$

The individual's optimization problem is subject to liquidity and short-selling constraints, and given the endogenous (A_0, B_0, and W_0) and exogenous (X_0) state variables. The liquidity and short-selling constraints are:

$$\begin{aligned} 0 \leq a_s(d) \leq & 1, & \\ 0 \leq w_\tau \leq & 1, & \text{for } \tau \geq 0, \\ C_\tau \leq & W_\tau + A_\tau, & \text{for } \tau \geq 0. \end{aligned}$$

The evolution of the endogenous state variable W_τ is given by:

$$W_{\tau+1} = \begin{cases} (W_\tau - C_\tau) \cdot (1 - a_s(d)) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau = s, \\ (W_\tau + A_\tau - C_\tau) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau \neq s, \end{cases}$$

The time-t value of the endogenous state variables A_t and B_t are given by:

$$A_{t+1} = \begin{cases} A_t, & \text{if } t \neq s + d - 1, \\ B_t, & \text{if } t = s + d - 1, d > 1, \\ \frac{a_s(1) \cdot (W_s - C_s)}{V_s\left(A_{x+s,s}^{(1)}\right)}, & \text{if } t = s, d = 1, \end{cases}$$

with $A_0 = 0$, and

$$B_{t+1} = \begin{cases} B_t, & \text{if } t \neq s, \\ \frac{a_s(d)\cdot(W_s - C_s)}{V_s\left(A^{(d)}_{x+s,s}\right)}, & \text{if } t = s, \end{cases}$$

with $B_0 = 0$. The state variable B_t does not play a role when $d = 1$. The time-t values of the probability that an x-years old individual will survive another τ years, ${}_\tau p_{x,t}$, for $\tau > 0$, are given by:

$${}_\tau p_{x,t} = \prod_{s=0}^{\tau-1} (1 - {}_1q_{x+s,t+s}) = \prod_{s=0}^{\tau-1} \frac{1}{1 + \exp\left(k^{(1)}_{t+s} + (x+s)\cdot k^{(2)}_{t+s} + \epsilon_{x+s,t+s}\right)}.$$

Systematic longevity risk affects the survival probabilities generating additional uncertainty in our life-cycle model. For the process of the survival probabilities we assume that there exists parameter risk in the distribution of future survival probabilities. We assume that the individual updates the parameters in the CBD model. The distribution of the parameters in the CBD model is given by:

$$\begin{aligned} V^{-1}|D \sim& \text{Wishart}\left(\tau + \bar{t} - \underline{t}, (\tau + \bar{t} - \underline{t} + 1)^{-1}\widehat{V}_\tau^{-1}\right), \\ \mu|V,D \sim& MVN\left(\widehat{\mu}_\tau, (\tau + \bar{t} - \underline{t} + 1)^{-1}V\right), \end{aligned}$$

where $\widehat{\mu}_\tau$ and $\widehat{V}_\tau = \widehat{C}'_\tau\widehat{C}_\tau$ are the maximum likelihood estimates of μ and V at time τ. The evolution of the individual's information on the exogenous state variables is given by:

$$\begin{aligned} \begin{pmatrix} k^{(1)}_{t+1} \\ k^{(2)}_{t+1} \end{pmatrix} =& \begin{pmatrix} k^{(1)}_{t} \\ k^{(2)}_{t} \end{pmatrix} + \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix} + \begin{pmatrix} C^{(1,1)} & C^{(1,2)} \\ 0 & C^{(2,2)} \end{pmatrix} \cdot \begin{pmatrix} N^{(1)}_t \\ N^{(2)}_t \end{pmatrix} \\ \widehat{\theta}_{t+1} =& \widehat{\theta}_t + \Delta_{\widehat{\theta}_t}, \end{aligned}$$

with $\Delta_{\widehat{\theta}_t} = f(t, \widehat{\theta}_t, N_t)$, which is equal to:

$$\begin{pmatrix} \frac{\widehat{C}_t^{(1,1)} \cdot N_t^{(1)}}{t+1+\bar{t}-\underline{t}} \\ \frac{\widehat{C}_t^{(2,2)} \cdot N_t^{(2)}}{t+1+\bar{t}-\underline{t}} \\ \frac{(t+\bar{t}-\underline{t})}{t+1+\bar{t}-\underline{t}} \cdot (\widehat{\mu}_t^{(1)})^2 - (\widehat{\mu}_t^{(1)} + \frac{\widehat{C}_t^{(1,1)} \cdot N_t^{(1)}}{t+1+\bar{t}-\underline{t}})^2 + \frac{(\widehat{\mu}_t^{(1)} + \widehat{C}_t^{(1,1)} \cdot N_t^{(1)})^2 - \widehat{V}_t^{(1,1)}}{t+1+\bar{t}-\underline{t}} \\ \frac{(t+\bar{t}-\underline{t})}{t+1+\bar{t}-\underline{t}} \cdot \widehat{\mu}_t^{(1)} \cdot \widehat{\mu}_t^{(2)} - (\widehat{\mu}_t^{(1)} + \frac{\widehat{C}_t^{(1,1)} \cdot N_t^{(1)}}{t+1+\bar{t}-\underline{t}}) \cdot (\widehat{\mu}_t^{(2)} + \frac{\widehat{C}_t^{(2,2)} \cdot N_t^{(1)}}{t+1+\bar{t}-\underline{t}}) + \frac{(\widehat{\mu}_t^{(1)} + \widehat{C}_t^{(1,1)} \cdot N_t^{(1)}) \cdot (\widehat{\mu}_t^{(2)} + \widehat{C}_t^{(2,2)} \cdot N_t^{(2)}) - \widehat{V}_t^{(1,2)}}{t+1+\bar{t}-\underline{t}} \\ \frac{(t+\bar{t}-\underline{t})}{t+1+\bar{t}-\underline{t}} \cdot (\widehat{\mu}_t^{(2)})^2 - (\widehat{\mu}_t^{(2)} + \frac{\widehat{C}_t^{(2,2)} \cdot N_t^{(1)}}{t+1+\bar{t}-\underline{t}})^2 + \frac{(\widehat{\mu}_t^{(2)} + \widehat{C}_t^{(2,2)} \cdot N_t^{(2)})^2 - \widehat{V}_t^{(2,2)}}{t+1+\bar{t}-\underline{t}} \end{pmatrix}.$$

The individual updated information at time t on the distribution of the survival probabilities is fully captured by $X_t = \left[k_t^{(1)} \; k_t^{(2)} \; \widehat{\theta}_t' \right]'$.

In Appendix A.1 we describe the method to obtain the currently optimal level of annuitized wealth, conditional on the purchase of a deferred annuity with a fixed deferral period. In Appendix A.2 we describe the method to obtain the optimal level of annuitized wealth, conditional on a fixed postponement period before purchasing an immediate annuity.

A.1 The optimal deferred annuity decision

The individual's lifetime investment and consumption problem is solved via backwards dynamic programming. In order to solve the problem we first determine an individual's lifetime choices (i.e., the yearly fraction of wealth in equities and the yearly consumption) conditional on $a_0(d)$ (i.e., conditional on currently purchasing a deferred annuity) for a grid of $a_0(d)$ between zero and one.

We first determine the optimal life-cycle choices, conditional on an annuity income stream. The individual is assumed to have no bequest motive, which implies that the individual's optimal consumption level at the last period is to consume all his, after annuity income, wealth. Hence, assuming $d \leq MA - x$ the individual's time-$(MA - x)$ value function is given by:

$$J_{MA-x}\left(MA, W_{MA-x}, A_{MA-x}, B_{MA-x}, X_{MA-x}\right) = \frac{\left(W_{MA-x} + A_{MA-x}\right)^{1-\gamma}}{1-\gamma}.$$

Similar to Koijen, Nijman, and Werker (2009), at any intermediate point in time the lifetime utility function satisfies the Bellman equation:

$$
\begin{aligned}
&J_t\left(x+t, W_t, A_t, B_t, X_t\right)= \\
&\max_{\{w_t, C_t\}}\left\{\frac{\left(C_t\right)^{1-\gamma}}{1-\gamma}+\beta \cdot \mathbb{E}_t\left[p_{x+t, t} \cdot J_{t+1}\left(x+t+1, W_{t+1}, A_{t+1}, B_{t+1}, X_{t+1}\right)\right]\right\}, \qquad (19)
\end{aligned}
$$

At each point in time there are two control variables, (C_t, w_t). Therefore, similar to Koijen, Nijman, and Werker (2009), to determine the optimal values of the control variables, at each point in time $t \geq 0$ we need to set the first order derivatives of Bellman equation with respect to the control variables equal to zero:

$$
0 = \mathbb{E}_t\left[p_{x+t,t} \cdot \left(C_{t+1}^{\star}\right)^{-\gamma} \cdot \left(r_t - r^{rf}\right)\right], \qquad (20)
$$

$$
C_t^{\star} = \left(\beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot \left(C_{t+1}^{\star}\right)^{-\gamma} \cdot \left(1 + r^{rf} + w_t^{\star} \cdot \left(r_t - r^{rf}\right)\right)\right]\right)^{\frac{1}{-\gamma}}. \qquad (21)
$$

Equation (20) is the first order condition of (19) with respect to the fraction of liquid wealth invested in equity and equation (21) is the first order condition of (19) with respect to the consumption level.

To solve the individual's lifetime investment and consumption problem we use a grid of the endogenous state variable after-consumption wealth, $\widetilde{W}_t = W_t + A_t - C_t$. Following Carroll (2006) we use an after-consumption wealth space with a triple exponential growth rate between the grid points between 0.001 and 100 and the grid point 0. To obtain results within reasonable time, we compute the expectations through a regression, similar to the simulation method proposed by Longstaff and Schwartz (2001) for pricing American-style options. First, we solve $w_t^{\star}$ using equation (20), next we solve $C_t^{\star}$ using equation (21). To solve $w_t^{\star}$ based on (20), we follow Koijen, Nijman, and Werker (2009). We define H "test portfolios" with different fractions of wealth invested in equities, i.e., each test portfolio is characterized by its fraction of wealth invested in equities. We use an equally spaced grid of the fraction of liquid wealth invested in equities, i.e., $w_t(h) \in \{0, 1/(H-1), \ldots, 1\}$. Then, for a given test portfolio, we parameterize the conditional expectation as a function of the exogenous state variables in order to solve equation (20).[22] Let $C_{t+1}^{\star}(h)$ be the optimal consumption level in the follow-

[22]Following Koijen, Nijman, and Werker (2009) we compute the expectation through regressions for a grid of the endogenous state variables. Even though it is theoretical possible to use a grid of exogenous

ing year, conditional on the corresponding fraction of liquid assets invested in equities ($w_t(h)$). The exogenous state variables at time t depend on the information known to the investor at time t. We simulate the equity return and the survival probabilities in the CBD model and we parameterize the following simulated expectation:

$$\mathbb{E}_t \left[p_{x+t,t} \cdot \left(C^{\star}_{t+1}(h)\right)^{-\gamma} \cdot \left(r_t - r^{rf}\right)\right] = X_t^P \cdot \theta_t \left(\widetilde{W}_t, A_t, B_t, w_t(h)\right), \; h = 1, \ldots, H, \quad (22)$$

where $\theta_t \left(\widetilde{W}_t, A_t, B_t, w_t(h), \right)$ are the parameters to be estimated using least squares, and X_t^P is a polynomial expansion in the exogenous state variables.[23] We use $N = 5,000$ simulated trajectories and $M = 130$ points. Since we know the optimal consumption policy at time $t+1$ only at the endogenous grid points, we interpolate the consumption policy linearly for intermediate values. Let K be the dimension of X_t^P, and let $\theta_{t,k} \left(\widetilde{W}_t, A_t, B_t, w_t(h)\right)$ be the regression coefficients of $\theta_t \left(\widetilde{W}_t, A_t, B_t, w_t(h)\right)$ corresponding to the k^{th} component of X_t^P. To solve the equality in (20) we parameterize the regression parameters on a polynomial basis of the second degree in the asset weights:

$$\theta_{t,k} \left(\widetilde{W}_t, A_t, B_t, w_t(h)\right) = X_t^h \cdot \psi_{k,t} \left(\widetilde{W}_t, A_t, B_t\right), \text{for } k = 1, \ldots, K, \quad (23)$$

where $\psi_{t,k} \left(\widetilde{W}_t, A_t, B_t\right)$ are the parameters to be estimated using least squares, and $X_t^h = \left[1 \; w_t(h) \; \left(w_t(h)\right)^2\right]$ is a polynomial basis in the fraction of liquid wealth annuitized in the test portfolios. Notice that we use a polynomial basis of the second degree instead of the first degree as is done in Koijen, Nijman, and Werker (2009), since this provides a better fit of the regression parameters. Given the parametrization in equation (23) we determine the optimal fraction of wealth invested in equities from setting the right hand side of equation (22) equal to zero, using projection coefficient of the polynomial expansion (i.e., using $\psi_{t,k} \left(\widetilde{W}_t, A_t, B_t\right)$).[24]

---

and endogenous state variables, in order to properly parameterize the conditional expectation it requires a very large regression, which makes it computationally unattractive.

[23] We use a polynomial expansion of the second degree in $\exp\left(\widehat{\mu}_t^{(1)}\right)$, $\exp\left(\widehat{\mu}_t^{(2)}\right)$, $\exp\left(\widehat{C}_t^{(1,1)}\right)$, $\exp\left(\widehat{C}_t^{(1,2)}\right)$, $\exp\left(\widehat{C}_t^{(2,2)}\right)$, $\exp\left(k_t^{(1)}\right)$, and $\exp\left(k_t^{(2)}\right)$, including cross-terms in the polynomial expansion, to capture all relevant information on the exogenous state variables. We tried several polynomial expansions of the state variables. This specification turned out to be the most accurate, i.e., had the lowest sum of squared error. We take the exponent of the state variables, because the future survival probabilities are exponential transformations of the state variables.

[24] Notice that using this method does not incorporate the short-selling constraints. The short-selling

Next, we determine the optimal consumption level by solving equation (21), given the obtained optimal fraction of wealth invested in equities, using the above described method. In order to avoid to make N simulations for each trajectory, we parameterize the conditional expectation in equation (21) as a polynomial expansion of the exogenous state variables X_t^P, which depends on the information known to the investor at time t:

$$\mathbb{E}_t\left[p_{x+t,t}\cdot\left(C_{t+1}^{\star}\right)^{-\gamma}\cdot\left(1+r^{rf}+w_t^{\star}\cdot\left(r_t-r^{rf}\right)\right)\right]=\exp\left(X_t^P\cdot\nu_t\left(\widetilde{W}_t,A_t,B_t\right)\right), \tag{24}$$

where $\nu_t\left(\widetilde{W}_t,A_t,B_t\right)$ are the parameters to be estimated using least squares. We use linear regression after taking the logarithm of the expectation, in order to ensure that the conditional expectation is strictly positive implying that the consumption is strictly positive. The wealth level at time t follows from $W_t=\widetilde{W}_t-A_t+C_t$.

Finally, we have to determine the optimal fraction of wealth invested in a deferred annuity. To determine the optimal fraction of wealth currently (at time $s=0$) invested in deferred annuities we calculate the expected utility for different portfolios, conditional on an individual's initial wealth level. The portfolios differ in the fraction of annuitized wealth, and hence their annuity income level. Conditional on the purchase of a deferred annuity with a deferral period of d years, $a_0^{\star}(d)$ the optimal fraction of annuitized wealth is then determined by:

$$a_0^{\star}(d)=\underset{a_0(d)}{\operatorname{argmax}}\; J_0\left(x,W_0(a_0(d)),A_0(a_0(d)),B_0(a_0(d)),X_0\right), \tag{25}$$

where $W_0(a_0(d))$, $A_0(a_0(d))$, $B_0(a_0(d))$ are the values of the endogenous state variables conditional on $a_0(d)$. The optimal fraction of annuitized wealth is thus obtained by comparing the expected lifetime utility of an individual, given the optimal consumption and investment choices, conditional on the fraction of annuitized wealth.

---

constraints implies that the fractions of wealth invested in equities should be between zero and one. Therefore, we investigate whether the corner solutions (i.e., $w_t^{\star}=0$ and $w_t^{\star}=1$) are optimal. In case both roots are in the $[0,1]$ interval, we select the one with the highest utility level.

A.2 The optimal postponed annuity level

In this appendix we describe the method to obtain the optimal annuity decision when the individual postpones the purchase of an immediate annuity. Contrary to currently purchasing a deferred annuity, when postponing the purchase of immediate annuities, the decision may depend on the state variables. Let s be the number of years until the annuity is purchased, hence the individual purchases an immediate annuity ($d = 1$) at time s. In this case B_t does not play a role, and hence the endogenous state variables at time t are fully captured by A_t and W_t.

To solve the optimal consumption and investment choices using the backwards induction algorithm, we distinguish three types of decision moments, namely:

i) *After time s.* The optimal decisions are obtained for a grid of endogenous state variables. Using the method described in Appendix A.1 we obtain optimal investment and consumption choices conditional on the wealth level, the annuity income level, and the exogenous state variables.

ii) *At time s.* At this time the individual has to determine the optimal annuity income level, besides the optimal consumption and investment choices. The method to obtain these optimal choices is described in the remainder of this section.

iii) *Before time s.* The optimal decisions are obtained for a given wealth level. Using the method described in Appendix A.1 the optimal investment and consumption choices are obtained, conditional on the exogenous state variables and the wealth level.

To determine the individual's optimal choices in the life-cycle we use a backwards induction algorithm to solve the optimal consumption and investment choices conditional on the exogenous state variables, annuity income and wealth level. At the moment of purchasing the annuity, we determine the optimal consumption, investment, and annuity decision, conditional on the future optimal consumption and investment decisions. Then using the backwards induction algorithm we solve the optimal consumption and investment decisions before the moment of the annuity purchase, conditional on the individual's optimal future consumption, investment, and annuity decisions.

After time s we would have M^2 grid points (for wealth and annuity income). In order to reduce the number of grid points after time s to M we normalize the state variables after time s such that the yearly annuity income stream equals one. This normalization can be done because for $t > s$ the yearly annuity income stream is a known constant, and is comparable to the fixed income stream in Horneff, Maurer, Mitchell, and Stamos (2007). At or before time s we have M grid points (only for wealth). In order to reduce the number of grid points at time $t \leq s$ from M to 1, we normalize the state variable W_t by its after-consumption wealth level i.e., we determine the optimal decisions conditional on an after-consumption wealth level of 1.[25] We recover the original state variables and $C_t^\star$ by multiplying the normalized variable with the corresponding after-consumption wealth level.

The approach to obtain the optimal decisions at time s is an extension of the method explained in Appendix A.1, by including the condition for the optimal fraction of annuitized wealth. To obtain the optimal decisions at time s we have three control variables, namely $(C_s, w_s, a_s(1))$. Let $\overline{C}_t$ for $t \geq s$ be the consumption level normalized at the time s after-consumption wealth level, and in order to avoid over-notation denote $J_{s+1} = J_{s+1}(x+s+1, W_{s+1}, A_{s+1}, X^P_{s+1})$. An increase in A_t leads to an increase in an individual's income level of the same size and thus impacts his wealth level at time t. Moreover, an increase in A_t leads to an increase in A_{t+1} of the same magnitude, i.e., the annuity income increases not only for the following year, but for all future years. Using $\frac{\partial J_t}{\partial W_t} = (C_t^\star)^{-\gamma}$ we have that:

$$\frac{\partial J_t}{\partial A_t} = (C_t^\star)^{-\gamma} + \beta \cdot \mathbb{E}_t \left[p_{x+t,t} \cdot \frac{\partial J_{t+1}}{\partial A_{t+1}} \right] \text{ for } t \geq s+1.$$

Let $W_s - C_s > 0$ be the normalization constant for the original problem[26], then we have

[25] Notice that, similar to Koijen, Nijman, and Werker (2009), and Horneff, Maurer, and Stamos (2007), the normalized decision variables are independent of a normalization of the state variables. Hence, we find that $a_s^\star(1)$ and $w_s^\star$ are independent of the wealth level at time s, because the state variables at time s are the wealth level and $A_s = 0$.

[26] Notice that $W_s - C_s = 0$ would imply that the individual has no wealth and no annuities at $s+1$. Hence, this would not be optimal since the utility at $t > s$ is minus infinity.

that:

$$\frac{\partial J_{s+1}}{\partial A_{s+1}} / \frac{\partial J_{s+1}}{\partial W_{s+1}} = \tag{26}$$
$$\mathbb{E}_{s+1}\left[1 + \left(p_{x+s+1,s+1} \cdot \beta \cdot \left(\frac{\overline{C}^{\star}_{s+2}}{\overline{C}^{\star}_{s+1}}\right)^{-\gamma} \cdot \left(1 + p_{x+s+2,s+2} \cdot \beta \cdot \left(\frac{\overline{C}^{\star}_{s+3}}{\overline{C}^{\star}_{s+2}}\right)^{-\gamma} \cdot (\dots)\right)\right)\right].$$

For both the normalized and the original problem, we set the first order derivatives at time s of the Jacobian with respect to each of the three control variables equal to zero:

$$\begin{aligned}
0 =& \mathbb{E}_s \left[p_{x+s,s} \cdot \left(\overline{C}^{\star}_{s+1}\right)^{-\gamma} \cdot (1 - a^{\star}_s(1)) \cdot \left(r_s - r^{rf}\right)\right], \\
0 =& \mathbb{E}_s [p_{x+s,s} \cdot \left(\overline{C}^{\star}_{s+1}\right)^{-\gamma} \cdot (\frac{1}{V_s(A^{(1)}_{x+s,s})} \cdot \frac{\partial J_{s+1}}{\partial A_{s+1}} / \frac{\partial J_{s+1}}{\partial W_{s+1}} \\
& - (1 + r^{rf} + w^{\star}_s \cdot \left(r_s - r^{rf}\right)))], \\
\left(\overline{C}^{\star}_s\right)^{-\gamma} =& \beta \cdot \mathbb{E}_s \left[p_{x+s,s} \cdot \left(\overline{C}^{\star}_{s+1}\right)^{-\gamma} \cdot \left(a^{\star}_s(1) \cdot \frac{1}{V_s(A^{(1)}_{x+s,s})} \cdot \frac{\partial J_{s+1}}{\partial A_{s+1}} / \frac{\partial J_{s+1}}{\partial W_{s+1}}\right.\right. \\
& \left.\left. + (1 - a^{\star}_s(1)) \cdot \left(1 + r^{rf} + w^{\star}_s \cdot \left(r_s - r^{rf}\right)\right)\right)\right],
\end{aligned}$$

where $\overline{C}^{\star}_s$, $w^{\star}_s$, and $a^{\star}_s(1)$ are the time-s optimal normalized consumption, fraction of liquid wealth invested in equity, and fraction of wealth invested in annuities, respectively. We determine equation (26) by simulating the paths of future optimal consumption, conditional on information up to time s. Notice that, because of the backwards induction algorithm, we know the optimal consumption policy at time $t > s$ only at the endogenous grid points, thus we interpolate the consumption policy linearly for intermediate values.

To solve the choices at time s we define H^2 test portfolios. These test portfolios are characterized by the fraction of after-consumption wealth invested in annuities ($a_s(1, h_1) \in \{0, 1/(H-1), \dots, 1\}$, for $h_1 \in H$), and the fraction of after-annuitized liquid wealth invested in equity ($w_s(h_2) \in \{0, 1/(H-1), \dots, 1\}$, for $h_2 \in H$). Hence, each test portfolio is characterized by $(a_s(1, h_1), w_s(h_2))$. Let $h = (h_1, h_2)$, and $\overline{C}^{\star}_{s+1}(h)$ be the optimal normalized consumption level at time $s+1$ corresponding to test portfolio h, and $J_{s+1}(h)$ the value of the Bellman at time $s+1$ corresponding to test portfolio h. We generalize equation (22), to solve the investment decision at time s we first parameterize for every $h \in H^2$:

$$X_s^P \cdot \theta_{s,k}^w \left(a_s(1,h_1), w_s(h_2)\right) = \mathbb{E}_s\left[p_{x+s,s} \cdot \left(\overline{C}_{s+1}^\star(h)\right)^{-\gamma} \cdot \left(1 - a_s(1,h_1)\right) \cdot \left(r_s - r^{rf}\right)\right],$$

$$X_s^P \cdot \theta_{s,k}^a \left(a_s(1,h_1), w_s(h_2)\right) = \mathbb{E}_s[p_{x+s,s} \cdot \left(\overline{C}_{s+1}^\star(h)\right)^{-\gamma} \cdot \left(-1 - r^{rf} - w_s^\star \cdot \left(r_s - r^{rf}\right)\right. + \frac{\partial J_{s+1}(h)}{V_s(A_{x+s,s}^{(1)}) \cdot \partial A_{s+1}} / \frac{\partial J_{s+1}(h)}{\partial W_{s+1}})].$$

Furthermore, we generalize equation (23) by parameterizing the regression coefficients on a polynomial basis:

$$\theta_{s,k}^w \left(a_s(1,h_1), w_s(h_2)\right) = X_s^h \cdot \psi_{s,k}^w,$$

$$\theta_{s,k}^a \left(a_s(1,h_1), w_s(h_2)\right) = X_s^h \cdot \psi_{s,k}^a,$$

where X_s^h is a polynomial expansion in the portfolio weights of the second degree.[27]

Finally, generalizing equation (24), we use the following parametrization to obtain the optimal consumption level:

$$\exp\left(X_s^P \cdot \nu_s\right) = \mathbb{E}_s[p_{x+s,s} \cdot \left(\overline{C}_{s+1}^\star\right)^{-\gamma} \cdot (a_s^\star(1) \cdot \frac{\partial J_{s+1}}{V_s(A_{x+s,s}^{(1)}) \cdot \partial A_{s+1}} / \frac{\partial J_{s+1}}{\partial W_{s+1}} + (1 - a_s^\star(1)) \cdot \left(1 + r^{rf} + w_s^\star \cdot \left(r_s - r^{rf}\right)\right))].$$

Given the above described technique we find the optimal annuity, consumption, and investment decision at time s, conditional on the exogenous state variables and a normalized wealth income. The optimal investment decisions ($w_s^\star$, and $a_s^\star(1)$) are independent of the normalization, and in order to recover the original optimal consumption level we multiply with the after-consumption wealth level at time s, i.e., $C_s^\star = \frac{\overline{C}_s^\star}{1+\overline{C}_s^\star} \cdot W_s$. Using the technique described in Appendix A.1 we obtain the individual's optimal consumption and investment choices for any time when he does not purchase an annuity.

[27]When the optimal $w_s^\star$ or $a_s^\star(1)$ is outsize [0,1]-interval we determine the optimal corner solutions, i.e., $a_s^\star(1) = 1$ and we determine $w_s^\star$ conditional on $a_s(1) = 0$, $a_s^\star(1)$ conditional on $w_s^\star = 1$, and $a_s^\star(1)$ conditional on $w_s^\star = 0$, and select the one with the highest utility level. Notice that when $a_s^\star(1) = 1$, there is no liquid wealth after consumption, and hence the first order condition with respect to w_t equals zero, for all w_t.