

A THEORY OF RISK CAPITAL

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Abstract

We present a theory of risk capital and of how tax and other costs of risk capital should be allocated in a financial firm. Risk capital is equity investment that backs obligations to creditors and other liability holders and maintains the firm's credit quality. Credit quality is measured by the ratio of the value of the firm's option to default to the default-free value of its liabilities. Marginal default values provide a full and unique allocation of risk capital. Efficient capital allocations maintain credit quality and preclude risk shifting. Our theory leads to an adjusted present value (APV) criterion for making investment and contracting decisions. We relate our results to capital budgeting procedures used in corporate finance. We also set out practical and policy applications, including implications for risk-based regulatory capital requirements.

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A THEORY OF RISK CAPITAL

1. Introduction

This paper presents a theory of risk capital and a general procedure for allocating the tax and other costs of risk capital to lines of business. “Risk capital” is equity investment that backs up obligations to creditors, counterparties and other liability holders. We review how a value-maximizing firm should trade off the costs and benefits of risk capital to determine the target credit quality of the firm’s obligations. We show how the costs affect the firm’s optimal portfolio of businesses, given the target credit quality, and how the costs should be “priced” and charged back to lines of business, based on efficient allocations of risk capital. Efficient allocations are required to assess profitability, to price products and services and to set compensation. We explore the practical and policy implications of our theory and allocation procedures.

We provide a more precise definition of “risk capital” in section 2. Definitions are important, because “capital” can mean so many things. For example, “economic capital,” which is widely used to distinguish economic from accounting and regulatory capital, is sometimes, but not always the same concept as risk capital.

Merton and Perold (1993) distinguish risk capital from “cash capital,” which is an amount available to invest. If a start-up firm issues \$50 million in debt securities and \$50 million in common stock, it has \$100 million of cash capital but at most \$50 million of risk capital. But investments of cash capital are *not* this paper’s starting focus. Allocating more of (the costs of) *risk* capital to a business reduces the business’s net value, but does not by itself determine how much cash should be invested. Also, this

paper is not about risk-based regulatory capital requirements, although we set out implications for regulation in Section 6.

The efficient risk-capital allocation for a business depends on its *marginal default value*, which is the derivative of the value of the firm's option to default (its *default put*) with respect to a change in the scale of the business. (The default put determines the firm's credit quality, which the firm must maintain at an optimal, or at least acceptable target level.) Marginal default values add up exactly and support a unique allocation. The businesses with the largest marginal default values should be allocated the most risk capital and charged the most for the costs of risk capital.

Efficient risk-capital allocations satisfy two requirements. First, *no risk-shifting*: risk capital should be allocated so that a marginal change in the composition of the firm's portfolio of businesses does not affect the credit quality of the firm's liabilities. Second, *no internal arbitrage*: risk capital should be allocated so that it is not possible to add value at the margin merely by shifting risk capital from one business to another. These requirements are general and require no restrictions on the joint probability distribution of returns.

One can imagine a tax-free Modigliani-Miller (MM) world in which equity financing is always available on fair ($NPV = 0$) terms. In this case risk capital would be free of charge and there would be no need to allocate it. Investment decisions would depend solely on market risks and returns. But increasing equity to provide risk capital is costly in practice, for at least two reasons. First, returns to equity are subject to corporate income tax. (Corporate finance would say that returns to equity do not generate interest

tax shields.) Second, additional capital may increase agency costs and monitoring costs borne by shareholders.

Of course risk capital has benefits too. More capital reduces possible debt-overhang and risk-shifting problems and it makes costs of bankruptcy or financial distress more remote. Costs incurred by creditors to monitor and to protect their interests are reduced.

All firms deploy risk capital, but our theory and procedures are especially important for financial firms dealing with customers, creditors and counterparties that are not prepared to bear significant default risk.¹ Such firms must put up enough risk capital to maintain an acceptable target credit quality for their obligations. They typically operate in both safe and risky businesses, and therefore must take care not to give the risky businesses “free passes” to expand. Expansion of risky businesses should consider the costs or consequences of (1) increasing the firm’s risk capital or (2) imposing additional default risk on customers, creditors and counterparties. Consequence (2) amounts to a decision to operate at a lower credit quality. The lower credit quality would then feed back to revised risk-capital allocations.

Whether a business, product or contract is a safe or risky call on the firm’s risk capital depends on the *firm’s* portfolio. For example, a forward contract for heating oil could be a speculative position for firm A and require a high offsetting allocation of (the costs of) risk capital. The same forward contract could act as a hedge for firm B and require a low or negative allocation.

¹ Merton and Perold (1993) refer to liability holders who demand high credit quality as “credit sensitive.”

We focus on capital allocation in the context of a firm operating in two or more lines of business, but this focus is for expositional convenience only. The risk-capital allocation problem arises any time a firm contemplates an investment or a commitment, even for a one-off transaction.

This paper does not calculate the optimal level of risk capital, either from a private or social point of view.² We review how a financial firm’s managers would trade off the costs and benefits of more or less risk capital and decide on a target level of credit quality. We define credit quality as the ratio of the value of a one-period default put to the default-free value of the firm’s debt and other liabilities—in other words, the fraction of the value of promised payments that liability holders expect *not* to realize over the next period. This fraction determines the credit spread demanded by lenders.

1.1 Preview

Section 2 of this paper defines risk capital and identifies the costs and benefits of deploying more or less of it. We explain why a financial firm should set risk capital to achieve a target level of credit quality and why credit quality is best measured using the value of the firm’s default put. We prove that marginal default values “add up” and support a unique allocation of (the costs of) risk capital.

Section 3 then shows how risk-capital allocations should be set. The key is to adjust allocations to offset differences in marginal default values, subject to the no-risk-shifting and no-internal-arbitrage conditions.

In Section 3 we consider marginal changes in a given portfolio of businesses. In Section 4 we allow both scale and composition of the portfolio to vary and derive

² See Kashyap et al. (2010), Miles et al. (2013) and Admati et al. (2013).

conditions for optimal portfolios when risk capital is costly. Our risk-capital allocations are consistent with the conditions for the optimum portfolio and could in principle be used to implement the optimum in a decentralized setting. The conditions for an optimum yield a two-step adjusted present value (APV) procedure for valuing expansion of a business. The first term of the APV is the pre-tax NPV of the investment, calculated as if risk capital were costless. The second term is a charge for the tax and other costs of risk capital. The cost depends on the amount of risk capital allocated to the business and therefore “supporting” it. The cost is *not* expressed as an interest rate or “cost of equity” on the allocated risk capital. It is a dollar charge, e.g., for taxes.

Section 5 presents allocation formulas assuming that returns are normally distributed. The formulas illustrate how our allocations could work in practice. A detailed numerical example is included as an appendix.

Section 6 considers applications and implications. We discuss the APV rule derived in Section 4 more specifically and compare it to the APV rule for non-financial corporations. We show how risk capital should be allocated to investments that do not require up-front cash capital, such as forward or swap contracts. We contrast our method with allocations based on VaR or “expected shortfall.” We also discuss the difficulties inherent in RAROC (risk-adjusted return on capital). RAROC applications try to solve a two-part problem—calculating market-value NPV *and* incorporating the costs of risk capital—by setting one hurdle rate for investment.

Section 6 also describes discouraging implications for risk-based bank capital requirements. Such requirements impose an additional constraint that inevitably distorts

investments and risk-capital allocations. The distortions are *not* from “regulatory arbitrage,” that is, from substitution of risky for safer assets within a line of business.

Section 7 concludes.

1.2 Literature Review

There is a large literature on risk management and investment decisions in banks and other financial corporations.³ Prior work specifically on allocation of risk capital is much more limited. Merton and Perold (1993) is the best place to start. They define risk capital as the present-value (PV) cost of acquiring complete insurance against negative returns on the firm’s net assets—the value of a one-period at-the-money forward put (a put with an exercise price equal to the current value of net assets plus one period’s interest at the risk-free rate). We start with the firm’s default put, which determines credit quality. The value of the default put equals the cost of insurance for the firm’s debt and other liabilities.

Merton and Perold (1993) identify the crucial distinction between cash capital and risk capital. For example, a swap contract requires no cash investment up front, but can absorb or release risk capital. A book of high-quality, floating-rate mortgage loans might require a large cash investment but very little risk capital.

Merton and Perold (1993) focus on decisions to add or subtract an entire business, and conclude that a financial firm should not attempt to allocate its total risk capital back to lines of business. We agree that our allocations, which are based on *marginal* changes in investment, cannot be used to evaluate such decisions. Allocations at the margin must be recalculated after a material discrete change in the firm’s portfolio of businesses. But

³ Texts include Hull (2010), Jorion (2006), Matten (2000), Saunders and Cornett (2008) and Stulz (2003).

allocations at the margin are essential in evaluating risk and profitability for an existing portfolio, which is usually stable in the short run. Risk-capital allocations are not for the long run, but should be updated frequently, perhaps every quarter for a firm with trading or market-making positions that can expand or contract rapidly. Perold (2005) endorses marginal allocations and arrives at an APV rule similar to ours in examples that assume normal return distributions. Our derivations hold for any joint probability distribution of returns within a business portfolio, although we too assume normally distributed returns in Section 5.

Some applications of risk-adjusted return on capital (RAROC) recognize this distinction between cash and risk capital. In Zaik et al. (1996), *economic capital* is allocated to a bank's activities depending on the activities' downside risks over a one-year horizon, with credits for risks diversifiable inside the bank. Economic capital is in effect risk capital, but computed in a VaR setting rather than from marginal default values. Each activity's rate of return on economic capital is then compared to the bank's cost of equity capital, that is, the expected rate of return demanded by the bank's shareholders.

RAROC is no doubt useful for some purposes, but the measure is not derived from a formal analysis of how risk capital should be allocated. Our APV valuation, which does follow from a formal analysis, does not charge an activity for the cost of equity capital, but for the tax or other costs of putting up risk capital.

RAROC applications are typically based on some version of value at risk (VaR). For example, Hull (2010, pp. 425-438) defines economic capital as the "amount of capital a financial institution needs in order to absorb losses over a certain time horizon with a

certain confidence level.” He proposes allocating economic capital to business units in proportion to their incremental requirements.

VaR is widely used to measure risk in financial firms. The extent to which it is used to allocate risk capital is less clear. Jorion (2006) says that VaR is used for resource allocation as well as information reporting, and he points out that VaR limits for higher-level business units can be less than the sum of VaR limits for lower-level business units. However, he does not say whether or how VaR can be used to allocate capital fully or uniquely. The difficulties in using VaR for this purpose are well-understood.⁴

Conceptual problems with VaR have been discussed extensively. The most extensive formal analysis is probably Artzner et al. (1999), who set out four properties required for a “coherent” risk measure. VaR is not coherent, essentially because it is a quantile-based measure and thus does not properly reflect the effects of diversification.

Contribution VaRs, which depend on the covariances or betas of line-by-line returns vs. returns for the firm as a whole, do add up. See Saita (1999) and Stulz (2003), for example.⁵ But allocations based on VaR or contribution VaR are not consistent with maximizing value. For example, capital allocations proportional to VaR would allocate zero capital to a risk-free asset. Our allocations, which are consistent with the conditions for value maximization, assign *negative* capital to risk-free and some low-risk assets. As

⁴ See Jorion (2006, pp.542-551) and Stulz (2003, pp. 617-625).

⁵ Contribution VaRs appear in Froot and Stein (1998, pp. 67-68), Saita (1999), Stoughton and Zechner (2007), Stulz (2003, pp. 99-103) and no doubt in other places. The label varies: synonyms for “contribution” include “marginal,” for example in Saita and in Stulz. Others refer to “incremental VaR,” which is not the same thing. Incremental VaR is the discrete change in VaR from adding or subtracting all of an asset or business from the firm’s overall portfolio. Merton and Perold (1993), Perold (2005) and Turnbull (2000) focus on incremental VaR.

we will demonstrate, such assets should not be charged for capital; they should be rewarded, because at the margin they improve its credit quality.

VaR depends on the probability of losses beyond some downside threshold, not on the value of losses below that threshold. “Expected shortfall” or “tail value at risk,” defined as the expected value of losses conditional on losses being positive, has been suggested as a VaR alternative.⁶ The expected shortfall for the firm as a whole can be close to the value of the firm’s default put. The expected shortfall for an individual asset is not the same as marginal default value, however.

Some papers on “capital allocation” focus on investment and risk management decisions but not specifically on allocating risk capital. For example, Froot and Stein’s (1998) main interest is how financial firms’ invest (cash) capital.⁷ They discuss contribution VAR and the problems of implementing RAROC. They do not consider default, however. Turnbull (2000) extends this line of research, introducing default risk. Stoughton and Zechner (2007) add a focus on information and agency costs internal to the firm. We do not address these issues, but admit that they will pose a challenge in practice.

Grundl and Schmeiser (2007) conclude that capital either need not or should not be allocated by insurance companies. Capital need not be allocated, they argue, if there are no taxes or other frictional costs. Of course we agree—in that special case. But they also argue that capital allocations are inherently arbitrary and thus potentially misleading for decision making, because the costs of risk capital amount to a fixed overhead cost.

⁶ Artzner et al. (1999) also discuss expected shortfall (“Tail conditional expectation” or “Tail VaR”) and other coherent risk measures.

⁷ Froot (2007) builds on this model to analyze risk management in the insurance industry.

That is incorrect. Capital must increase as the scale of the firm expands if credit quality is held constant. This is true of an across-the-board expansion or an expansion of the firm's riskiest lines of business. If the amount of capital is fixed in the short run, then expansion of these riskiest lines can only be accommodated if risk capital is taken away from safer lines, forcing them to contract, or if credit quality is degraded.

Therefore the amount of risk capital must be a decision variable, possibly constrained, but not an exogenously fixed amount. It cannot be fixed if credit quality is important and customers, creditors and counterparties are rational. If capital is fixed and credit quality is unconstrained, then the firm would have a free pass to expand with no regard to default risk.

This paper extends Myers and Read (2001), who analyze capital (surplus) allocation for insurance companies.⁸ Principles carry over, although proofs are more general and not limited by the special characteristics of insurance. This paper also covers several important topics not considered by Myers and Read:

1. Myers and Read did not fully develop the economics of risk capital or why the required amount of (equity investment to provide) risk capital depends on the firm's target credit quality, that is, the ratio of default-put value to total liabilities.
2. Myers and Read assumed a fixed portfolio of lines of insurance. Here we examine risk-capital allocations for a financial firm that chooses its *optimal* (value-maximizing) portfolio of businesses when (1) the amount of risk capital is an unconstrained decision variable or (2) is constrained in the short run. Efficient

⁸ We are not focusing on insurance applications, and therefore do not include a full review of recent articles on capital allocation in insurance. But see, for example, Cummins, Lin, and Phillips (2006), Grundl and Schmeiser (2007), Zanjani (2010), Panjer (2012), and Bauer and Zanjani (2013).

- capital allocations satisfy the no-risk-shifting and no-internal-arbitrage requirements both cases. The APV rule for allocating the costs of holding risk capital follows from the conditions for an optimal portfolio.
3. Myers and Read worked exclusively with joint lognormal or normal distributions. Our main results, including our basic formula for risk-capital allocation, are derived here for any joint probability distribution.
 4. Myers and Read showed how to allocate capital (for insurance companies) but not how to allocate the *costs* of holding risk capital back to lines of insurance.⁹ Here we show how the costs should be priced and charged against the NPV of an investment in a business or activity undertaken by a bank or other financial firm. New results are arrived at. For example, NPV should be calculated *pre-tax*. This result may seem surprising at first, because NPVs are almost always computed after-tax by non-financial corporations. We also show how the APV rule applies to specific cases, including swap and forward positions and hedging transactions. We explain why efficient capital allocations are negative for safe and low-risk businesses.
 5. Myers and Read did not discuss RAROC. The allocations of risk capital that are implicit in RAROC cannot be efficient except in special cases.
 6. We also analyze the portfolio optimization problem when the firm is subject to risk-based regulatory capital requirements. We show why and how such requirements distort portfolio choices, even if regulators could control “regulatory arbitrage.”

⁹ This oversight has been noted in the insurance literature by Grundl and Schmeiser (2007, p. 307) and Venter (2004, p. 96).

2. Risk Capital, the Default Put and Credit Quality

2.1 Defining Risk Capital

Start with a financial firm's market-value balance sheet:

<u>Assets</u>	<u>Liabilities and equity</u>
Assets (A_i)	Debt or other liabilities (L)
Default Put (P)	Equity (E)
Franchise value (G)	

The line-of-business assets A_i are assumed marked to market. The firm's "franchise value," which includes intangible assets and the present value of future growth opportunities, is entered as G . We assume for simplicity that franchise value disappears ($G = 0$) if the firm defaults.¹⁰

The *default-risk free* value of debt or other liabilities, including deposits if the firm is a bank, is L . Other liabilities could include insurance contracts, letters of credit, and short positions in swaps, forwards or options. We do *not* assume that debt or other liabilities are default-risk free, however. We have moved default risk to the left side of the balance sheet as the default-put value P . We assume no third-party financial guarantees or other credit backup.¹¹

¹⁰ We could generalize by introducing a residual franchise value G_{\min} in default. The availability of G_{\min} to satisfy creditors would reduce the value of the default put.

¹¹ Third-party credit backup does not complicate our analysis if acquired at market value, that is, in zero-NPV transactions. The backup absorbs default risk otherwise borne by creditors or other counterparties. The firm could pay for the backup from the additional cash raised by issuing debt on more favorable terms. Equity value and capital would not change. Deposit insurance creates two problems, however. First, the insurance has been offered at low, fixed premiums, thus subsidizing risky banks and encouraging investment in risky lines of business. Second, deposit insurers are government agencies that may not be as well-equipped as private investors to monitor and prevent risk-shifting.

Therefore the default put value P translates directly to the credit spread demanded by lenders and counterparties.

We define the maturity of the default put as one period. If the firm defaults, the payoff to the put equals the shortfall of the end-of-period asset value from the end-of-period payment due to lenders and counterparties, including interest. Defining the length of the period is an open issue for practice. A financial firm would probably update capital allocations frequently, which suggests a period length of, say, a quarter or year. On the other hand, the firm may issue longer-term debt or enter longer-term transactions with counterparties. These transactions may require a longer view of the firm's credit quality and a longer-term put.¹²

Equity (E) is the market value of equity, defined as common stock plus issues of preferred stock or subordinated debt that count as capital. The firm's risk capital C is not the same thing as its equity, however. The capital-account balance sheet is:

<u>Assets</u>	<u>Liabilities and capital</u>
Assets (A)	Debt or other liabilities (L)
	Capital (C)

Capital is $C = A - L$, the difference between the market value of the firm's assets and the *default-risk free* value of its liabilities. (In practice L would be approximated by the book or face value of debt or other liabilities, with no haircut for default risk.)

Capital C is the “cushion” of assets A over promised payments to creditors or

¹² Our analysis works for any put maturity, with a qualification. We assume that the amount of capital is set at the start of the period and not added to or withdrawn during the period. If the put maturity is long, say three years, then the value of the put will depend on whether the firm is able and willing to raise additional capital if asset value declines in years one or two. Valuing the put becomes a much more complicated dynamic problem if the firm also has to decide on the optimal policy for replacing possible losses in its capital account.

counterparties. Capital is not a pot of cash or money-market securities held in a reserve account. It is a measure of how much equity investment is at risk to protect debt and other liabilities.¹³

The capital-account balance sheet is close to a book balance sheet, because it does not show default value P or the intangible assets or future growth opportunities in G . Thus capital cannot be equity at market value. If it were, a firm could increase its capital simply by increasing asset risk and the risk of default and thus forcing down the market value of the firm's liabilities.

Thus risk capital is not the same thing as the market value of equity. But increases in risk capital require additional equity *investment*. Suppose that the financial firm expands a risky trading desk and therefore decides to put up more risk capital. It could provide the capital by retaining additional net income or by issuing stock. The returns to the additional equity investment are subject to corporate tax. Thus a marginal increase in risk capital means a marginal increase in the present value of future taxes.

2.2 The Default Put and the Credit-Quality Constraint

As Merton and Perold (1993) explain, financial firms often have *credit-sensitive* customers and counterparties. Their customers and counterparties are also their liability holders, who avoid doing business with firms with doubtful credit quality or demand additional collateral. In these cases the credit quality of the firm's products and services is especially important. We define credit quality as the ratio of the value of the default put to the default-risk-free value of one-period unsecured debt. This is the fraction of the

¹³ We say "equity investment," but recognize that the dividing line between capital C and liabilities L may be fuzzy in practice. For example, regulators may include some loan-loss reserves or some types of subordinate debt in their definition of capital.

firm's promised payments or services that liability holders expect *not* to realize. Notice that this ratio translates directly to the credit spread in the market interest rate on the debt.

We assume that the firm has traded off the costs and benefits of more or less capital and decided on a target credit quality. The firm adds or subtracts risk capital (or reallocates it across businesses) to maintain this target.

A value-maximizing firm would decide on the target by trading off the costs and benefits of more or less risk capital, as in the tradeoff theory of capital structure. The costs of holding more risk capital include taxes and agency costs that depend on the amount of equity investment. The costs of holding *less* capital include:

1. Creditors and counterparties face more default risk and therefore spend more on monitoring and may impose more covenants or other restrictions on the firm. They may also increase their own risk-capital balances, which is costly to them for the same reasons that risk capital is costly for the financial firm. The creditors and counterparties pass their increased costs to the financial firm, or may take their business elsewhere.
2. Debt overhang problems increase, as do other incentive problems arising from shareholder-creditor conflicts of interest.
3. The probability-weighted costs of reorganization or liquidation, including fire sales of illiquid assets, increases.

The costs of holding less risk capital are more closely linked to the value of the default put than to the probability of default. The extent of losses in default matters. Also the incentive problems in cost category (2) are directly proportional to the default-

put value.¹⁴ Thus default-put value is the natural measure of default risk for a value-maximizing firm and for its value-maximizing creditors and counterparties.

The tradeoff of the costs and benefits of more or less risk capital leads to an optimal credit quality, which we assume the firm takes as a target and a constraint.¹⁵ We do *not* assume that risk capital is constant: that would give the firm a “free pass” to expand risky businesses, imposing the increased default risk on its creditors, customers and counterparties. Risk capital is a decision variable, not a constant, so long as the credit-quality constraint is binding.¹⁶ If the amount of risk capital is fixed in the short run, then the credit-quality constraint allows expansion of risky businesses only if risk capital is taken from other businesses or if safe businesses are simultaneously expanded.

3. Allocating Risk Capital

In the rest of this paper, “capital” and “risk capital” mean the same thing, although we will sometimes say “risk capital” for emphasis.

¹⁴ The incentive problems arise because common equity behaves like a call option. Put-call parity in our balance-sheet notation is $E = A + P - L$. Therefore any change in default-put value P translates dollar for dollar to equity value E if assets A and liabilities L are constant. Debt overhang means that $dE/dA = 1 + dP/dA$ is less than one because dP/dA is negative. The temptation to risk-shift at lenders’ expense, say by increasing the standard deviation of asset returns σ , arises because $dE/d\sigma = dP/d\sigma > 0$.

¹⁵ If the composition and risks of the firm’s asset portfolio were constant, we could also express the target as an optimal ratio of risk capital to assets. But the problem of risk-capital allocation arises because a financial firm operates businesses with differing risks and must decide which businesses to expand and which to contract. The optimal capital ratio changes when portfolio composition changes and credit quality is held constant. Thus the capital ratio is not a fixed target for an optimizing firm. Corporate finance often posits an optimal target debt ratio, but in that context business risk is implicitly assumed constant.

¹⁶ Therefore we disagree with Grundl and Schmeiser (2007, p. 314), who argue that the cost of holding equity capital is a fixed, common cost, for which allocations must be arbitrary.

3.1 Value of the Default Put

The asset portfolio consists of two or more assets (businesses) with start-of-period values A_i . Thus $A = \sum_i A_i$. The value of the default put is:¹⁷

$$P = PV\left[\max\{0, (R_L L - R_A A)\}\right], \quad (1)$$

where R_L is the gross return to a dollar of debt or other liabilities (one plus the promised interest rate) and R_A is the uncertain gross return on the firm's assets. All returns are assumed to be uncertain except for R_L .¹⁸ The end-of-period promised payoff to liability holders, including interest, is $R_L L$. With complete markets, the present value of the default put is:

$$P = \int_Z [R_L L - R_A A] \pi(z) dz \quad (2)$$

where $\pi(z)$ is a state-price density in the default region Z . This region consists of all outcomes where assets fall short of liabilities and the put is in the money.

Each state z is a unique point in the default region Z . Each point is a combination of returns on the assets (R_i), which generate a portfolio return of $R_A A$. The valuation Eq. (2) sums across the continuum of states, with the payoff in each state z multiplied by the state-price density $\pi(z)$. Note that the states are identified by asset returns and that the state prices $\pi(z)$ are fixed. Therefore an extra dollar delivered in state z by asset A_i has

¹⁷ We simplify by assuming that the firm will default if the put is in the money at maturity. Strategies for optimal default could be more complicated. For example, the firm might raise additional financing in order to avoid default and loss of franchise value G .

¹⁸ We take L as fixed. $R_L L$ is the exercise price of the default put. We could allow for uncertain liabilities, for example insurance contracts, as in Myers and Read (2001), who define marginal default values with respect to liabilities rather than assets. But in our paper it's easier to think of a risky liability as a short position in a risky asset.

exactly the same present value as an extra dollar delivered by A_j . The valuation formula sums across states.

3.2 Allocating Risk Capital Based on Marginal Default Value

Define the *marginal default value* of asset (business) i as $p_i \equiv \frac{\partial P}{\partial A_i}$, the partial derivative of overall put value P with respect to A_i . Marginal default-option values add up exactly and uniquely. The sum of the products of each asset and its marginal default value equals the default value of the firm as a whole. As also shown in Myers and Read (2001), the default value P can be expressed as an asset-weighted sum of marginal default values p_i :

$$P = \sum_i p_i A_i \quad (3)$$

This adding-up result works no matter how finely lines of business are subdivided. For example, if A_i is split into A_{i1} and A_{i2} , the marginal default values p_{i1} and p_{i2} add up exactly as in Eq. (3).

Eq. (3) requires no assumptions about the probability distribution of returns. It does assume sufficiently complete markets, so that the assets A_i have well-defined market values, and also that an across-the-board expansion of a portfolio of businesses does not change the joint probability of rates of return.¹⁹

The capital ratio for the firm as a whole is $c \equiv \frac{C}{A}$. Therefore, $L = (1 - c)A$ and

Eq. (2) can be modified as:

¹⁹ This assumption is of course wrong in some settings, for example if expansion of a business means adding diversifying assets within the business. In this case, the adding-up result would still work for the individual assets, however.

$$P = \int_Z A[R_L(1-c) - R_A] \pi(z) dz \quad (4)$$

This valuation formula says that an across-the-board expansion of assets and liabilities (with c constant) will result in a proportional increase in overall default value. Given c , $\frac{\partial P}{\partial A}$ is a constant for any proportional change.

Expansion of a single line of business will also affect P , but not proportionally. Therefore we also allow capital ratios to vary by line. Define the capital ratio for line i as c_i . Default value is:

$$\begin{aligned} P &= \int_Z \left[\left(\sum_i (1-c_i) A_i \right) R_L - \left(\sum_i A_i R_i \right) \right] \pi(z) dz \\ &= \sum_i \int_Z A_i [(1-c_i) R_L - R_i] \pi(z) dz \end{aligned} \quad (5)$$

The default value per unit of assets is

$$p = \frac{P}{A} = \sum_i \int_Z x_i [(1-c_i) R_L - R_i] \pi(z) dz, \quad (6)$$

where $x_i \equiv \frac{A_i}{A}$. The marginal default values are

$$p_i \equiv \frac{\partial P}{\partial A_i} = \int_Z [(1-c_i) R_L - R_i] \pi(z) dz \quad (7)$$

Our adding-up result still holds: $P = \sum_i A_i p_i$ and $p = \sum_i x_i p_i$. Also, an increase in the marginal capital allocation c_i always decreases the exercise price of firm's default put and reduces its value. Therefore, we can offset differences in p_i by compensating changes in c_i .

If risk-capital allocations are constant ($c_i = c$), marginal default values p_i will vary across lines of business. A firm that allocates capital in proportion to assets, despite varying marginal default values, is forcing some businesses to cross-subsidize others, which contaminates investment decisions, performance measurement, incentives and pricing. The remedy is to vary capital allocation depending on marginal default values, so that each business's capital-adjusted contribution to default value is the same. In other words, capital should be allocated to satisfy the *no risk-shifting* principle: a marginal change in the composition of the firm's portfolio of lines of business does not affect the credit quality of the firm's liabilities.

We derive optimal portfolios and resulting capital-allocation formulas in the next section. But Eq. (7) gives a preview of one result. For a risk-free asset, where $R_i = R_L$, marginal default value is negative at any positive capital allocation c_i .

$$p_i = \int_z [-c_i R_L] \pi(z) dz < 0. \quad (8)$$

We will show that optimal capital-adjusted marginal default values must be all positive and equal across lines. Thus risk-free assets must be given a *negative* capital allocation c_i . Other low-risk assets may also get negative allocations. Note the contrast to allocations based on VaR or contribution VaR. For example, the contribution VaR for a safe asset is zero, since the covariance of the safe return with the firm's overall return is zero.

Before moving in the next section to optimal portfolios, we note two further results. First, marginal default values can be expressed as the sum of a scale term and a business-composition term:

$$p_i = p + \frac{\partial p}{\partial x_i} (1 - x_i) \quad (9)$$

The first term p is the change in default value due to an increase in A , the overall scale of the firm's assets, ignoring any change in the composition of its assets. The second term captures the change in p due to a change in the composition of the asset portfolio $\frac{\partial p}{\partial x_i}$.

The partial derivatives of the default value p and the marginal default values p_i with respect to the allocations c_i are:

$$\frac{\partial p}{\partial c_i} = - \int_Z x_i R_L \pi(z) dz = x_i \left(\frac{\partial p}{\partial c} \right) \quad (10)$$

$$\frac{\partial p_i}{\partial c_i} = - \int_Z R_L \pi(z) dz = \frac{\partial p}{\partial c} \quad (11)$$

Second, the valuation expressions can be simplified by defining $\Pi_Z(R_x) \equiv \int_Z R_x \pi(z) dz$. For example, $\Pi_Z(R_L)$ is the present value of a safe asset's return (but only in the in-the-money region Z , like the payoff on a cash-or-nothing put triggered by default). Write marginal default value p_i as:

$$p_i = (1 - c_i) \Pi_Z(R_L) - \Pi_Z(R_i) \quad (12)$$

The overall default value is

$$p = (1 - c) \Pi_Z(R_L) - \Pi_Z(R_A) \quad (13)$$

Here $(1-c)\Pi_Z(R_L)$ is the present value of the exercise price of the default put, received only if the put is exercised. $\Pi_Z(R_A)$ is the present value of the asset given up if the put is exercised. The difference between these two values is the value of the put.

The present values $\Pi_Z(R_x)$ have exact analytic solutions if returns are normally distributed—see Section 5. However, our results and procedures do not depend on specific probability distributions, so we use this more general notation.

Combining Eqs. (12) and (13), the relationship between the marginal default value for a line of business and the default value for the firm as a whole is

$$p_i - p = -(c_i - c)\Pi_Z(R_L) - (\Pi_Z(R_i) - \Pi_Z(R_A)) \quad (14)$$

We derive capital allocation formulas from Eq. (14) and from the first principle of capital allocation (no risk-shifting): capital should be allocated so that a marginal change in the composition of the firm's business portfolio does not affect the credit quality of the firm's liabilities. We measure credit quality by the ratio of the value of the default put to

the value of default-free liabilities, so this condition is $p_i = \left(\frac{\partial P}{\partial L}\right)\left(\frac{\partial L}{\partial A_i}\right) = \left(\frac{P}{L}\right)(1 - c_i)$.

Putting these results together gives the following formula for allocating capital:

$$c_i = c + \frac{\Pi_Z(R_A) - \Pi_Z(R_i)}{\Pi_Z(R_L) - P/L} \quad (15)$$

Eq. (15) has a clear economic interpretation: *the risk capital allocated to asset i depends on whether the present value of its gross return in default is larger or smaller than the present value of the gross portfolio return in default.*

The term P/L measures credit quality. The firm has to set a target value for P/L in order to allocate capital. It could make a judgment about optimal credit quality, based on

a tradeoff of the costs and benefits of risk capital. Or it could set a target level of credit quality that is acceptable to creditors and counterparties.

Next we consider how a financial firm should (in principle) choose its optimal portfolio of lines of business. Capital-allocation formulas will follow from the conditions for the optimum.

4. Portfolio Optimization and Capital Allocation

Assume that the management of a financial firm calculates the optimum portfolio of lines of business. Financial firms solve this problem implicitly when they set strategy, launch new lines of business or force major restructurings.

We assume that the firm has decided on a target level of credit quality, defined as the ratio of the value of the default put to the value of default-free liabilities. The firm then optimizes subject to the *credit-quality constraint* $P(A,C)/L \leq \alpha$. This constraint is critical. If we did *not* impose it, the optimization would have a “free pass” to shift credit risk to customers, creditors and counterparties.

For example, a financial firm may decide that it needs a single-A credit rating in order to transact with counterparties. It could set α accordingly. (Notice that α determines put value and the credit spread on unsecured borrowing.) Or management may solve the optimization problem at several different levels of credit quality, and then decide which credit quality provides the greatest value.

We assume for simplicity that debt can be raised at market rates in zero-NPV transactions.²⁰ That is, debt markets are assumed competitive and lenders and depositors

²⁰ Therefore, we leave out a sources-equal-uses-of-cash or an investment-equals-financing constraint, because the shadow price of the constraint would be zero.

fully informed. Thus the firm must pay interest rates that fairly compensate lenders for the default risk they bear.²¹ We also assume that headquarters has full information on the types of assets that divisions are investing in and can determine the joint probability distribution of line-by-line returns.

Capital has a tax cost of τ per dollar per period. (We will just refer to tax costs, but τ could also cover agency or other costs of contributing and maintaining capital.) Notice that τ is not the corporate tax rate, but the present value of the corporate tax paid on the expected return per dollar of capital over the next period, that is, over the life of the default put. If the marginal tax rate is $T = 30\%$ and the expected return is 10%, τ is roughly 3%. The cost of allocated capital is *not* an interest rate, a risk-adjusted discount rate or a cost of equity.

The objective is to maximize the market value of the firm. For simplicity we assume the optimal portfolio is chosen once or for one period only.²²

4.1. Portfolio Optimization with a Credit-Quality Constraint

The decision variables are the amount of risk capital C and A_i , the amount of assets held in line i . Total assets are $A = \sum_i A_i$.

²¹ The interest rate paid to depositors includes the value of transaction and other services provided “free of charge” by the bank.

²² We hold franchise value and growth opportunities G constant. Dynamics are more complicated. Froot and Stein (1998) introduce some dynamics of capital structure decisions.

The marginal NPV for line i is $npv_i(A_i)$, with decreasing returns to scale.^{23 24}

Capital is to be allocated line by line at rates c_i , with $C = \sum_i c_i A_i$. The capital ratio for the firm is a weighted average of the line-of-business capital ratios: $c = \sum_i c_i x_i$

where $x_i \equiv \frac{A_i}{A}$. We know that $P = \sum_i p_i A_i$ and $L = \sum_i (1-c_i) A_i$. If we assume the

credit quality constraint is binding, the Lagrange function is:

$$V(\bar{A}, \bar{c}, \lambda) = \sum_i \left[\int npv_i(A_i) dA_i - \tau c_i A_i \right] + \lambda \left(\alpha \sum_i (1-c_i) A_i - \sum_i p_i A_i \right) \quad (16)$$

Notice that the amount of capital C is not fixed. There is no constraint requiring capital allocations to add up to a fixed amount. The conditions for an optimum are:

$$(16a) \quad \frac{\partial V}{\partial A_i} = npv_i(A_i) - \tau c_i + \lambda(\alpha(1-c_i) - p_i) = 0$$

$$(16b) \quad \frac{\partial V}{\partial c_i} = -A_i \left[\tau + \lambda \left(\alpha + \frac{\partial p_i}{\partial c_i} \right) \right] = 0$$

$$(16c) \quad \frac{\partial V}{\partial \lambda} = \left(\alpha \sum_i (1-c_i) A_i - \sum_i p_i A_i \right) = 0$$

²³ Marginal NPV may depend on credit quality, for example because of costs or covenants imposed by nervous counterparties. But credit quality is held constant in this optimization so long as the credit quality constraint is binding. Therefore we do not express $npv_i(A_i)$ as a function of α .

²⁴ We have assumed that investing an additional dollar in line i does not change the joint probability distribution of line-by-line rates of return. This assumption would be violated if the mean return on line i decreased as investment in line i expanded. Therefore we assume that decreasing returns to investment come from an increasing *cost* of achieving a *fixed* mean and probability distribution of R_i . For example, we could specify marginal NPV as $npv_i(A_i) = \Pi(R_i) - e_i A_i$, where $\Pi(R_i)$ is the present value of returns in all states of nature, not just the default region Z , and e_i is a positive cost of expanding line i . In this case the total NPV of investing in line i is quadratic. Capital would be measured after $e_i A_i$ is paid for.

The first condition (16a) tells us that, at an optimum, the marginal line-of-business NPVs are equal to the product of the cost of capital τ and the line-of-business capital allocation c_i : $npv_i(A_i^*) = \tau c_i^*$, where asterisks indicate optimum values. (We show below that the last term in condition (16a) is zero at the optimum.) Thus the ratio of the marginal NPV to the capital allocation rate is the same for all lines.

Condition (16b) is the *no internal arbitrage* principle of capital allocation: if capital allocation rates are set correctly, it will not be possible to add value simply by reallocating capital from one line of business to another. The marginal product of capital is the same in all lines. To see this, recall that $\frac{\partial p_i}{\partial c_i} = \frac{\partial p}{\partial c}$. Therefore, (16b) can be

written as

$$\frac{\partial V}{\partial c_i} = -A_i \left[\tau + \lambda \left(\alpha + \frac{\partial p}{\partial c} \right) \right] = 0 \quad (17)$$

We can also find the shadow price on the credit constraint from Eq. (16b):

$$\lambda = - \frac{\tau}{\alpha + \frac{\partial p}{\partial c}}$$

Condition (16c)—the credit quality constraint—implies that the marginal default values by line of business bear the same relationship to the line-of-business capital allocation rates as the default value for the firm bears to the capital ratio: $p_i = \alpha(1 - c_i)$ and $p = \alpha(1 - c)$. This is our no risk-shifting condition with the firm's credit quality set to α .

Therefore, efficient line-by-line capital allocation rates require:²⁵

$$\frac{p_1(c_1)}{1-c_1} = \frac{p_2(c_2)}{1-c_2} = \dots = \frac{p(c)}{1-c} = \alpha \quad (18)$$

Eq. (18) is not yet a recipe for calculating capital allocations, because the marginal default value p_i depends on c_i , the capital allocation to line i , which we have not yet determined. Eq. (18) just says that capital-adjusted marginal default values must all be the same when expressed as a fraction of liabilities. (Dividing p_i by $1-c_i$ gives the ratio of marginal default value to the debt used at the margin to finance assets in line of business i .)

4.2. Portfolio Optimization with Credit-Quality and Capital Constraints

Capital is a decision variable for long-run planning. But capital is likely to be fixed over the short run. We can add a capital constraint $C \leq \bar{C}$ to the Lagrange function:

$$\begin{aligned} V(\bar{A}, \bar{c}, \lambda, \kappa) = & \sum_i \left[\int npv_i(A_i) dA_i - \tau c_i A_i \right] + \lambda \left(\alpha \sum_i (1-c_i) A_i - \sum_i p_i A_i \right) \\ & + \kappa \left(\bar{C} - \sum_i c_i A_i \right) \end{aligned} \quad (19)$$

The conditions for an optimum with this additional constraint are:

$$(19a) \quad \frac{\partial V}{\partial A_i} = npv_i(A_i) - (\tau + \kappa)c_i + \lambda(\alpha(1-c_i) - p_i) = 0$$

$$(19b) \quad \frac{\partial V}{\partial c_i} = -A_i \left[\tau + \kappa + \lambda \left(\alpha + \frac{\partial p_i}{\partial c_i} \right) \right] = 0$$

²⁵ This result may appear inconsistent with Myers-Read (2001), who conclude that marginal default values in all lines of business must be equal to avoid cross subsidies. But they define marginal default values with respect to liabilities, not assets. Our result is equivalent to the condition that marginal default values with respect to *liabilities* are the same in all lines.

$$(19c) \quad \frac{\partial V}{\partial \lambda} = \left(\alpha \sum_i (1-c_i) A_i - \sum_i p_i A_i \right) = 0$$

$$(19d) \quad \frac{\partial V}{\partial \kappa} = \bar{C} - \sum_i c_i A_i = 0$$

If all constraints are binding, the marginal NPV in each line of business is equal to the product of the line's capital allocation rate c_i and the “all-in” cost of risk capital $\tau + \kappa$:

$$npv_i(A_i^*) = (\tau + \kappa) c_i^*. \quad \text{The shadow price on the capital constraint is } \kappa = \frac{npv(A^*)}{c^*} - \tau.$$

The shadow price on the credit constraint is $\lambda = - \frac{\tau + \kappa}{\alpha + \frac{\partial p}{\partial c}}$. As before, $p_i = \alpha(1 - c_i^*)$.

Notice that “assets” in these optimizations correspond to cash capital. They are the amounts required to purchase assets at their current (mark-to-market) values. Thus it would be straightforward to extend our set-up to incorporate a constraint on cash capital as well as a constraint on risk capital.

4.3. Capital Allocation

We know from Section 2 that capital allocations should be set line by line using Eq. (15).

Eq. (18) requires $\frac{p_i(c_i)}{1-c_i} = \frac{p(c)}{1-c} = \alpha$. The formula for efficient capital allocation

becomes:

$$c_i = c + \frac{\Pi_Z(R_A) - \Pi_Z(R_i)}{\Pi_Z(R_L) - \alpha} \quad (20)$$

Here the constrained level of credit quality α substitutes for the general credit quality measure P/L in Eq. (15).

Thus the capital allocated to line of business i depends on the firm's overall capital ratio c , on credit quality α , and on the difference in default payoff values for the overall firm vs. the business. The allocation does not depend directly on the investment in business i , but only indirectly, because decisions about investments, capital and capital allocation are made jointly when the firm optimizes. But for capital allocation, the joint optimization need deliver only the overall capital ratio c , the overall default payoff values $\Pi_Z(R_A)$ and $\Pi_Z(R_L)$, and line i 's default payoff value $\Pi_Z(R_i)$.

Marginal capital allocations for an asset or business therefore depend on the present value of its returns in default, that is, on the present value of its returns as distributed across the default region Z . If its returns are "riskier" than the overall portfolio return R_A in region Z —that is, worth *less* than the overall portfolio return in that region—then $c_i > c$. If its returns are relatively "safe" in region Z —worth *more* than the overall return in default—then $c_i < c$. The capital ratio for line i does *not* depend on the line's marginal effect on the probability of default. It depends on the *value* of the line's payoff in default.

Thus capital can be allocated depending on the marginal default value of each line of business, where marginal default value is the derivative of the value of the firm's default put with respect to a change in the scale of the business. Marginal default values give a unique allocation that adds up exactly. Differences in marginal default values can be offset by differences in marginal capital allocations. Cross-subsidies are avoided if capital allocations are set so that capital-adjusted marginal default values are the same for all lines, as in Eq. (18). Each line's capital ratio should depend on the value of the line's payoffs in default. The procedure of setting marginal capital requirements to equalize

capital-adjusted marginal default values is a condition for optimization of the firm's portfolio of businesses.

4.4 Calculating Marginal APV

Eqs. (16a) and (19a) define marginal adjusted present value (APV).²⁶ The marginal APV for line of business i is

$$\text{APV}_i \equiv \frac{\partial V}{\partial A_i} = \text{npv}_i(A_i) - (\tau + \kappa)c_i \quad (21)$$

The first term is the marginal NPV of business i , calculated as if risk capital were costless. The second term is a charge for the tax and other costs of risk capital, plus a shadow price κ if the constraint on risk capital is binding. The charge is proportional to the capital allocation rate c_i . The charge is *not* expressed as a rate of return on allocated capital. It is not an interest rate, risk-adjusted rate of return or cost of equity.

Of course a risk-adjusted rate of return may be used as a discount rate for calculating the first term (NPV) in the APV formula. That rate should depend on the market risks of future cash flows, but not on the amount or cost of allocated risk capital. A risk-free interest rate would be used for risk-neutral valuation of derivative positions, for example.

The APV formula demonstrates that the amount invested depends on both (a) profitability and market risks, which determine NPV, and (b) firm-specific risks, which determine capital allocations and risk-capital charges $c_i(\tau + \kappa)$. The risk-capital charge changes net value but does not by itself determine the optimal scale of the business. For example, when $c_i < 0$ for safe assets, marginal APV increases, because incremental

²⁶ The third term in these equations equals zero. See Eq. (18).

investment in safe assets loosens the credit-quality constraint and frees up risk capital for riskier business in the financial firm's portfolio. But $c_i < 0$ does not mean investment in the safe assets is negative.²⁷

5. Default Values and Capital Allocation for Normal Distributions

If asset returns are normally distributed, the return to a portfolio of assets is normally distributed too. This allows closed-form formulas for marginal default values and capital allocations.²⁸ The formulas illustrate how our allocations work and what the allocations depend on. The formulas also make it relatively easy to construct and interpret numerical examples. A detailed example showing how capital allocations follow from the conditions for an optimal portfolio is in the Appendix.

The default value P depends on the market value of assets A , the market value of default-free liabilities L , and on σ_A , the standard deviation of end-of-period asset returns per unit of assets. The present value of the default option is:

$$P(A, L, \sigma_A) = (L - A)N\{y\} + \sigma_A A N'\{y\} \quad (21)$$

where N is the cumulative distribution function for a standard normal variable and

$y = \frac{L - A}{\sigma_A A}$. Capital is defined as the market value of assets less the market value of

default-free liabilities, so the present value of the default option can also be expressed as

²⁷ But $c_i < 0$ could be interpreted as negative equity investment, that is, the ability to finance safe assets *at the margin* with more than 100% debt. Of course a firm holding *only* safe assets could not borrow more than 100% of the assets' value.

²⁸ The assumption that returns are normally distributed, which we make for illustrative purposes, may strike some readers as inconsistent with limited liability. Whereas shareholders have limited liability, however, this is not necessarily true for the business units of a corporation. Consider the desks of a trading firm. See Merton (1997), who begins with the assumption that the surplus of the firm's assets is normally distributed and shows that the value of the firm's equity is log-normally distributed.

a function of assets and capital: $P(A, C, \sigma_A) = -C N\{y\} + \sigma_A A N'\{y\}$, where

$y = -\frac{C}{\sigma_A A}$. The default value per unit of assets is a function of the capital ratio:

$$p(c, \sigma_A) = -c N\{y\} + \sigma_A N'\{y\}, \quad (22)$$

where $y = -\frac{c}{\sigma_A}$.

Remember from Eq. (9) that $p_i = \frac{\partial P}{\partial A_i} = p + \frac{\partial p}{\partial x_i}(1-x_i)$. The change in p due

to a change in the composition of the portfolio ($\frac{\partial p}{\partial x_i}$) can also be written as

$$\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial c} \frac{\partial c}{\partial x_i} + \frac{\partial p}{\partial \sigma_A} \frac{\partial \sigma_A}{\partial x_i}. \quad \text{But } \frac{\partial c}{\partial x_i} = \frac{c_i - c}{1 - x_i} \text{ and } \frac{\partial \sigma_A}{\partial x_i} = \frac{\sigma_{iA} - \sigma_A^2}{\sigma_A(1 - x_i)}, \text{ so the marginal}$$

default value for each line of business is:

$$p_i = p + \frac{\partial p}{\partial c}(c_i - c) + \frac{\partial p}{\partial \sigma_A} \left(\frac{\sigma_{iA} - \sigma_A^2}{\sigma_A} \right) \quad (23)$$

where σ_{iA} is the covariance of the return on line of business i with the portfolio return.

The option delta ($\frac{\partial p}{\partial c}$) and vega ($\frac{\partial p}{\partial \sigma_A}$) are:

$$\frac{\partial p}{\partial c} = -N\{y\} \quad (24a)$$

$$\frac{\partial p}{\partial \sigma_A} = N'\{y\} \quad (24b)$$

The option delta is negative, so the higher the capital allocation, the lower the marginal default value. The option vega is positive, so the higher the covariance of returns, the higher the marginal default value.

We combine the constraint on credit quality from Eq. (18) and the expression for marginal default value in Eq. (23) and solve for the capital allocation for line i :

$$c_i = c - \left(\frac{\partial p}{\partial c} + \alpha \right)^{-1} \left(\frac{\partial p}{\partial \sigma_A} \right) \left[\frac{(\sigma_{iA} - \sigma_A^2)}{\sigma_A} \right] \quad (25)$$

Thus the marginal capital allocations in the normal case depend on the delta and vega of the default put and on the difference between σ_{iA} , the covariance of asset i 's return with the overall return, and the variance of the overall return σ_A^2 . Riskier assets ($\sigma_{iA} > \sigma_A^2$) must be allocated extra capital ($c_i > c$). Safer assets ($\sigma_{iA} < \sigma_A^2$) require less capital ($c_i < c$). Safe or low-risk assets have negative capital allocations.

6. Applications and Implications

We recommend that financial firms allocate capital to lines of business based on marginal default values. One can think of the allocation procedure in two steps. First identify each line's marginal impact on the value of the firm's default put. (The value of the default put is small for well-capitalized firms, but nevertheless positive. The credit spreads demanded by lenders prove that even a small default put is material.) Some lines of business will have larger marginal impacts than others. Second, calibrate the marginal capital allocated to each line of business so that capital-adjusted marginal default values are the same for all lines. See Eq. (15) or Eq. (20), which hold for any joint probability distribution of returns, and Eq. (25) for the joint normal distribution.

6.1 APV

If risk capital is fixed in the short run, the “all-in” cost of a dollar of allocated risk capital equals $\tau + \kappa$, that is, the sum of the tax or other costs of risk capital (τ) and the shadow price of the constraint on risk capital (κ), if the constraint is binding.

The all-in cost is *not* an interest rate or “cost of equity.” Suppose that the managers of business 1 are considering expanding assets by \$1 million. First they should evaluate the NPV of the expansion using a discount or hurdle rate matched to the market risk of future returns from business 1. In other words, they should respect the corporate-finance principle that the discount rate is an opportunity cost of capital, which depends on the market risk(s) of the investment, not on the risk of the firm’s overall portfolio, on the interest rate that the firm pays to borrow, nor on the allocated capital amount. The opportunity cost of capital depends on the use of funds, not the source. Suppose NPV, calculated according to these principles, is +\$0.1 million. Second, the managers should determine a risk-capital allocation for the additional assets. Say it is \$0.2 million, 18.2% of the PV of \$1.1 million ($c_1 = 0.182$). Assuming a 4% all-in cost for risk capital, the APV is $0.1 - 0.04 \times 0.182 = +\0.093 million.

The tax charge is the present value of taxes paid in the next period on net income to equity. The certainty-equivalent formula for this present value is $PV = Tr_f / (1 + r_f)$ per dollar of equity,²⁹ where T is the marginal corporate rate and r_f is the risk-free interest rate.

The optimal scale for business i is reached when marginal $APV_i = 0$. The business as a whole can be valued by the same APV method, using the marginal rate c_1

²⁹ This is the “Myers theorem” in Derrig (1994). Strictly speaking, the formula here applies to a marginal dollar of additional equity invested to provide risk capital.

and the mark-to-market PV of the all the business's assets. Of course the capital-allocation rate c_1 will change if the business grows to a substantially larger fraction of the firm's overall portfolio. A large investment or disinvestment decision requires comparison of APVs with versus without the decision, calculated using different risk-capital allocations.

Our capital-allocation procedures apply for any joint probability distribution of returns. Computing marginal default values and capital allocations is straightforward in principle. But the default put is a deep out-of-the-money option, and the lower tail is the part of the joint distribution of returns about which the least information is available. Thus information, not computation, is probably the biggest challenge for practice. Internal agency issues are a further, and perhaps more severe, practical challenge facing any capital-allocation method. (See Stoughton and Zechner (2007)).

6.2 Negative allocations of risk capital

We have shown that risk-free or low-risk businesses should get *negative* capital allocations. The reason is that marginal expansion of low-risk assets reduces the value of the default put, freeing up risk capital.

Suppose that negative allocations are ruled out and zero capital is allocated to safe assets ($c_i = 0$). Then expansion of safe businesses would decrease default-put value, violating the no-risk-shifting principle that risk capital should be allocated so that expansion of a line of business does not affect credit quality. The principle is restored by allocating negative capital to the safe asset.

The following thought experiment may be helpful. Suppose a financial firm holding a portfolio of risky businesses borrows \$100 million by an unsecured one-period

loan and invests the proceeds in one-period risk-free assets. There is no additional equity investment and no net effect on P , the total value of the default put, because the return from the \$100 million of safe assets exactly covers the additional liability. Thus it may appear that the risk-capital requirement for the additional safe assets is zero. That appearance is misleading, however, because *credit quality* P/L improves: P is constant but liabilities L have expanded. If the firm was operating at its target credit quality before the transaction, it can now achieve the same credit quality with less capital. Therefore it could borrow more than \$100 million and use the additional proceeds to reduce equity investment and risk capital.

Of course the firm holds the risk-free assets in a portfolio with other risky assets. Suppose the firm continues to borrow more and more and to invest more and more in safe assets. The average risk of the firm's portfolio falls, allowing the firm to operate with less and less risk capital. As the fraction of the firm's portfolio invested in the risk-free asset increases, the marginal capital allocation for the safe asset remains negative, but declines in absolute value, approaching zero from negative territory.

Do not confuse stand-alone risk-capital requirements with marginal requirements in a portfolio context. The risk capital required for a stand-alone position in risk-free assets is zero. But the marginal risk capital required for a safe asset held in a portfolio with risky assets is negative.

Allocations based on VaR or contribution VaR and thus the allocations normally underlying RAROC allocate zero capital to risk-free assets. The implicit argument is that any asset or activity with uncertain returns requires risk capital. Notice that this argument takes the target level of risk as zero. But there is no reason in theory or practice

to set a zero-risk target. An optimal balance of the costs and benefits of risk capital does not eliminate all credit risk. Financial firms in practice seek to maintain credit quality, but do not add more and more risk capital in an attempt to force their credit spreads to zero. Therefore we assume that the firm sets a target level of credit quality that is acceptable but not perfect ($P/L = \alpha > 0$). The capital allocation for business i depends on whether a marginal expansion of the business improves or degrades credit quality.

VaR or RAROC methods may encourage *over*-investment in the riskiest assets. If negative allocations are ruled out and the firm's overall capital C is fully allocated, then the riskiest assets are cross-subsidized, because they are allocated less capital than required by their marginal default values.

6.3 RAROC

The APV formula clarifies the difficulties built in to RAROC as a measure of risk-adjusted profitability. RAROC is an after-tax rate of return on the “economic” capital allocated to (and implicitly invested in) a business or bundle of assets. We have noted the problems in calculating risk capital based on VaR or contribution VaR. But suppose that risk capital as defined here is deployed as economic capital for RAROC – and suppose that risk capital is positive.³⁰ Economic capital could then be interpreted as equity investment, with debt financing for the rest of any required investment. Income would be computed after interest and tax. What then is the hurdle rate or “cost of equity”? The hurdle rate should be defined so that RAROC falls equal to the hurdle rate

³⁰ RAROC is not useful if economic capital is interpreted as risk capital and risk capital is negative. Negative risk capital works in an APV calculation because the “reward” for “freeing up” risk capital is separate from the calculation of NPV.

when marginal APV = 0. (Recall that the APV measure is consistent with the conditions for an optimal portfolio of businesses for a value-maximizing firm.)

We have found no tractable formula for a correct RAROC hurdle rate, although it is clear that a single “cost of equity” cannot be used for businesses that differ in market risk(s) and in risk-capital requirements.

A correct single hurdle rate on economic capital would somehow combine two different concepts of risk: (1) market risk and (2) firm-specific risk. The hurdle rate should incorporate market risk premiums based on exposures to priced risk factors in financial markets. Market risk premiums determine the opportunity cost of capital, which should not depend on asset- or firm-specific risks that are diversifiable by outside investors. The hurdle rate should also incorporate the costs of allocated risk capital. But risk capital depends on all risks, including diversifiable risks, that contribute to the risk of the firm’s overall portfolio and therefore to its default put. The problem with RAROC is that it attempts to combine type (1) and (2) risks in one hurdle rate of return. The APV formula is straightforward and general because NPV depends on type (1) risks only and the allocated costs of risk capital depend on type (2) risks only.

Of course there is always a hurdle rate that values assets or a business correctly in a RAROC setting, but finding the correct one rate will usually require calculating APV first. Therefore use APV.

6.4 Derivatives

Our formula gives capital allocations as a fraction of assets. How then can risk capital be allocated to a forward contract, swap agreement, or other derivative with zero net value? The solution is straightforward: unbundle the contract into components—

each of which has non-zero value—and calculate the capital allocation for each component. The capital allocation for the contract is the sum of the capital requirements for the components. For example, the value of a long one-period forward contract for jet fuel equals (1) the present value of the commitment to receive jet fuel at date 1 less (2) the present value of the commitment to pay the forward price at date 1. The risk capital allocated to leg (1) depends on the uncertainty about the future spot price of jet fuel and correlations of the spot price with returns on the firm’s other assets and liabilities. Leg (2) of the contract amounts to a short position in a safe asset. Safe assets are allocated negative capital, so the short position should be allocated positive capital. (The negative capital ratio $c_i < 0$ is multiplied by the negative short position, giving a positive allocation of the costs of risk capital.)³¹

The same unbundling method can be used to allocate risk capital to an option. For example, the value of a call over the next short interval equals (1) the option delta times the value of the underlying minus (2) risk-free borrowing. The capital allocated to leg (1) equals delta times the capital that would be allocated to the underlying. Leg (2) is a short position in a safe asset.

Capital allocations for derivatives can shift dramatically depending on in the composition of the firm’s portfolio. Suppose firm A’s proprietary trading desk takes a naked long position in jet-fuel forwards with a very large positive marginal default value.

³¹ Here we follow standard derivative-pricing models and assume that the obligation to pay is default-risk free. In practice collateral is often required to insure against default. But providing collateral does not change the firm’s overall credit quality; it simply creates a class of senior creditors, here the forward counterparty. All other creditors are junior. The split of senior from junior creditors does not affect the firm’s default put value P or its credit quality overall. We admit, however, that the management of collateral positions in practice may encounter transaction costs, borrowing-lending spreads and other frictions not considered here. Understanding the role of collateral when such frictions are present and risk capital is costly is a worthwhile topic for further research.

The position should be allocated positive risk capital at $c_i > c$. But the same long position could be a hedge for firm B if B is short jet fuel. In this case the long position has negative marginal default value and should be allocated negative capital. Again recall that negative allocated risk capital does not mean negative investment of cash capital. The cash capital required at the start of a fairly priced forward contract is zero.

6.5 Risk-Capital Allocation for Non-financial Corporations

In corporate finance, the tax-adjustment term in APV is usually expressed as a tax advantage of debt rather than a tax cost of equity. NPV is calculated after tax at an opportunity cost of capital, as if the investment were all-equity financed, and the present value of interest tax shields is then added to get APV. The interest tax shields depend on the amount of debt supported by the investment.³² In our setting, where the firm is allocating capital, NPV should be calculated as if the investment were 100% financed by “tax-free” financing. (Thus our analysis explains why financial firms typically value derivatives pre-tax.) The tax cost of the allocated capital required to support the investment is then subtracted. After-tax NPVs would double-count taxes.

These setups are of course equivalent, two sides of the same coin, but this paper’s APV setup is probably a better fit for financial firms, which are usually highly leveraged and regard risk capital as the scarce resource.

This paper’s APV setup is in some ways simpler than the corporate-finance APV, because NPV is pretax, and taxes enter only as a tax charge on allocated capital. Of

³² Under certain conditions the APV of an investment can be calculated in a single step by discounting all-equity after-tax cash flows at a “weighted average cost of capital” (WACC). In other words, there are conditions under which a single “hurdle rate” correctly captures both the market risk of the cash flows and the (tax) costs of risk capital. WACC assumes, however, that business risk and the debt ratio is constant over time (and across assets if the firm is using WACC for all investments). See Myers (1974) and Brealey, Myers and Allen (2013), Ch. 19.

course our APV formula values the asset or business for one period only, and capital allocations are likely to change. APVs in corporate finance are calculated for longer-lived, illiquid investments, where it is often important to model taxes in project-specific detail.

The theory of optimal capital structure for non-financial corporations usually seeks the target debt-to-value ratio that maximizes firm value. Our analysis says that the firm should not target a debt ratio, but rather its credit quality, defined as a ratio of default-put value to liabilities, or in practice as a credit spread or debt rating. The debt ratio required to meet the credit-quality target will then vary as the risks of lines of business change and as the firm's portfolio of businesses evolves. Therefore tests of the tradeoff theory that assume a constant target debt ratio are mis-specified unless business risk is constant.

Our theory identifies a pure financial motive for corporate diversification. As Lewellen (1971) argued, the capital required to achieve a given credit quality will be lower for a firm that operates in two or more imperfectly correlated lines of business than it will be for a firm that operates in fewer lines of business. But this argument for diversification may apply better to financial than non-financial corporations. The non-financial sector is not dominated by conglomerates.

6.6 Risk-Based Regulatory Capital Requirements

Regulators do not allocate risk capital, but they set risk-based capital requirements, which sounds like nearly the same thing. Therefore we consider whether a regulator could set risk-based capital requirements that (1) limit the size of the regulated firm's default put and (2) still allow the firm to operate at its portfolio optimum. We will refer to banks,

although our conclusions apply to any financial firm subject to prudential regulation. We are *not* here concerned with “regulatory arbitrage,” which is the substitution of risky for safer assets *within* a risk category defined by regulators. Regulatory arbitrage is a difficult but distinct problem.³³

Assume that the bank can raise additional capital if necessary. The regulator sets out to enforce better credit quality than the bank would choose on its own. This can be done in two ways. First, the regulator could require a lower ratio of default-put value to the bank’s debt, including deposits, and other liabilities. This means setting $\hat{\alpha} < \alpha$ in Eq. (16). (The “hat” indicates a parameter or variable determined by the regulator.) The bank would then choose its optimal portfolio subject to the tighter credit-quality constraint.

This regulatory strategy seems ideal, because there would be no distortion of the bank’s investments or internal capital allocations. The practical problem is for the regulator to monitor and observe credit quality and to demand additional capital if the credit-quality constraint (using $\hat{\alpha}$) is not met. The credit spreads demanded by the bank’s creditors and counterparties could assist the regulator, but monitoring credit quality cannot be outsourced to creditors and counterparties if the bank is too big to fail or if creditors and counterparties will be bailed out in a crisis. In that case observed credit spreads will understate the value of the put absorbed by the regulator or government.

Second, the regulator could set risk-based capital requirements business by

³³ Palia and Porter (2003) include a good description of risk-weighted capital requirements and regulatory capital arbitrage.

business. This intervention changes the firm's optimization to:

$$V(\bar{A}, \bar{c}, \lambda, \hat{\kappa}) = \sum_i \left[\int npv_i(A_i) dA_i - \tau c_i A_i \right] + \lambda \left(\alpha \sum_i (1 - c_i) A_i - \sum_i p_i A_i \right) + \hat{\kappa} \left(\sum_i \hat{c}_i A_i - \sum_i c_i A_i \right) \quad (26)$$

Notice that the risk-based capital requirements do not replace the bank's internal capital allocations c_i . The bank still wants to allocate capital efficiently, subject to the regulatory constraint.

The conditions for an optimum with risk-based capital requirements are:

$$(26a) \quad \frac{\partial V}{\partial A_i} = npv_i(A_i) - \tau c_i + \lambda(\alpha(1 - c_i) - p_i) + \hat{\kappa}(\hat{c}_i - c_i) = 0$$

$$(26b) \quad \frac{\partial V}{\partial c_i} = -A_i \left[\tau + \hat{\kappa} + \lambda \left(\alpha + \frac{\partial p_i}{\partial c_i} \right) \right] = 0$$

$$(26c) \quad \frac{\partial V}{\partial \lambda} = \left(\alpha \sum_i (1 - c_i) A_i - \sum_i p_i A_i \right) = 0$$

$$(26d) \quad \frac{\partial V}{\partial \hat{\kappa}} = \sum_i \hat{c}_i A_i - \sum_i c_i A_i = 0$$

This optimization is similar to the capital-constrained optimization in Eq. (19) except that the regulatory capital requirement $\hat{C} = \sum_i \hat{c}_i A_i$ replaces the fixed capital amount \bar{C} . The

“all-in” cost of capital is now $\tau + \hat{\kappa}(\hat{c}_i - c_i)$. The shadow price on the capital constraint is

$$\hat{\kappa} = \frac{npv(A^*)}{c^*} - \tau. \quad \text{The shadow price on the credit constraint is } \lambda = -\frac{\tau + \hat{\kappa}}{\alpha + \frac{\partial p}{\partial c}}. \quad \text{As}$$

before, $p_i = \alpha(1 - c_i^*)$.

The bank will choose investment in line i by solving for the marginal APV:

$$apv_i \equiv \frac{\partial V}{\partial A_i} = npv_i(A_i) - \tau c_i + \hat{\kappa}(\hat{c}_i - c_i) = 0 \quad (27)$$

The bank will under-invest in line i where $\hat{c}_i > c_i$ and over-invest where $\hat{c}_i < c_i$, depending on how tight the regulatory constraint is and the size of its shadow price $\hat{\kappa}$.

This regulatory strategy will *always* distort the bank's investments, unless the regulator could set the capital requirements exactly equal to the internal capital allocations that the bank would choose if subject to a tighter regulatory constraint on credit quality. This ideal result seems impossible: The regulator would have to understand the bank's businesses, including possible new businesses; profit margins $npv_i(A_i)$ by line, and the joint probability distribution of line-by-line returns. But that omniscient regulator could achieve the same ideal result more simply and directly by tightening the credit-quality constraint and then tracking credit quality. In that case the risk-adjusted capital requirements would be at best redundant.

The idea that risk-based capital requirements could be non-distorting becomes more incredible when one realizes that efficient risk-capital allocations depend on the composition of the bank's portfolio of businesses and the correlations of the businesses' returns.³⁴ Custom capital requirements would be required for each bank. Also regulators would have to set *negative* risk-based capital requirements for safe and many low-risk assets.

It may be that risk-based capital requirements are "better than nothing." The real issue is whether they are better than setting constraints on overall credit quality or

³⁴ Gordy (2003) derives conditions in a VaR setting for risk-based capital requirements that do not depend on the composition of the bank's portfolio of lines of business. The conditions are extremely restrictive, however. For example, there must be a single common factor driving line-by-line returns.

assuring bulletproof credit quality by requiring all banks to hold much more capital than has been customary. The tax cost of holding additional capital could easily be offset by allowing banks to hold matching amounts of low-risk assets tax-free.

7. Conclusions

We argue that capital can and should be allocated based on the marginal default value of each line of business, where marginal default value is the derivative of the value of the firm's default put with respect to a change in the scale of the business. Capital allocations are relevant for pricing, performance measurement, incentives, compensation, and trading and hedging decisions.

Capital allocations based on marginal default values add up exactly. This adding-up result requires sufficiently complete markets, complete enough that the firm's assets and default put option have well-defined market values, but does not require any restrictions on the joint probability distributions of returns. Allocations will be sensitive to distributional assumptions, however.

Differences in marginal default values across lines of business should be cancelled out by offsetting differences in marginal capital allocations. We calculate the resulting capital allocations. The allocations are systematically different from allocations proportional to VaR or contribution VaR.

We believe our allocations are correct. For example, they can be derived from the conditions for the optimal portfolio of businesses chosen by a value-maximizing firm. Of course no bank or financial firm solves an explicit mathematical program to determine its optimal portfolio at the start of every period. Usually the firm takes its existing portfolio as fixed or considers gradual marginal changes. But capital allocations are typically set

for short periods, say a quarter or a year. It seems reasonable to take the existing portfolio as given for the next short period.

Sometimes a bank or financial firm has to decide whether to add or subtract a line of business or a significant block of assets—see Merton and Perold (1993). The decision hinges on whether the bank is better off with or without the business or assets. Capital allocations “with” are not the same as “without.” All capital allocations can change after a discrete investment. The only general way to evaluate discrete changes is to compare value with versus without, using different capital allocations.

If the discrete change is small relative to the bank’s overall assets, allocations for existing lines can be a good approximation if the bank has many existing businesses and if business that is changed is not too large. Allocations for a business that is expanded or contracted can be very sensitive to the magnitude of the change, but allocations to existing businesses can be much more stable and for practical purposes may not have to be adjusted frequently.³⁵

Consider a proposal to add an entirely new business. The new business’s present value is reduced by the cost of the capital allocated to it. The investment is worthwhile if its APV is positive, taking the mix of existing businesses as constant. APV equals NPV minus the all-in cost of allocated capital, which includes the tax or other costs of holding capital and a shadow price if the amount of capital is constrained.³⁶ The amount of capital allocated increases steadily as the scale of the new business increases. Thus

³⁵ Myers and Read (2001) perform experiments showing that allocations to existing lines change slowly when new lines are added and subtracted. Of course these results are reassuring only if portfolio composition changes gradually.

³⁶ If raising equity capital is feasible but incurs transaction costs, the marginal transaction costs should be charged against APV in place of the shadow price on the capital constraint.

capital allocation is a source of decreasing returns to investment.³⁷ Optimal scale for a new business (holding existing assets constant) is reached when APV, net of the all-in cost of allocated capital, is zero at the margin.

We have assumed tax costs of holding capital. Capital is also said to be costly because of agency costs of equity, including information and monitoring costs borne by outside shareholders.³⁸ See Merton and Perold (1993) and Perold (2005), for example. These costs surely exist, but it is unclear whether the costs are proportional to the amount of capital, as we have implicitly assumed.

An MM-style fallacy may be lurking here. Saying that financing from outside equity introduces agency costs does not necessarily mean that the costs are reduced with marginally less equity and more debt. The agency costs of equity for financial firms may come primarily from the asset side of their balance sheets. If so the value lost because of agency could be roughly constant over a range of capital ratios.

³⁷ Line-by-line APVs could not be used to construct the optimal overall mix of business, however. The APV of each business would depend on the order in which candidate businesses were evaluated. This problem is highlighted by Merton and Perold (1993).

³⁸ Risk capital may also be costly if “MM doesn’t apply to banks,” so that a bank’s weighted-average cost of debt and equity increases with ratio of equity to debt even when taxes and agency costs are set aside. For example, DeAngelo and Stulz (2013) argue that high leverage is optimal in a model of bank capital where there is a market premium for liquid financial claims. Their argument could identify another contributor to the cost τ of risk capital. See on the other hand Admati and Hellwig (2013).

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Appendix: Numerical Examples

Consider a firm selecting a portfolio of businesses from scratch. The firm can invest in either one or both of two lines of business. Let $x_1 = x$ and $x_2 = 1 - x$ be the proportions of total assets invested in the two lines. Assume that asset returns are normally distributed. The firm's liabilities are riskless except for the possibility of default.

The goal is to maximize APV, defined here as the NPV of the investments less the cost of risk capital. The NPV of each line $i = 1, 2$ is equal to $\int npv_i(A_i) dA_i$, where $npv_i(A_i)$ is the marginal NPV. Both lines of business are subject to decreasing returns to scale. We assume that the relationship between the NPV and scale of each line of business has the quadratic form $NPV(A_i) = \frac{1}{2}a_i A_i^2 + b_i A_i + c_i$, with $a < 0$, $b > 0$, and $c = 0$, so that the marginal NPV is $npv_i(A_i) = a_i A_i + b_i$ and the average NPV is $\overline{npv}_i(A_i) = \frac{1}{2}a_i A_i + b_i$.

We have to impose a constraint on credit quality because our objective function includes the *cost* of capital but not any *benefits*. An unconstrained solution would set capital to zero. But if the optimal solution includes positive investment in at least one line ($A^* > 0$), then the credit quality constraint will be binding.

Suppose the firm wishes to operate with a default value that does not exceed a fraction α of the present value of liabilities: $P \leq \alpha L$. The optimal capital ratio—the minimum capital ratio consistent with this credit quality constraint—is completely determined by the minimum credit quality α and the risk of the portfolio σ . The risk of the portfolio is determined by the standard deviations of asset returns σ_1 and σ_2 , the

correlation of asset returns ρ , and the asset allocation (x). Therefore, the optimal capital ratio is $c^* = c^*(\alpha, \sigma(x))$.

Total assets at any fixed combination of the two lines of business can be found by solving for the point at which the marginal NPV is equal to the cost of capital:

$$npv_x(A^*, x) = x npv_1(xA^*) + (1-x) npv_2((1-x)A^*) = \tau c^*(\alpha, \sigma(x)) \quad (28)$$

Given assets A^* and capital ratio c^* , we can calculate capital C , default-free liabilities L , and the default value P :

$$C = c^*(\alpha, \sigma(x)) A^*(x) \quad (29a)$$

$$L = (1 - c^*(\alpha, \sigma(x))) A^*(x) \quad (29b)$$

$$P = \alpha L \quad (29c)$$

The marginal default values (p_i) and capital allocation rates (c_i) by line of business can be calculated using Eqs. (23) and (25).

Let x^* be the asset allocation at which the APV of the portfolio is a maximum, where APV is $A \left[x \overline{npv}_1(xA) + (1-x) \overline{npv}_2((1-x)A) - \tau c(x) \right]$. At the optimum, the ratio of the marginal NPV to capital is the same in both lines of business and is equal to the cost of capital: $\frac{npv_1(A_1)}{c_1^*(x^*)} = \frac{npv_2(A_2)}{c_2^*(x^*)} = \tau$. Also, the marginal APV in both lines

is zero:

$$npv_1(A_1) - \tau c_1^*(x^*) = npv_2(A_2) - \tau c_2^*(x^*) = 0. \quad (30)$$

To obtain numerical results, we assume that the standard deviations of asset returns for Line 1 and 2 are 10% and 30%, respectively, and the returns are uncorrelated.

The NPVs are 2% minus 0.0001% of Line 1 assets and 3% minus 0.0001% of Line 2 assets. The tax cost of capital is $\tau = 3\%$. Assume the firm seeks to maintain credit quality $\alpha = 1\%$. Results are summarized in Appendix Table 2. The first column contains the results at the optimal asset allocation, where the APV of the firm's investments is at the maximum. The columns to the right give results for allocations ranging from 100% in Line 1 (0% in Line 2) to 100% in Line 2 (0% in Line 1). In all cases the constraint on credit quality is binding.

The optimum asset allocation calls for investing 54.46% of assets in Line 1, which is relatively safe, and 45.54% in Line 2. Portfolio asset risk is 14.71% and the capital ratio is 17.66%. The firm has assets of \$38,205 and requires capital of \$6,749. Capital allocations are - 2.69% (yes, that is a minus) of assets in Line 1 and 42.00% of assets in Line 2. This means - \$559 of capital is allocated to Line 1 and \$7,308 of capital is allocated to Line 2. The APV of the portfolio is \$368, which is equal to the \$570 NPV minus the cost of capital (3% of \$6,749). The market value of the firm is lower at all other asset allocations.

At the optimal asset allocation, the higher gross profitability of Line 2 is exactly offset by its higher capital cost. Thus the marginal profitability of Line 1 and the marginal profitability of Line 2 are zero at the optimum.

How could headquarters implement the optimal portfolio? It could simply allocate -\$559 of capital to Line 1 and \$7,308 of capital to Line 2. Each line would be charged the 3% cost on any additional capital sought by either line. Neither line would want to expand, because expansion would push marginal profitability into negative territory. If the firm started with a non-optimal asset mix, headquarters could simply

charge the cost of capital shown in the appropriate non-optimal column of Appendix Table 2. One line or the other would face negative marginal profitability and move to relinquish capital, which would free up the other line to expand. Both lines would be content only at the optimum. Thus the optimum could, in principle, be achieved in a decentralized setting.

Appendix Table 1 – Variable Definitions

This table describes each line item in Appendix Table 2.

Asset allocation	Proportion of total assets invested in Line 1 ("x")
Asset risk	Standard deviation of returns on a portfolio with proportion x of assets invested in Line 1 & balance invested in Line 2
Capital ratio	Ratio ("c") of capital ("C") to assets ("A") required to achieve minimum credit quality (default-to-liability ratio).
Assets	Value of total assets in portfolio
Line 1	Value of assets invested in Line 1 ("A ₁ " where A ₁ = x*A)
Line 2	Value of assets invested in Line 2 ("A ₂ " where A ₂ = (1-x)*A)
Liabilities	Present value of default-free liabilities ("L" where L = A - C)
Capital	Total capital (C = c*A)
Default value	Present value of option to default ("P"); obtained via risk-neutral valuation under the assumption of normal return distributions
APV	Present value of portfolio after deducting the cost of risk capital (APV = NPV– τ*C)
NPV	Net present value of investments
Line 1	Net present value of investment in Line 1
Line 2	Net present value of investment in Line 2
Default-to-liability ratio	The present value of the default put expressed as a ratio to the present value of default-free liabilities
Default-to-asset ratio	The present value of the default put expressed as a ratio to the present value of assets
Default-to capital ratio	The present value of the default put expressed as a ratio to capital
Variance A	The variance of returns on the portfolio
Covariance 1,A	The covariance of returns on Line 1 with returns on the portfolio
Covariance 2,A	The covariance of returns on Line 2 with returns on the portfolio
Marg. default value Line 1	Marginal default value for Line 1 ("p ₁ ") calculated using Eq. (23)
Marg. default value Line 2	Marginal default value for Line 2 ("p ₂ ") calculated using Eq. (23)
Capital allocation Line 1	Capital allocation rate for Line 1 ("c ₁ ") calculated using Eq. (25)
Capital allocation Line 2	Capital allocation rate for Line 2 ("c ₂ ") calculated using Eq. (25)
Capital	Equal to the product of the minimum capital ratio and total assets (C = c*A)
Line 1	Equal to the product of the Line 1 capital allocation rate and Line 1 assets (C ₁ = c ₁ *A ₁)
Line 2	Equal to the product of the Line 2 capital allocation rate and Line 2 assets (C ₂ = c ₂ *A ₂)
Capital charge Line 1	Equal to the product of the market cost of capital and Line 1 capital (= τ*C ₁)
Capital charge Line 2	Equal to the product of the market cost of capital and Line 2 capital (= τ*C ₂)
APV Line 1	Adjusted present value of Line 1 is net present value of Line1 less the cost of allocated capital (APV ₁ = NPV ₁ - τ*C ₁)
APV Line 2	Adjusted present value of Line 2 is net present value of Line1 less the cost of allocated capital (APV ₂ = NPV ₂ - τ*C ₂)
Marginal profit Line 1	Marginal profit reflects the all-in cost of capital (including shadow price)
Marginal profit Line 2	Marginal profit reflects the all-in cost of capital (including shadow price)

Appendix Table 2 - Capital Allocation with Constraint on Credit Quality

This table presents numerical examples for a firm selecting a two-line portfolio with a constraint on credit quality (default-to-liability ratio). We assume that the firm maintains a default-to-liability ratio of 1%. The standard deviations of asset returns for Lines 1 and 2 are 10% and 30%, respectively, and the returns are uncorrelated. The marginal net present values are 2% minus 0.0001% of assets in Line 1 and 3% minus 0.001% of assets in Line 2, respectively. The cost of capital is $\tau = 3\%$. See Appendix Table 1 for a description of the variables in each row.

Asset allocation	54.46%	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
Asset risk	14.71%	10.00%	9.49%	10.00%	11.40%	13.42%	15.81%	18.44%	21.21%	24.08%	27.02%	30.00%
Capital ratio	17.66%	9.57%	8.76%	9.57%	11.85%	15.32%	19.74%	24.93%	30.82%	37.38%	44.67%	52.85%
Assets	38,205	17,130	22,404	28,133	33,528	37,315	38,157	35,616	30,612	24,686	19,024	14,146
Line 1	20,806	17,130	20,164	22,506	23,470	22,389	19,079	14,247	9,184	4,937	1,902	0
Line 2	17,399	0	2,240	5,627	10,058	14,926	19,079	21,370	21,428	19,749	17,122	14,146
Liabilities	31,457	15,492	20,441	25,442	29,556	31,598	30,626	26,737	21,178	15,459	10,527	6,671
Capital	6,749	1,639	1,963	2,691	3,972	5,717	7,531	8,880	9,434	9,227	8,498	7,476
Default value	315	155	204	254	296	316	306	267	212	155	105	67
NPV	570	196	265	350	445	534	590	596	555	484	403	324
Line 1	200	196	200	197	194	197	200	183	142	87	36	0
Line 2	371	0	65	153	251	336	390	413	413	397	367	324
APV	368	147	206	269	326	362	364	330	272	207	148	100
Default-to-liability ratio	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%	1.00%
Default-to-asset ratio	0.82%	0.90%	0.91%	0.90%	0.88%	0.85%	0.80%	0.75%	0.69%	0.63%	0.55%	0.47%
Default-to-capital ratio	4.66%	9.45%	10.41%	9.45%	7.44%	5.53%	4.07%	3.01%	2.24%	1.68%	1.24%	0.89%
Variance A	0.0216	0.0100	0.0090	0.0100	0.0130	0.0180	0.0250	0.0340	0.0450	0.0580	0.0730	0.0900
Covariance 1,A	0.0054	0.0100	0.0090	0.0080	0.0070	0.0060	0.0050	0.0040	0.0030	0.0020	0.0010	0.0000
Covariance 2,A	0.0410	0.0000	0.0090	0.0180	0.0270	0.0360	0.0450	0.0540	0.0630	0.0720	0.0810	0.0900
Marginal default value Line 1	1.03%	0.90%	0.91%	0.94%	0.97%	1.01%	1.04%	1.08%	1.13%	1.18%	1.25%	1.34%
Marginal default value Line 2	0.58%	1.06%	0.91%	0.78%	0.68%	0.61%	0.56%	0.53%	0.51%	0.49%	0.48%	0.47%
Capital allocation Line 1	-2.69%	9.57%	8.76%	6.40%	3.07%	-0.60%	-4.39%	-8.35%	-12.73%	-17.90%	-24.58%	-34.39%
Capital allocation Line 2	42.00%	-6.27%	8.76%	22.24%	32.33%	39.21%	43.87%	47.12%	49.48%	51.20%	52.36%	52.85%
Capital	6,749	1,639	1,963	2,691	3,972	5,717	7,531	8,880	9,434	9,227	8,498	7,476
Line 1	-559	1,639	1,767	1,440	720	-135	-838	-1,190	-1,169	-884	-468	0
Line 2	7,308	0	196	1,251	3,252	5,852	8,369	10,070	10,602	10,111	8,965	7,476
Capital charge Line 1	-17	49	53	43	22	-4	-25	-36	-35	-27	-14	0
Capital charge Line 2	219	0	6	38	98	176	251	302	318	303	269	224
APV Line 1	216	147	147	154	172	201	225	219	177	113	50	0
APV Line 2	151	0	59	115	154	161	139	111	95	94	98	100
Marginal profit Line 1	0.00%	0.00%	-0.28%	-0.44%	-0.44%	-0.22%	0.22%	0.83%	1.46%	2.04%	2.55%	3.03%
Marginal profit Line 2	0.00%	3.19%	2.51%	1.77%	1.02%	0.33%	-0.22%	-0.55%	-0.63%	-0.51%	-0.28%	0.00%