INTEREST RATE RISK MODELING: AN OVERVIEW

Sanjay K. Nawalkha
University of Massachusetts

Banca y Finanzas Cuantitativas
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- Based on: **Interest Rate Risk Modeling : The Fixed Income Valuation Course.** Sanjay K. Nawalkha, Gloria M. Soto, Natalia A. Beliaeva, 2005, Wiley Finance.
  - **Chapters 1, 2, 4, 5, 9, 10**

- Goals:
  - Understand interest rate Risk: What it is? How to manage it?
  - Explain the main models for interest rate risk management and discuss some empirical evidence.

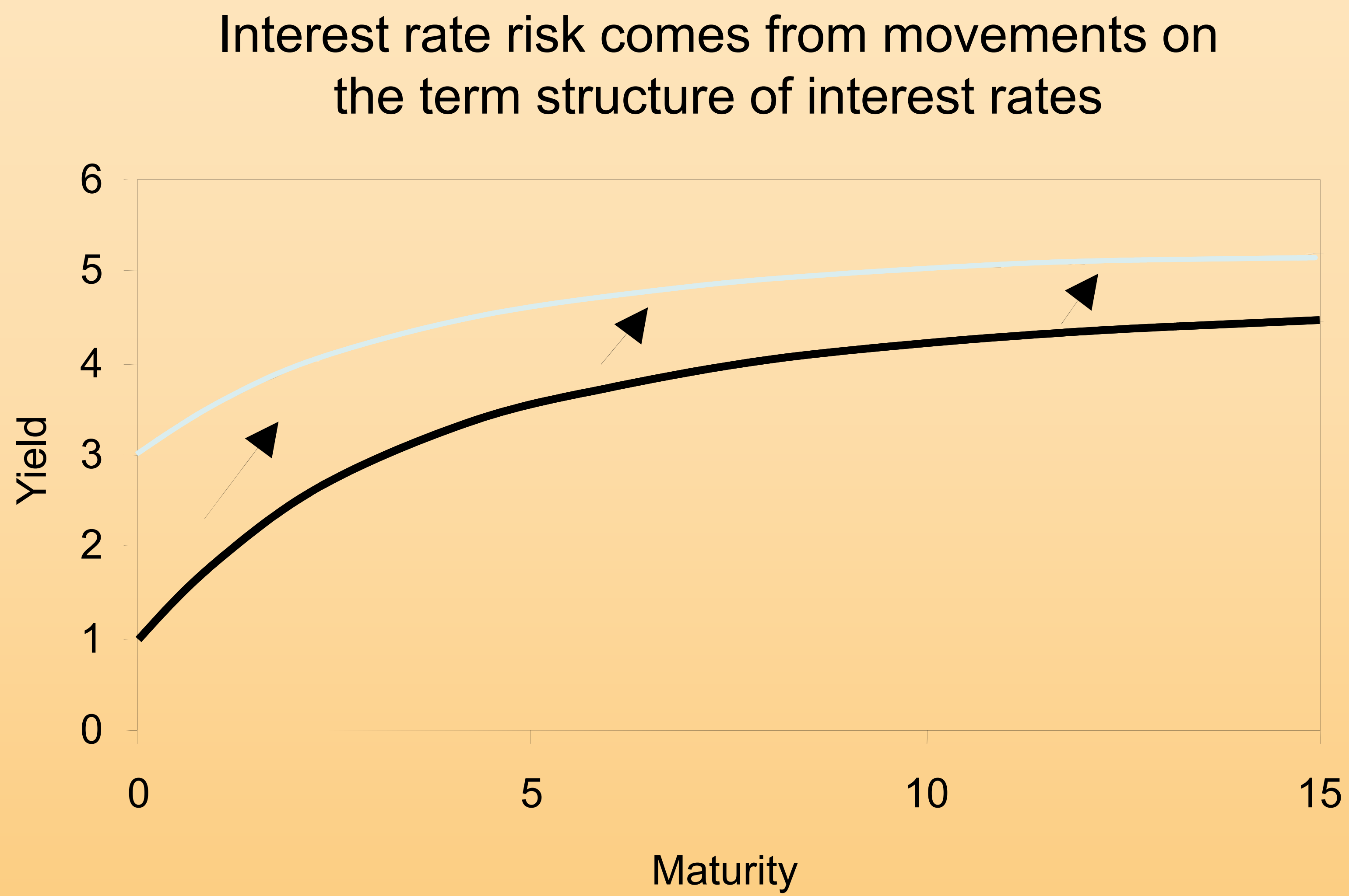
Interest Rate Risk Modeling: An Overview

- Introduction
- Duration and Convexity
- M-Square and M-Absolute Models
- Duration Vector Models
- Key Rate Duration Model
- Principal Component Duration Model
- Summary

Introduction

- Interest rate risk is the risk that the value of an interest-dependent asset, such as a loan or a bond, will worsen due to interest rate movements.
- Interest risk management is very important for financial institutions, because most of their assets and liabilities are affected by changes in interest rates.
- Financial institutions then measure and manage interest rate risk. But: how they do it?
- The exposition addresses this question by explaining the most popular models in the area of interest rate risk management over the past two decades.

Our goal is to understand interest risk management



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Duration and Convexity Model

- *Duration* is the most commonly used risk measure for measuring the interest rate risk exposure of a security.

- Convexity usually complements duration, providing a closer approximation to interest rate risk.

- Consider a bond with cash flows C_t, payable at time t. The bond sells for a price P, and is priced using a term structure of continuously compounded zero-coupon yields given by $y(t)$.

- Duration is the weighted-average maturity of a bond, where weights are the present values of the bond's cash flows, given as proportions of bond's price:

$$D = \sum_{t=t_1}^{t=t_N} t \; w_t \,, \quad \text{with} \quad w_t = \left\{ \frac{C_t}{e^{ty}} \right\} / P$$

Duration

- The traditional duration model can be used to approximate the percentage change in the bond price as follows:

$$\frac{\Delta P}{P} \cong -D\Delta y$$

- This expression assumes that:
  - The change in the yield, Δy, is *equal* for all bonds regardless of their coupons and maturities. → *The shift in the term structure is assumed to be parallel.*
  - The yield curve experiences infinitesimal shifts → For non-infinitesimal shifts, a second order effect (convexity), is needed.

Convexity

- Convexity is given as the weighted-average of maturity-squares of a bond, where weights are the present values of the bond's cash flows, given as proportions of bond's price:

$$CON = \sum_{t=t_1}^{t=t_N} t^2 w_t$$

- For large changes in the interest rates, duration and convexity are used to derive a two-term Taylor series expansion for approximating the percentage change in the bond price as follows:

$$\frac{\Delta P}{P} \cong -D\Delta y + \frac{1}{2} CON (\Delta y)^2$$

- Convexity measures the gain in a bond's value due to the second order effect of a large and parallel shifts in the term structure of interest rates. This suggests that for bonds with identical durations, higher convexity is always preferable. *Is this really true?*

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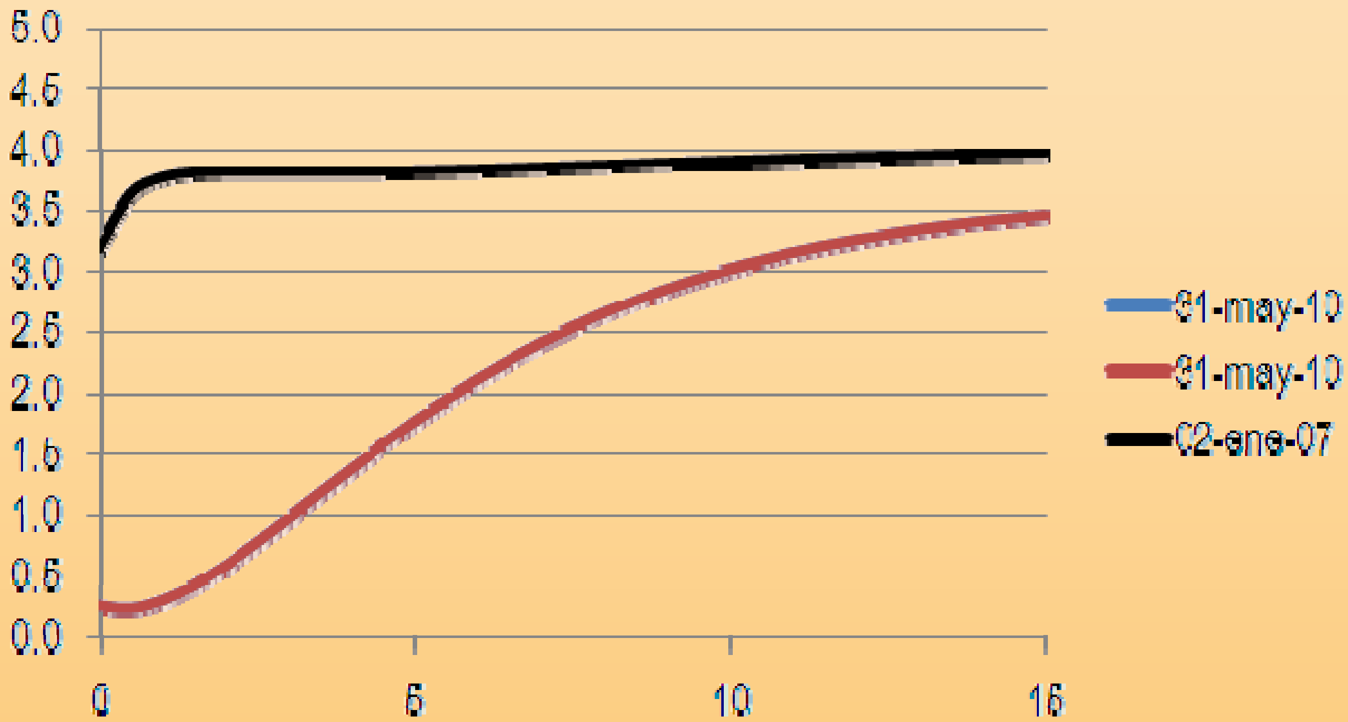
M-Square

- Higher convexity may not be desirable when the assumption of parallel yield curve shifts implicit in the traditional model does not hold.

Euro yield curve (Jan. 2007 – May 2010)



M-Square

- An alternative view of convexity, which is based upon a more realistic economic framework, relates convexity to slope shifts in the term structure.

- This view of convexity was proposed by Fong and Vasicek [1983, 1984] and Fong and Fabozzi [1985] through the introduction of M-square, which is a linear transformation of convexity.

- For a better understanding of this measure and the rest of models, consider how to express the shifts in the term structure of interest rates. Two main approaches are:
  – Define shifts in the term structure of zero-coupon yields
  – Define shifts in term structure of instantaneous forward rates

Measuring Shifts: Zero-Coupon Yields

- When expressed in terms of the term structure of zero-coupon yields, the bond price is given as follows:

$$P = \frac{C}{e^{y(t_1)t_1}} + \frac{C}{e^{y(t_2)t_2}} + \frac{C}{e^{y(t_3)t_3}} + \ldots + \frac{C}{e^{y(t_N)t_N}} + \frac{F}{e^{y(t_N)t_N}}$$

  where each cash flow is discounted by the zero-coupon yield $y(t)$ corresponding to its maturity t.

- For example, we can assume a simple polynomial form for the term structure of zero-coupon yields, as follows:

$$y(t) = A_0 + A_1 \cdot t + A_2 \cdot t^2 + A_3 \cdot t^3 + \ldots$$

  where parameters A_0, A_1, A_2, and A_3, are the height, slope, curvature, and the rate of change of curvature (and so on) of the term structure.

Measuring Shifts: Zero-Coupon Yields

- In this case, a change in the term structure of zero-coupon yields can be expressed as:

$$y'(t) = y(t) + \Delta y(t)$$

where,

$$\Delta y(t) = \Delta A_0 + \Delta A_1 \cdot t + \Delta A_2 \cdot t^2 + \Delta A_3 \cdot t^3 + \ldots$$

- The shift in the term structure of zero-coupon yields is then defined similarly as a function of the changes in height, slope, curvature, and other parameters.

Measuring Shifts: Forward Rates

- In many instances, it is easier to work with instantaneous forward rates, as certain interest rate risk measures and fixed income derivatives are easier to model using forward rates.

- The relationship between the term structure of zero-coupon yields and the term structure of instantaneous forward rates can be given as follows:

$$y(t) \cdot t = \int_0^t f(s)ds$$

where $y(t)$ is the zero-coupon-yield for term t, and $f(t)$ is the instantaneous forward rate for term t (which is the same as the forward rate that can be locked-in at time zero for an infinitesimally small interval t to $t + dt$)

Measuring Shifts: Forward Rates

- Given the term structure of zero-coupon yields, it is possible to obtain the term structure of instantaneous forward rates by taking the derivative of both sides of the previous equation as follows:

$$f(t) = t \cdot \partial y(t) / \partial t + y(t)$$

- As can be seen, if the term structure of zero-coupon yields is rising (falling), then $\partial y(t)/\partial t>0$ (<0), and instantaneous forward rates will be higher (lower) than zero-coupon yields.

- The bond price can be expressed in terms of the instantaneous forward rates as follows:

$$P = \frac{C}{e^{\int_0^1 f(s)ds}} + \frac{C}{e^{\int_0^2 f(s)ds}} + \frac{C}{e^{\int_0^3 f(s)ds}} + \ldots + \frac{C}{e^{\int_0^N f(s)ds}} + \frac{F}{e^{\int_0^N f(s)ds}}$$

Measuring Shifts: Forward Rates

- When the term structure of zero-coupon yields is expressed as a simple polynomial, the term structure of instantaneous forward rates has also a polynomial form, given as:

$$f(t) = t(A_1 + 2A_2 \cdot t + 3A_3 \cdot t^2 + ... +)$$
$$+ (A_0 + A_1 \cdot t + A_2 \cdot t^2 + A_3 \cdot t^3 + ...)$$

or

$$f(t) = A_0 + 2A_1 \cdot t + 3A_2 \cdot t^2 + 4A_3 \cdot t^3 + ...$$

- As shown, both the term structure of zero-coupon yields and the term structure of instantaneous forward rates have the same height, but the later has twice the slope, and three times the curvature (and four times the rate of change of curvature, and so on) of the term structure of zero coupon yields. This illustrates that the term structure of forward rates is more volatile, especially for longer maturities.

Measuring Shifts: Forward Rates

- Under the polynomial form, the shift in the term structure of instantaneous forward rates is given as follows:

$$\Delta f(t) = \Delta A_0 + 2\Delta A_1 \cdot t + 3\Delta A_2 \cdot t^2 + 4\Delta A_3 \cdot t^3 + ...$$

where the new term structure is given as:

$$f'(t) = f(t) + \Delta f(t)$$

M-Square

- The M-square risk measure is defined as the weighted average of the squared differences between cash flow maturities and the planning horizon *H*, where weights are the present values of the bond's (or a bond portfolio's) cash flows, given as proportions of bond's (or the bond portfolio's) price:

$$M^2 = \sum_{t=t_1}^{t=t_N} (t - H)^2 \cdot w_t$$

- Unlike convexity, the M-square measure is specific to a given planning horizon.

M-Square

- Given an arbitrary shift in the term structure of instantaneous forward rates, there exists a lower bound on the future value V_H of the bond portfolio at the planning horizon *H*, given as (no short position):

$$\frac{\Delta V_H}{V_H} \geq -(D-H)\cdot \Delta f(H) - \frac{1}{2} K_4 \cdot M^2$$

where $K_4 \geq \partial[\Delta f(t)] / \partial t$ for all *t* such that, $0 \leq t \leq t_N$.

- The minimum bound depends on two risk measures which are under the control of the portfolio manager: traditional duration and M-square. Ceteris paribus, the smaller the magnitude of M-square, the lower the risk exposure of the bond portfolio.

- The M-square model selects the bond portfolio that minimizes the M-square of the bond portfolio, subject to the duration constraint (i.e., duration = planning horizon *H*).

M-Square versus Convexity

- A linear relationship exists between M-square and convexity:

$$M^2 = CON - 2 \cdot D \cdot H + H^2$$

- For the special case when H=0, M-square converges to the convexity of the bond.
- If duration is kept constant, then M-square is an increasing function of convexity.
- This last result leads to the *convexity-M-square paradox*: in traditional duration analysis higher convexity is beneficial since it leads to higher returns; however, according to M-square model, M-square should be minimized in order to minimize immunization risk.

M-Square versus Convexity

- The “convexity view” and the “M-square view” have exactly opposite implications for bond risk analysis and portfolio management:
  - Convexity emphasizes the *gain* in the return on a portfolio, against large and parallel shifts in the term structure of interest rates.
  - On the other hand, M-square emphasizes the *risk exposure* of a portfolio due to slope-shifts in the term structure of interest rates.

- Which view is valid depends upon the extent of the violation of the parallel term structure shift assumption.

- Actually, the convexity view is not consistent with bond market equilibrium, while the M-square view is consistent with equilibrium conditions and it requires no specific assumptions regarding the shape of the shifts in the term structure.

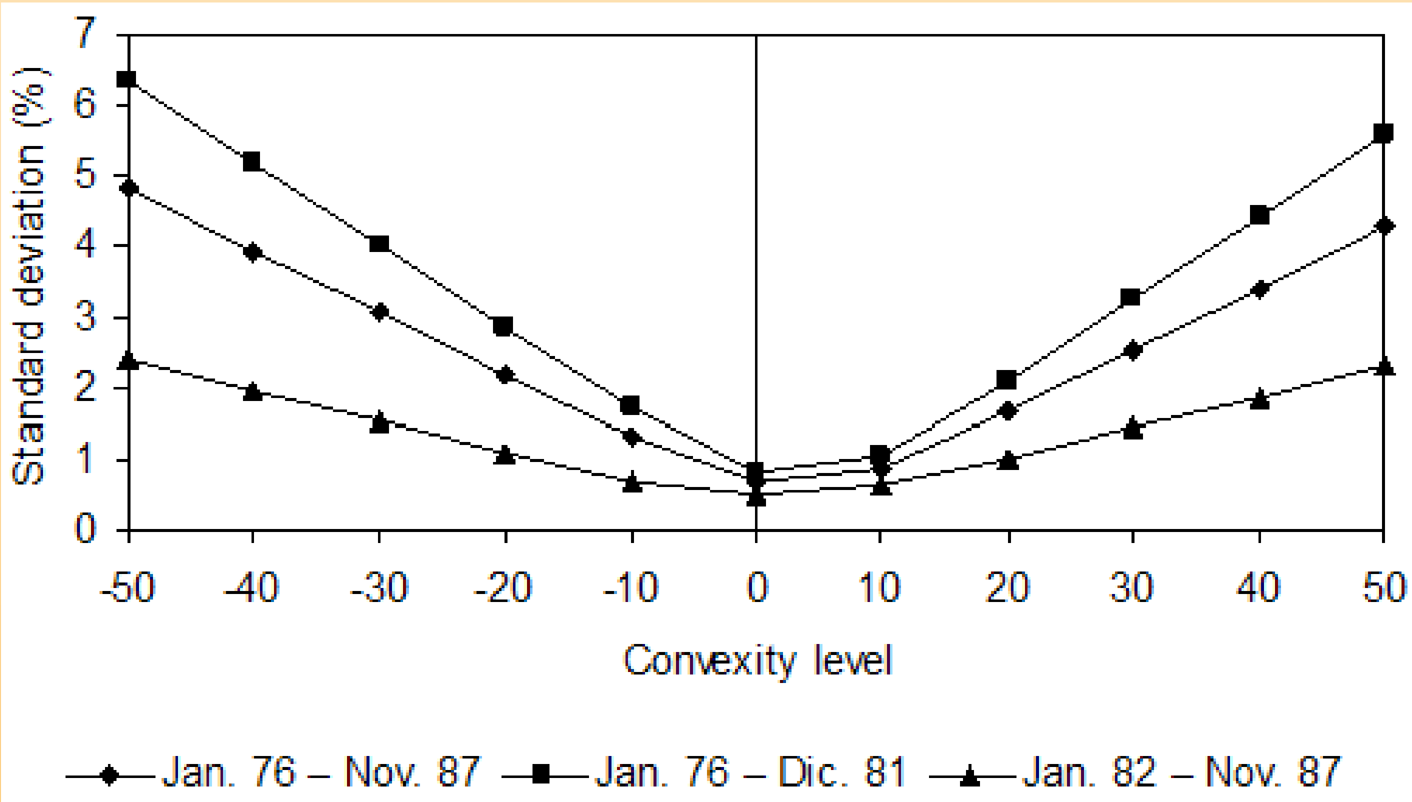
An Empirical Investigation of M-square

- Among others, **Lacey and Nawalkha [1993, JFI]** have analyzed empirically whether convexity adds risk or extra return.
- Data: CRSP government bond data over the period January 1976 through November 1987.
- Exercise 1: Test the sign of γ_2 in $R(H) - R_F(H) = \beta_0 + \beta_1 \cdot D + \gamma_2 \cdot CON + \varepsilon$
- Results: High positive-convexity is not associated with positive excess returns over the riskless return, a conclusion that rejects the "convexity view".

Test period	β_1	γ_2	Number of observations
Jan. 1976 - Nov. 1987	0.00039	-0.000026	3,881
Jan. 1976 - Dec. 1981	-0.00083	-0.000048	1,553
Jan. 1982 – Nov. 1987	0.00126***	-0.000034	2,328
Jan. 1977 - Dec. 1982	0.00029	-0.000043	1,682
Jan. 1978 - Dec. 1983	-0.00004	-0.000057	1,808
Jan. 1979 - Dec. 1984	0.00057	-0.000075*	1,942
Jan. 1980 - Dec. 1985	0.00109*	-0.000071*	2,097
Jan. 1981 - Dec. 1986	0.00120**	-0.000041	2,263

An Empirical Investigation of M-square

- Exercise 2: Compute volatility of excess returns for duration-matching bond portfolios with different convexity exposure.
- Results: Holding duration constant, and increasing the absolute size of convexity leads to higher immunization risk for bond portfolios.



M-Absolute

- Unlike M-square model, that requires two risk measures for hedging (i.e., both duration and M-square), Nawalkha and Chambers [1996] derive the M-absolute model, which only requires one risk measure for hedging against non-parallel yield curve shifts.

- M-absolute is defined as the weighted average of the absolute differences between cash flow maturities and the planning horizon, where the weights are the present values of the bond’s (or a bond portfolio’s) cash flows, given as proportions of bond’s (or the bond portfolio’s) price:

$$M^A = \sum_{t=t_1}^{t=t_N} \left| t - H \right| \cdot w_t$$

- Unlike duration but similar to M-square, the M-absolute measure is specific to a given planning horizon.

M-Absolute

- Given an arbitrary shift in the term structure of instantaneous forward rates, there exists a lower bound on the future value V_H of the bond portfolio at the planning horizon H, given as (no short position):

$$\frac{\Delta V_H}{V_H} \geq -K_3 \cdot M^A$$

where $K_3 = \text{Max}\left(\left|K_1\right|, \left|K_2\right|\right)$ and $K_1 \leq \Delta f\left(t\right) \leq K_2$ for all t such that, $0 \leq t \leq t_N$.

- That is, K_3 depends on the term structure movements and gives the maximum absolute deviation of the term structure of the initial forward rates from the term structure of the new forward rates.

M-Absolute

- A portfolio manager can control the portfolio's M-absolute but not K_3 .

- The smaller the magnitude of M-absolute, the lower the immunization risk of the portfolio.

- Therefore, the immunization objective of the M-absolute model is to select a portfolio that minimizes the portfolio's M-absolute:

$$\text{Min}\left[\sum_{i=1}^{J} p_i \, M^A{}_i\right]$$

$$s.t.$$

$$\sum_{i=1}^{J} p_i = 1,\ p_i \geq 0, \qquad \text{for all } i = 1, 2, \ldots, J$$

where $M^A{}_i$ defines the M-absolute of the ith bond.

M-Absolute versus Duration

- The relative desirability of the duration model or the M-absolute model depends on the nature of term structure shifts expected:
  - If height shifts completely dominate slope, curvature, and other higher order term structure shifts, then the traditional duration model will outperform the M-absolute model.
  - If, however, slope, curvature, and other higher order shifts are relatively significant in comparison with height shifts, then the M-absolute model may outperform the traditional duration model.

Duration, M-square and M-absolute (summary)

- The traditional duration model (duration + convexity) completely immunizes against the height shifts but ignores the impact of slope, curvature, and other higher order term structure shifts.

- M-square measures the immunization risk of duration-hedged portfolios, and hence, is able to provide significant enhancement in the immunization performance over the traditional duration model.

- The M-absolute model only requires one risk measure for hedging against non-parallel yield curve shifts, and its relative desirability over the duration model depends on the nature of term structure shifts expected.

An Empirical Investigation of M-absolute

- **Nawalkha and Chambers [1996, FAJ]** test M-absolute against duration.
- Data: McCulloch's term structure data over the period 1951 through 1986.
- Exercise: Compute absolute deviations between effective and target returns of the M-Absolute strategy and the traditional duration strategy (with maximum diversification among bonds) for bond portfolios with a holding period of 4 years (32 overlapping four-year periods within the period of study).
- Result: The M-absolute strategy reduces the immunization risk inherent in the duration model by more than half.

An Empirical Investigation of M-absolute

	Sum of absolute deviations	As a percentage of the duration strategy
Observation period 1951-1970		
Duration strategy	0.10063	100.00
M-absolute strategy	0.03560	35.37
Observation period 1967-1986		
Duration strategy	0.28239	100.00
M-absolute strategy	0.11604	41.09

- Implication: Changes in the height of the term structure of instantaneous forward rates must be accompanied by significant changes in the slope, curvature, and other higher order term structure shape parameters.

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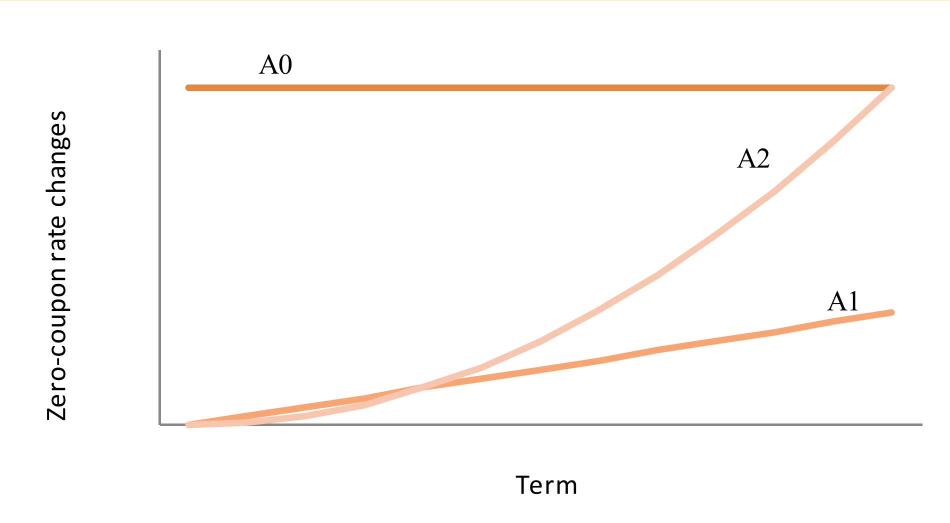
Duration Vector Models

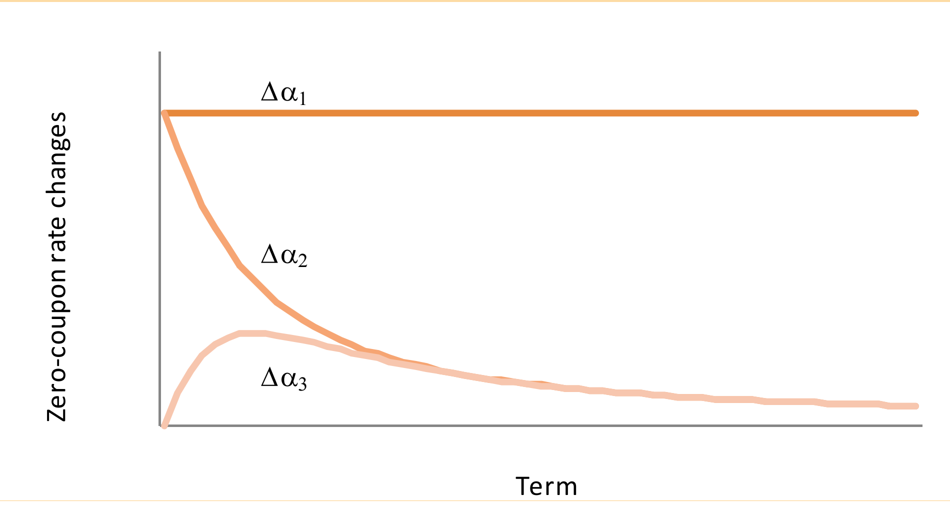
- Once it is recognized that height changes in the term structure are accompanied by changes in in the slope, curvature, and other higher order term shape parameters, **multifactor models** are required in order to measure and manage interest rate risk.
- The most usual approach has been to assume a parametric fitting of the term structure and then derive the corresponding duration vector:
  - Granito [1984, JAI Press], Chambers, Carleton, and McEnally [1988, JFQA], Prisman and Shores [1988, JBF], Prisman and Tian [1994, JFQA], Nawalkha [1995] and Grandville [2001, MIT Press] consider a **polynomial function**.
  - Willner [1996, JFI] and Diebold, Ji and Li [2006, Edward Elgar] propose the **exponential function** given by Nelson and Siegel [1987, JBus].

Duration Vector Models



Polynomial function

$$y(t) = A_0 + A_1 \times t + A_2 \times t^2 + A_3 \times t^3 + \ldots$$



Exponential function

$$y(t) = \alpha_1 + (\alpha_2 + \alpha_3)\frac{\beta}{t}\left(1 - e^{-t/\beta}\right) - \alpha_3 e^{-t/\beta}$$

Duration Vector Models

- In the simplest version (only lineal effects, polynomial function for zero-coupon rates), the polynomial duration vector model expresses the instantaneous percentage change in the current value of a portfolio as:

$$\frac{\Delta V_0}{V_0} = -D(1)\times\Delta A_0 - D(2)\times\Delta A_1 - D(3)\times\Delta A_2 - D(4)\times\Delta A_3 - \ldots$$

- The risk of the portfolio is captured by a vector of risk measures, given as *D(1)*, *D(2)*, *D(3)*, *D(4)*, etc., defined as:

$$D(m) = \sum_{t=t_1}^{t=t_N} w_t \cdot t^m, \text{ and } w_t = \left\{\frac{C_t}{e^{y(t)t}}\right\} / V_0$$

- The shift vector expresses the change in the height ΔA_0, the slope ΔA_1, the curvature ΔA_2, the rate of change of curvature $\Delta A_{3\,,}$ and so on of the term structure of zero-coupon rates.

Duration Vector Models

- Moreover, based on the derivation of the M-vector of Nawalkha and Chambers [1997, JPM] and Nawalkha, Soto and Zhang [2003, JBF], it can be demonstrated that even not restricting the term structure shifts to be of a polynomial function form, a Taylor expansion of the bond return function also gives this vector of polynomial-type duration measures.

Duration Vector Models

- In particular, given a instantaneous shift in the term structure of forward rates from *f*(*t*) to *f'*(*t*) such that *f'*(*t*) = *f*(*t*) + Δ*f*(*t*), the instantaneous percentage change in the current value of the portfolio can be expressed as:

$$
\begin{aligned}
\frac{\Delta V_0}{V_0} = & -D(1)\,\left[\Delta f(0)\right] \\
& -D(2)\,\left[\frac{1}{2}\left(\frac{\partial\left(\Delta f(t)\right)}{\partial t} - \left(\Delta f(0)\right)^2\right)\right]_{t=0} \\
& -D(3)\,\left[\frac{1}{3!}\left(\frac{\partial^2(\Delta f(t))}{\partial t^2} - 3\cdot\Delta f(0)\frac{\partial\left(\Delta f(t)\right)}{\partial t} + \left(\Delta f(0)\right)^3\right)\right]_{t=0} \\
& \quad \vdots \\
& -D(Q)\,\left[\frac{1}{Q!}\left(\frac{\partial^{Q-1}(\Delta f(t))}{\partial t^{Q-1}} + \ldots + \left(\Delta f(0)\right)^Q\right)\right]_{t=0}
\end{aligned}
$$

Duration Vector Models

- In this general case:
  - The first element of the duration vector is the traditional duration measure given as the weighted-average time to maturity, and the first shift vector element captures the change in the level of the forward rate curve for the instantaneous term, given by $\Delta f(0)$;
  - The second shift vector element captures the difference between the square of this change and the slope of the change in the forward rate curve (given by $\partial \Delta f(t)/\partial t$ at $t = 0$);
  - The third shift vector element captures the effect of the third power of the change in the level of the forward rate curve, the interaction between the change in the level and the slope of the change in the forward rate curve, and the curvature of the change in the forward rate curve (given by $\partial^2 \Delta f(t)/\partial t^2$ at $t = 0$);
  - and so on.

Duration Vector Models

- The model converges to the traditional duration model with duration and convexity when only shifts in the height of the term structure are considered.
- Similar to the controversy between “convexity view” and the “M-square view”, it has been found that the magnitudes of the higher order derivatives in the shift vector elements (shifts in the shape parameters of the term structure) dominate the magnitudes of higher powers (second order effects of the shifts and beyond).
- For this reason, the polynomial model is a good approximation of this general model.
- Generally, the first three to five duration vector measures are sufficient to capture **all of the interest rate risk** inherent in bond portfolios.

Duration Vector Models

- To immunize a portfolio for a planning horizon of *H* years, the duration vector model requires setting the portfolio duration vector to the duration vector of a hypothetical default-free zero coupon bond maturing at *H*.

- Since the duration vector elements of a zero-coupon bond are given as its maturity, maturity squared, maturity cubed, etc., the immunization constraints are given as follows:

$$D(1) = \sum_{t=t_1}^{t=t_N} w_t \cdot t = H, \qquad D(2) = \sum_{t=t_1}^{t=t_N} w_t \cdot t^2 = H^2$$

$$D(3) = \sum_{t=t_1}^{t=t_N} w_t \cdot t^3 = H^3 \ldots \; D(Q) = \sum_{t=t_1}^{t=t_N} w_t \cdot t^Q = H^Q$$

Empirical Investigations

- **Nawalkha and Chambers [1997, JPM]** perform an empirical analysis to determine an appropriate value of Q for the duration vector model.
- Data: McCulloch term structure data for the period 1951 through 1986.
- Exercise: Compute the absolute deviations of actual portfolio values from target values of simulated bond portfolios immunized for planning horizons of 4 years.
- Results: Hedging performance improves steadily as the length of the duration vector increases. About three to five duration vector constraints (i.e., Q = 3 to 5) have shown to almost perfectly immunize against the risk of non-parallel yield curve shifts.

Empirical Investigations

	$Q=1$	$Q=2$	$Q=3$	$Q=4$	$Q=5$
Panel A: Observation period 1951-1970					
Sum of absolute deviations	0.10063	0.03445	0.01085	0.00305	0.00136
As percentage of $Q=1$	100.00	34.23	10.78	3.03	1.35
Panel B: Observation period 1967-1986					
Sum of absolute deviations	0.28239	0.06621	0.03228	0.01412	0.01082
As percentage of $Q=1$	100.00	23.45	11.43	4.99	3.83

- The results of the tests are similar over two sub-periods, providing empirical confirmation that the duration vector model is independent of the particular stochastic processes for term structure movements.

Empirical Investigations

- **Soto [2001, JBF]** perform an empirical analysis to determine the performance of the model using real (not simulated) bond data.
- Data: Daily prices for Spanish government bonds for the period 1992 through 1995.
- Exercise: Compute the mean absolute deviations between the realized return and the initial zero-coupon rate of real bond portfolios immunized for holding periods of 1 and 2 years.
- Results: Hedging performance improves as the length of the duration vector increases. In general, a minimum of three duration constraints (*D(1)*, *D(2)* and *D(3)*) are required to guarantee immunization, with the only exception of portfolios including a bond maturing at the end of the horizon, which perform well under traditional duration in short holding periods.

Empirical Investigations

Strategies	Madiff (bp.)	Imadiff	Realloc. (assets)
One-year holding periods			
D1	88.303	100.000	0.029
D2	58.482	66.228	0.053
D3	19.875	22.508	0.062
D4	22.527	25.511	0.086
Maturity	183.608	207.928	–
Mat-mat	38.769	43.905	–
Two-year holding periods			
D1	119.331	100.000	0.022
D2	90.520	75.857	0.045
D3	50.984	42.725	0.058
D4	40.282	33.757	0.085
Maturity	146.776	123.000	–
Mat-mat	45.614	38.225	–

Empirical Investigations

- **Ventura and Pereira [2006, JBF]** perform an empirical analysis to determine the performance of the M-vector model using real (not simulated) bond data.
- Data: Daily prices for Portuguese government bonds for the period August 1993 through September 1999.
- Exercise: Compute the deviations between the realized return and the initial zero-coupon rate of real bond portfolios immunized for holding periods of 2 years (also sensibility analysis for 4 years)
- Results: Hedging performance improves as the length of the duration vector increases. Three duration constraints are sufficient to guarantee a return close to the target. Additional constraints beyond third contributes little to increasing hedging performance and can even cause the results to deteriorate (more frequent reallocations). The only exception are duration-matching portfolios including a bond maturing at the end of the holding period, for which traditional duration is enough.

Empirical Investigations

Immunization results with semi-annual rebalancing over two-year horizons, 1993–1999

	$r^H_{r,i} - r^H_{T,i}$ (Basis points)			RMSD (B.p.)	I_{RMSD}	Percentage of Simulations Duration-Matching outperforms
	Average	Maximum	Minimum			
Maturity	−31	107	−152	75.1	100.0	N/A
Maturity-Bullet	−20	117	−104	72.3	96.4	42.9%
Maturity-Barbell	−21	50	−68	50.2	66.8	51.0%
$M(1)$	−58	28	−157	79.5	105.9	42.9%
$M(2)$	−54	23	−142	72.2	96.2	42.9%
$M(3)$	−43	43	−127	55.3	73.7	55.1%
$M(4)$	−21	69	−133	56.9	75.8	46.9%
$M(5)$	−16	94	−113	49.8	66.4	59.2%***
$M(6)$	−16	106	−129	62.5	83.2	44.9%
$M(7)$	−7	95	−106	52.5	69.9	42.9%

This table summarizes different measures of the performance of the immunization strategies over 49 horizons of 2 years in the sample period 1993–1999 with semi-annual rebalancing. The second, third and fourth columns show, respectively, the average, maximum and minimum deviation of the actual return of a given immunization strategy from the expected return. Duration matching is said to outperform maturity matching when its absolute difference between promised and actual return is smaller. Asterisks accompanying the *percentage of cases duration-matching outperforms* indicate that the strategy performs statistically different from the naïve maturing strategy at 1% (*), 5% (**) or 10% (***) levels by means of a normal approximation to the binomial sign test.

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Key Rate Duration Model

- The key rate duration model describes the shifts in the term structure as a discrete vector representing the changes in the key zero-coupon rates of various maturities.

- Key rate durations are defined as the sensitivity of the portfolio value to the given key rates at different points along the term structure.

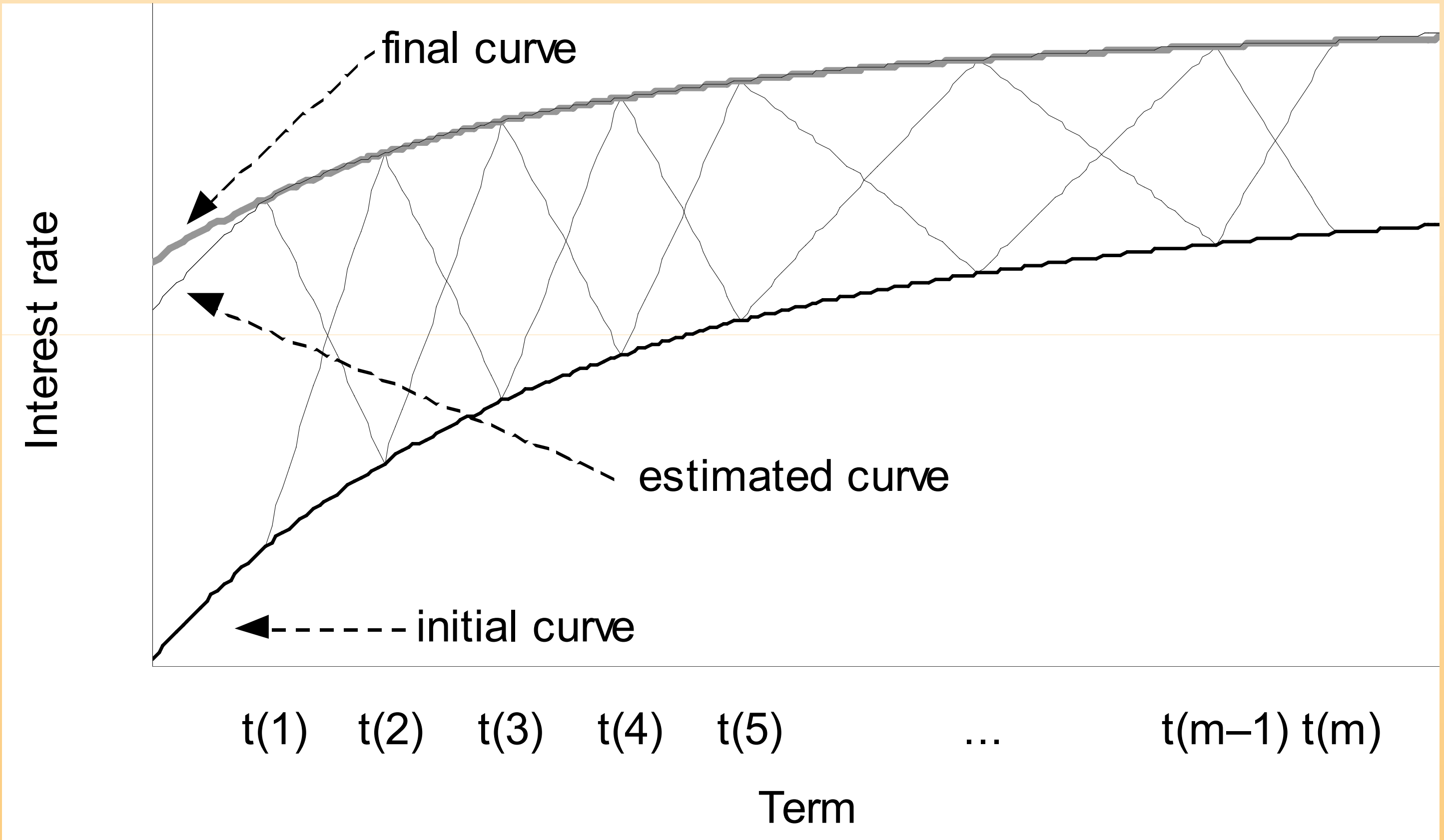
- Any smooth change in the term structure of zero-coupon yields can be represented as a vector of changes in a number of properly chosen key rates:

$$TSIR\ shift = \left(\Delta y(t_1), \Delta y(t_2) \ldots, \Delta y(t_m) \right)$$

where $y(t_i)$ is the zero-coupon rate for term t_i and $y(t_1)$, $y(t_2)$, … , $y(t_m)$, define the set of m key rates.

Key Rate Changes

- The changes in all other interest rates are approximated by <u>linear interpolation</u> of the changes in the adjacent key rates, thus obtaining a <u>piecewise linear approximation</u> for the shift in the term structure.



Key Rate Durations

- The set of key rate shifts can be used to evaluate the change in the price of any fixed-income security.

- An infinitesimal and instantaneous shift in a specific key rate, $y(t_i)$, results in an instantaneous price change given as:

$$\frac{\Delta P_i}{P} = -KRD(i) \cdot \Delta y(t_i)$$

where $KRD(i)$ is the i-th key rate duration, defined as the (negative) percentage change in the price resulting from the change in the i-th key rate:

$$KRD(i) = -\frac{1}{P}\frac{\partial P}{\partial y(t_i)}$$

Key Rate Durations

- The set of *KRD*s forms a vector of m risk measures, representing the first-order price sensitivities of the portfolio to the m key rates:

$$KRD = \left[KRD(1) \quad KRD(2) \quad \ldots \quad KRD(m) \right]$$

- The total percentage change in price due to an infinitesimal shift in the term structure can be obtained as the sum of the effect of each key rate shift on the security price.

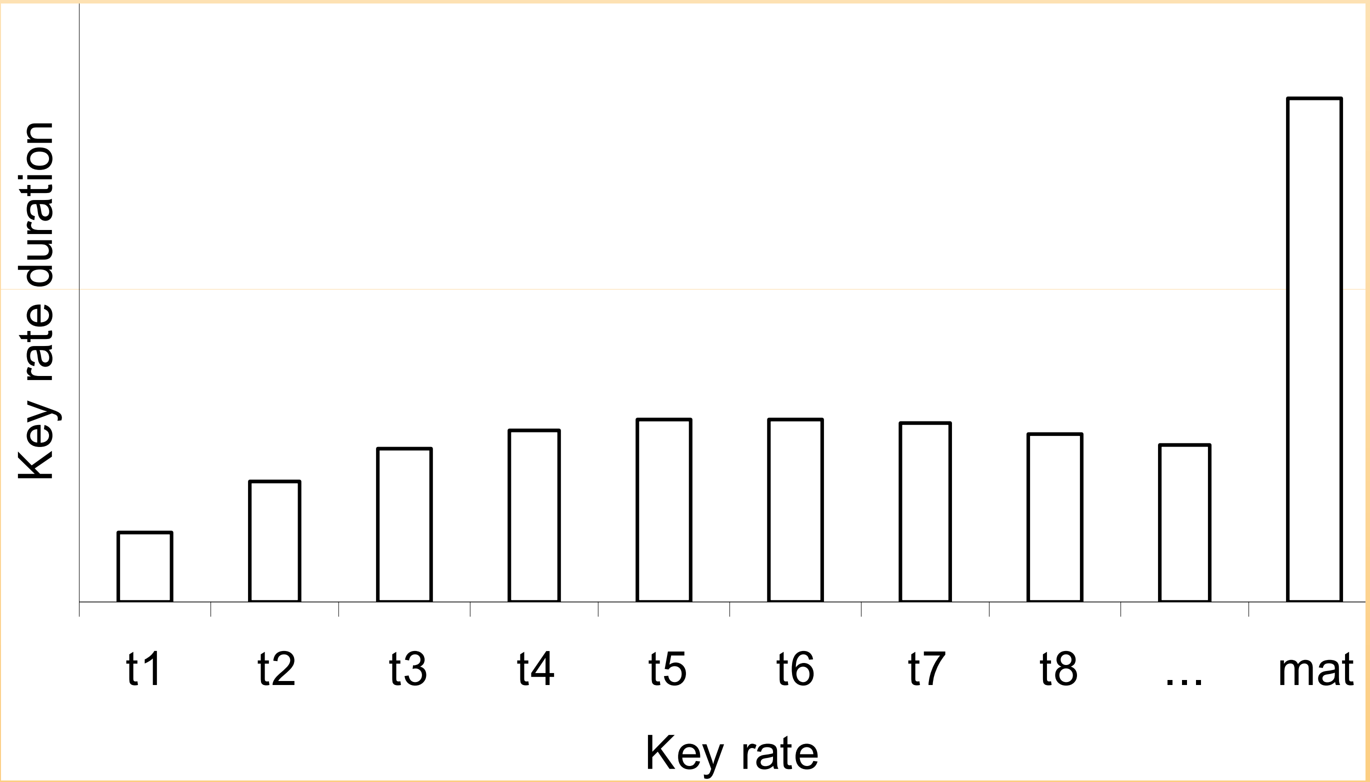
$$\Delta P = \Delta P_1 + \Delta P_2 + \ldots + \Delta P_m$$

- The total percentage change in price in terms of the *KRD* and key rate shifts is:

$$\frac{\Delta P}{P} = -\sum_{i=1}^{m} KRD(i) \cdot \Delta y(t_i)$$

Key Rate Durations

- Key rate durations give the risk profile of fixed-income portfolios across the whole term structure.
- For example, for a coupon-bearing bond:



Limitations of the Key Rate Model

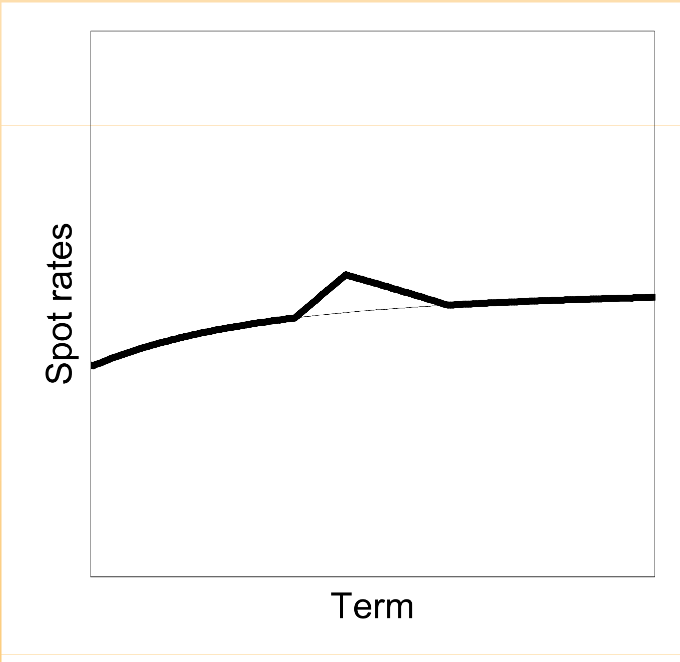
- There are three limitations of the key rate model:
  - Arbitrary choice of the key rates.
  - Unrealistic shapes of the individual key rate shifts.
  - Loss of efficiency caused by not modeling the history of term structure movements.

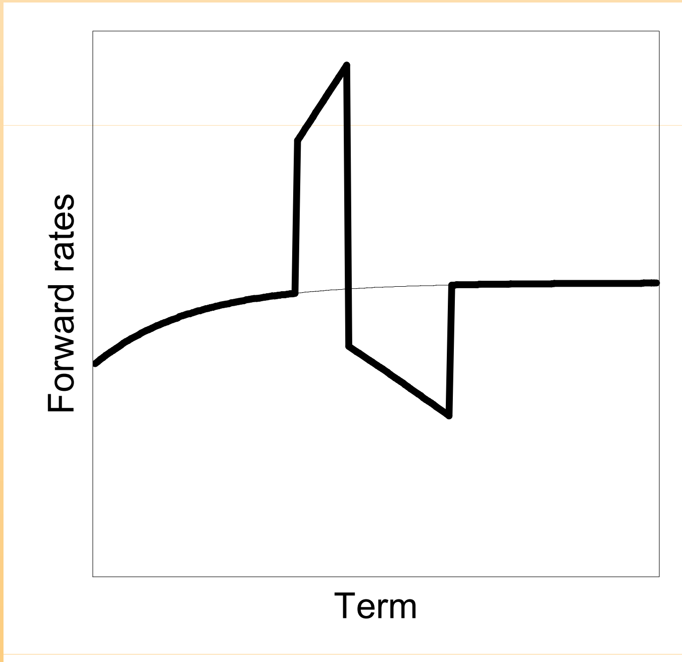
Limitations of the Key Rate Model
The Choice of Key Rates

- The model allows for any number of key rates.
- Therefore, interest rate risk can be modeled and hedged to a high degree of accuracy.
- However, the number of key rates durations to be used and the corresponding choice of key rates remain quite arbitrary.
- Despite the choice of the risk factors is important, the key rate model offers no guidance about how to make the choice.
- When the model was first introduced by Ho [1992, JFI], he suggested using as many as <u>11 key rates</u>.
- In order to reduce the number of risk factors, the manager could narrow the choice based upon the maturity structure of the portfolio.

Limitations of the Key Rate Model
The Shape of Key Rate Shifts

- Each individual key rate shift has a historically implausible shape.
- Each key rate shock implies a kind of forward rate saw-tooth shift which is unrealistic.



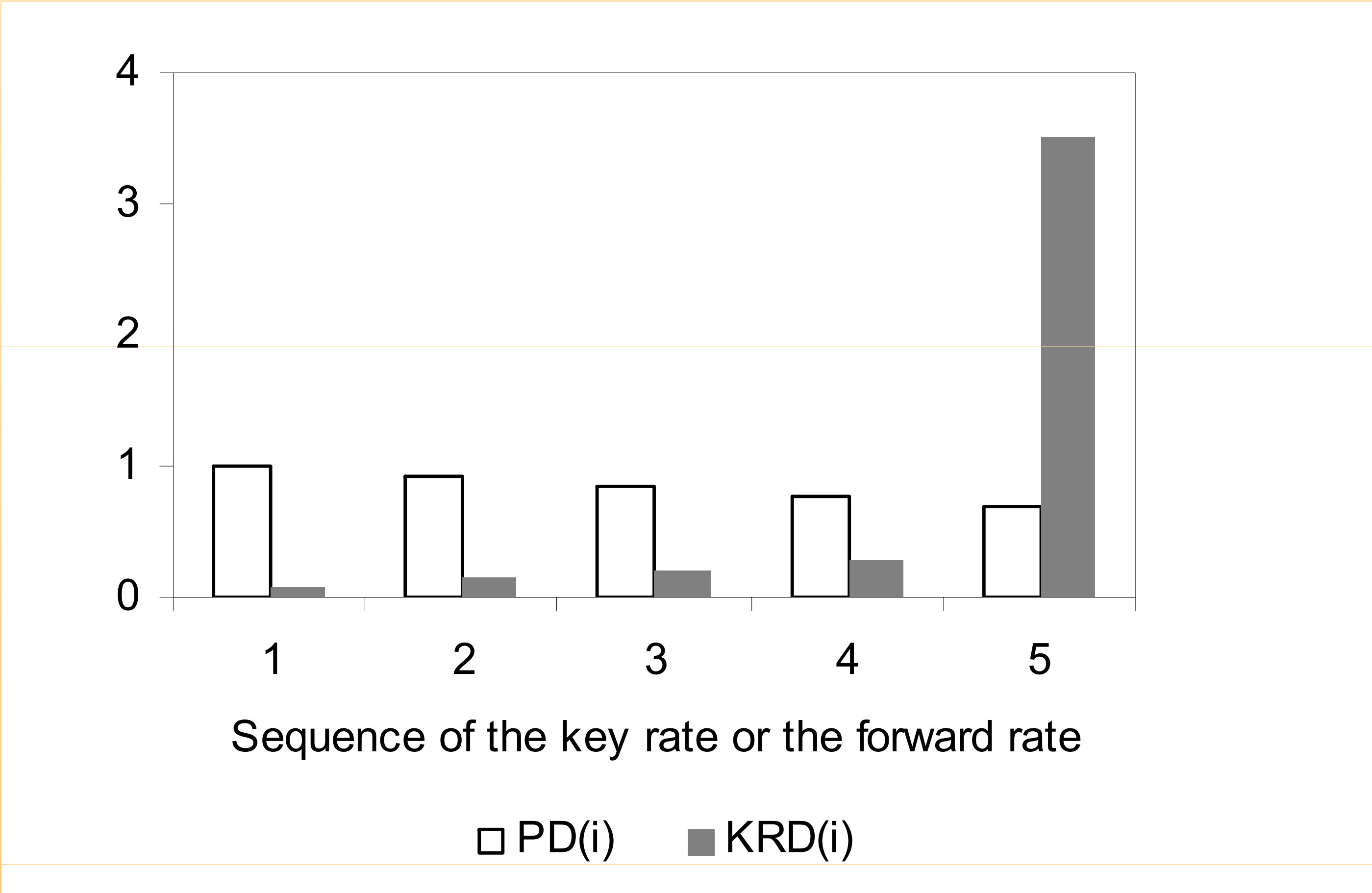


Limitations of the Key Rate Model
The Shape of Key Rate Shifts

- In order to address this shortcoming, a natural choice is to focus on the forward rate curve instead of the zero-coupon curve.
- Johnson and Meyer [1989, FAJ] first proposed this methodology and called it the **partial derivative approach** or **PDA**.
- According to PDA, the forward rate structure is split up into many linear segments and all forward rates within each segment are assumed to change in a parallel way.
- While under the key rate model each key rate only affects the present value of the cash flows around the term of the rate, under the PDA approach each forward rate affects the present value of all cash flows occurring within or after the term of the forward rate.

Limitations of the Key Rate Model
The Shape of Key Rate Shifts

Partial durations versus key rate durations of a coupon-bearing bond



Limitations of the Key Rate Model

Loss of Efficiency

- Some assert that the key rate model is not an efficient one in describing the dynamic of the term structure.
- This is because historical volatilities of interest rates provide useful information about the behavior of the different segments of the term structure, and the key model disregards this information.
- Since each key rate change is assumed to be independent of the changes in the rest of key rates, the model deals with movements in the term structure whose probabilities may be too small to worry about.
- As a result, the use of the key rate model for interest rate risk management imposes too severe restrictions on portfolio construction that lead to increased costs and a loss of degrees of freedom when managing or hedging.

Limitations of the Key Rate Model
Loss of Efficiency

- A number of variations of the key rate model that try to deal with this undesirable consequence have gone through the inclusion of the covariance of interest rate changes into the analysis. For example:
  - Falkenstein and Hanweck [1996, JFI] propose a covariance-consistent key rate hedging, which consists on finding the portfolio that minimizes the variance of the portfolio returns.
  - Reitano [1996, JPM] proposes a stochastic immunization program, which searches for the portfolio that minimizes a risk measure defined as a weighted average of the portfolio's return variance and the worst case risk given by the magnitude of the key rate durations.

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Principal Component Duration Model

- The principal component model assumes that term structure movements can be summarized by a few composite variables.
- These new variables are constructed by applying a statistical technique called **principal component analysis (PCA)** to the past of interest rate changes.
- By construction, the first principal component explains the maximum percentage of the total variance of interest rate changes. The second component is linearly independent of the first, and explains the maximum percentage of the remaining variance, and so on.
- TSIR shifts are now expressed in terms of the principal components:

$$TSIR\ shift = (\Delta c_1, \Delta c_2 \ldots, \Delta c_m)$$

Principal Component Duration Model

- Principal components are linear combination of interest rate changes:

$$\Delta c_j = \sum_{i=1}^{m} u_{ji} \Delta y(t_i) \qquad j = 1, \ldots, m$$

where u_{ji} are called principal component coefficients.

- And inversely, the changes in the *m* interest rates are a lineal combination of the principal components:

$$\Delta y(t_i) = \sum_{j=1}^{m} u_{ji} \Delta c_j \qquad i = 1, \ldots, m$$

Principal Component Duration Model

- When the principal components are standardized to have a unit variance, interest rate changes are expressed as:

$$\Delta y(t_i) = \sum_{j=1}^{m} l_{ij} \Delta c_j^* \qquad i = 1, \dots, m$$

where l_{ij} are called factor loadings.

- Since principal components are ordered according to their explanatory power, retaining only the first components does not implies a significant losing of information.

- This not only helps obtaining a low-dimensional parsimonious model, but also reduces the noise in the data due to unsystematic factors.

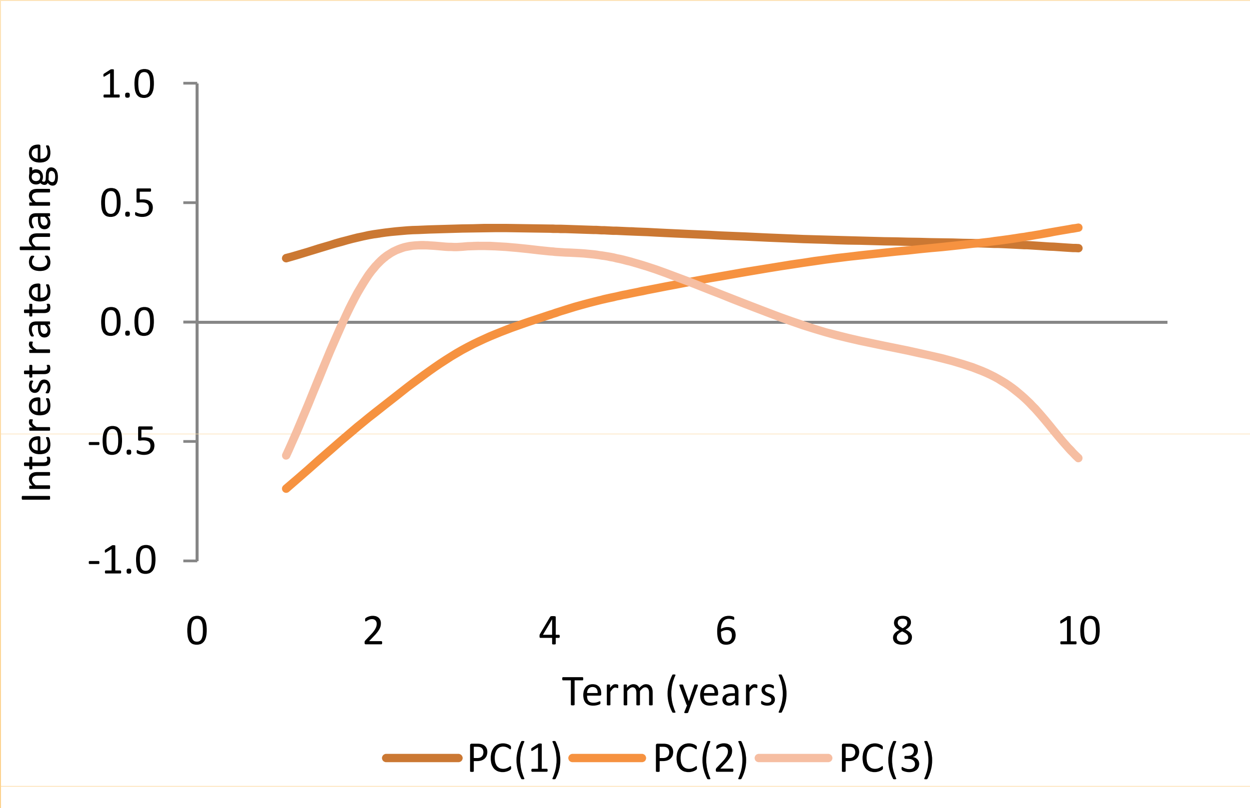
Principal Component Duration Model

- Numerous studies for different periods and countries reveal that three principal components are sufficient in explaining the variation of interest rates in Treasury bond markets.
- The table shows the eigenvectors and eigenvalues of the covariance matrix of monthly changes in the U.S. zero-coupon rates from Jan. 2000 to Dec. 2002.

Rate	PC(1)	PC(2)	PC(3)	PC(4)	PC(5)	PC(6)	PC(7)	PC(8)
1	0.270	-0.701	-0.565	0.292	-0.138	-0.085	0.060	-0.026
2	0.372	-0.385	0.227	-0.423	0.459	0.445	-0.240	0.132
3	0.396	-0.120	0.315	-0.328	-0.037	-0.605	0.244	-0.442
4	0.395	0.028	0.296	0.103	-0.411	-0.182	-0.054	0.735
5	0.382	0.124	0.243	0.346	-0.415	0.476	-0.166	-0.483
7	0.350	0.252	-0.031	0.344	0.444	0.149	0.682	0.102
9	0.332	0.334	-0.225	0.266	0.397	-0.348	-0.614	-0.047
10	0.312	0.393	-0.576	-0.556	-0.270	0.162	0.085	0.022
Eigenvalues	0.605	0.057	0.009	0.001	0.001	0.000	0.000	0.000
% of variance explained	89.8%	8.5%	1.3%	0.2%	0.1%	0.1%	0.0%	0.0%
Cumulative % of variance	89.8%	98.3%	99.6%	99.8%	99.9%	100.0%	100.0%	100.0%

Principal Component Duration Model

- The figure shows the shape of the eigenvectors corresponding to the first three components.



- Accordingly, PC (1) is a level or height factor, PC(2) is a slope or twist factor, and PC(3) is a curvature factor.

Principal Component Duration Model

- Assuming that these first three components are retained, interest rate changes can be expressed as:

$$\Delta y(t_i) \approx l_{ih}\Delta c_h + l_{is}\Delta c_s + l_{ic}\Delta c_c \qquad i = 1,\ldots,m$$

  where *h* refers to the height factor, *s* to the slope factor and *c* to the curvature factor.

- Principal component durations and convexities can be computed from the first and the second partial derivatives of portfolio value with respect to the three factors as follows:

$$PCD(i) = -\frac{1}{P}\frac{\partial P}{\partial c_i} \qquad i = h,s,c$$

$$PCC(i,j) = \frac{1}{P}\frac{\partial^2 P}{\partial c_i \partial c_j} \qquad i,j = h,s,c$$

Principal Component Duration Model

- Using a second-order Taylor series approximation, the percentage change in portfolio value is given as:

$$\frac{\Delta P}{P} = -\sum_{i=h,s,c} PCD(i) \cdot \Delta c_i + \frac{1}{2} \sum_{i=h,s,c} \sum_{j=h,s,c} PCC(i,j) \cdot \Delta c_i \cdot \Delta c_j$$

- Since the principal components are orthogonal, cross effects can be disregarded, which gives:

$$\frac{\Delta P}{P} = -\sum_{i=h,s,c} PCD(i) \cdot \Delta c_i + \frac{1}{2} \sum_{j=h,s,c} PCC(i,i) \cdot \Delta c_i^2$$

Principal Component Duration Model

- Immunizing a portfolio for a given horizon requires choosing a portfolio whose three principal component durations equal those of a zero-coupon bond maturing at the end of the planning horizon:

$$PCD(i)_{PORT} = PCD(i)_{zero} = H \cdot l_{H\,i} \qquad\qquad i = h, s, c$$

where H is the length of the planning horizon and l_{Hi} is the loading of the i-th principal component on the zero-coupon rate for term H.

Principal Component Duration Model
Main advantages

- The benefits of using the PCA are:
  - A significant **reduction in dimensionality** when compared with other models.
  - It is able to produce **orthogonal risk factors**, which makes interest rate risk measurement and management a simpler task, because each risk factor can be treated independently.

Principal Component Duration Model
Main shortcomings

- The principal component model has a couple of shortcomings:
  - The static nature of the technique is unable to deal with the non-stationary time-series behavior of interest rate changes.
  - Principal components are purely constructions that summarize information in correlated systems, but do not always lead to an economic interpretation.

Limitations of the Principal Component Model

Static Factors Arising from a Dynamic Structure

- Application of PCA to term structure movements implies that:
  - The covariance structure of interest rate changes is constant
  - The shape of the principal components are stationary

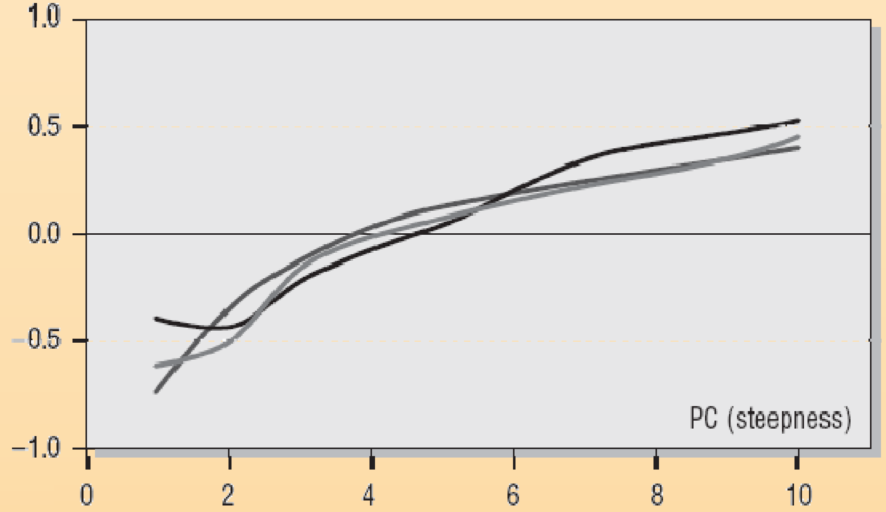
- These are critical, because if the shapes of the principal components change frequently, then these components cannot explain the future volatility of interest rates.

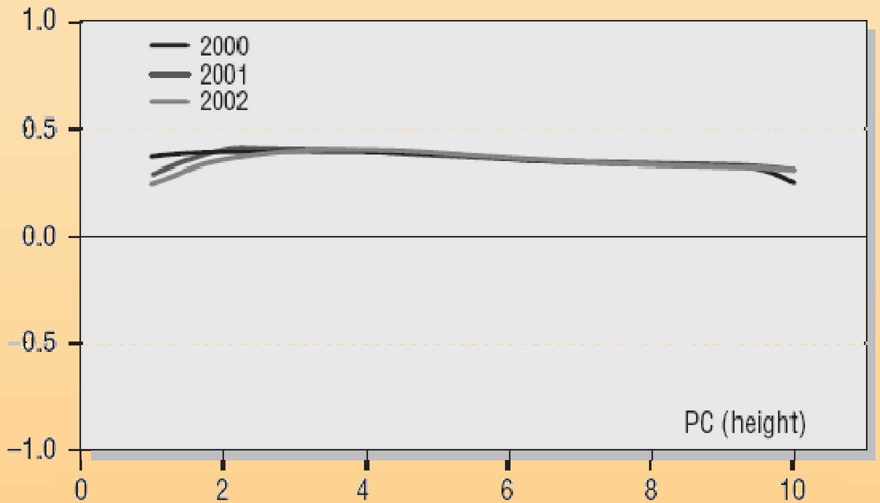
- According to **Bliss[1997] or Soto [2004b]**, the dynamic pattern in the volatility of interest rates affects the stability of the principal components.

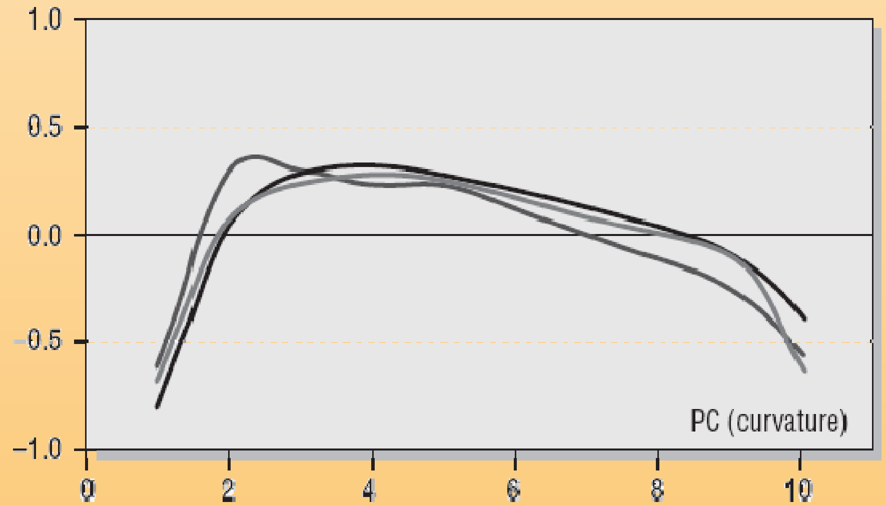
Limitations of the Principal Component Model

Static Factors Arising from a Dynamic Structure

- An example: Monthly changes of U.S. zero-coupon rates for years 2000, 2001, 2002.







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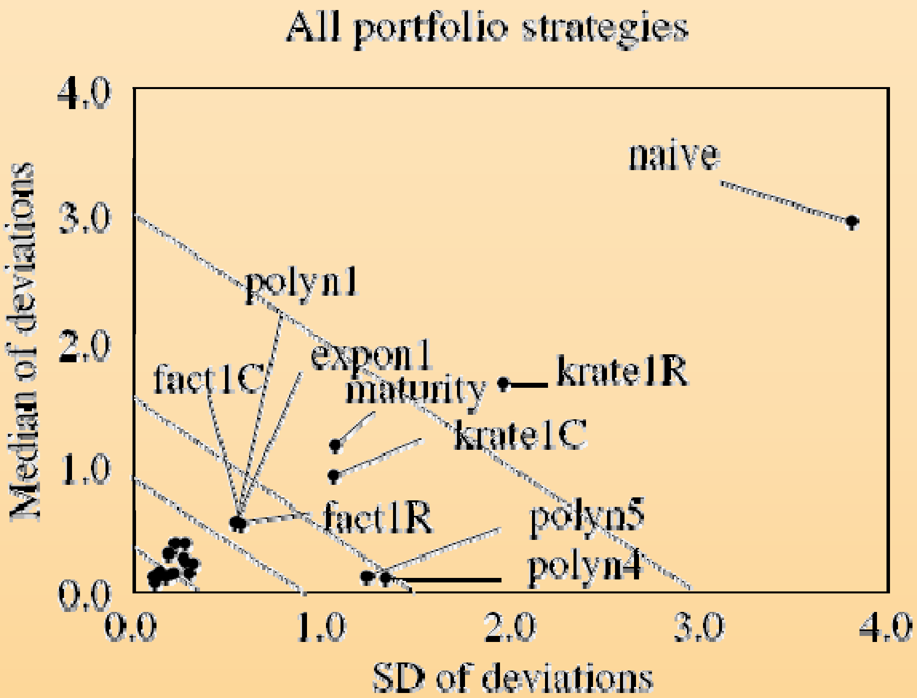
A conceptual comparison of the models

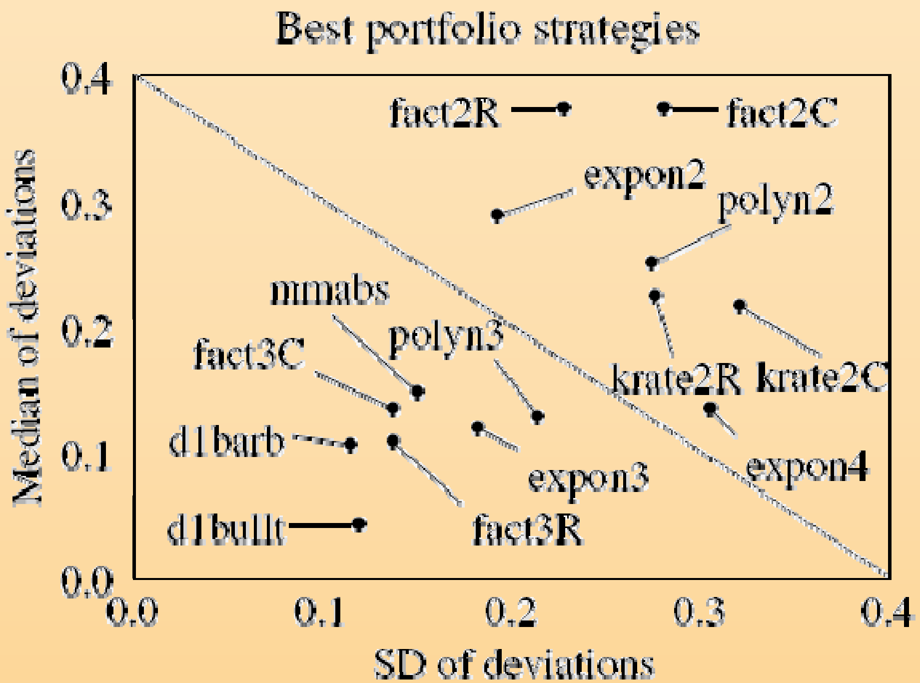
Polynomial Model	Key Rate Model	PCA Model
Unknown dimension	High dimension	Small dimension
Easy interpretation	Easy interpretation	Lack of interpretation
Stability	Stability	Inestability
Dependence	Dependence	Independence

An empirical comparison of the models

- **Soto [2004a]** analyzes the hedging performance of different strategies, including multiple factor models, trying to determine if performance is primarily attributable to the particular model chosen or to the number of risk factors considered.
- Data: Daily prices for Spanish government bonds for the period 1992 through 1999.
- Exercise: Compute the deviations between the realized return and the initial zero-coupon rate of real bond portfolios immunized for holding periods of 1, 2 and 3 years, and also the variability of these deviations.
- Results: (i) traditional immunization is easily bettered by more realistic strategies; (ii) the number of risk factors considered has a greater influence on the result than the particular model chosen; and (iii) three-factor immunization strategies offer the highest immunization benchmarks.

An empirical comparison of the models





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