KPMG

cutting through complexity

Dr. Dirk Schubert

The Financial Economics of Hedge Accounting of Interest Rate Risk according to IAS 39

The Financial Economics of Hedge Accounting of Interest Rate Risk according to IAS 39

by Dr. Dirk Schubert

Foreword

I am pleased to introduce the paper on "The Financial Economics of Hedge Accounting of Interest Rate Risk according to IAS 39". Hedge accounting of interest rate risk according to IAS 39 is applied to avoid P & L volatility resulting from accounting mismatch, and it is a common practice. However, issues concerning hedge accounting have not ceased to be items of topical interest. This analysis will show that hedge accounting of interest rate risk relies upon modern approaches of financial economics which are related to the pricing of interest rate derivatives. Derivative markets play a fundamental role for hedge accounting and are used to derive the valuation model used under IAS 39.

This paper provides a setup for interest rate hedging and demonstrates the connection of hedge accounting under IAS 39, valuation practices and risk management. The alignment of hedge accounting and risk management is also a principle advocated in the Exposure Draft for hedge accounting (ED 2010/13) published last year; therefore the analysis leads over to current discussions on hedge accounting.

I therefore hope that you will find this paper informative und useful in your daily work.

Sincerely,

Klaus Becker
Member of the Managing Board
Head of Financial Services
KPMG in Germany

Table of Contents

1 Introduction

Though hedge accounting of interest rate risk according to IAS 39 is applied in practice, the application has revealed that the existing rules under IAS 39 need to be augmented with respect to valuation or risk management practice in order to be meaningful or implementable.

The aim of the following paper is to provide a setup for interest rate hedging which relies upon financial economics, especially Arbitrage Pricing Theory. It will be shown that the rules concerning hedge accounting of interest rate risk rely upon modern approaches of financial economics and therefore provide a framework for the hedging of interest rate risk. The rules for hedge accounting of IAS 39 (IAS 39.88) imply a specific valuation concept in context with the definition of the portion, effectiveness measurement and determination of booking entries. This result also holds for the Exposure Draft (ED 2010/13) for hedge accounting published at the end of last year. The derived economic reasoning for IAS 39 is similar for US GAAP. Following the derived framework for IAS 39 in identifying the portion of the hedged risk separately and showing its reliable measurability, it can basically be extended to the hedge accounting of foreign currency hedging and under adequate conditions to hedge accounting of credit risk.

Interest rate hedge accounting is applied in order to avoid P&L volatility resulting from accounting mismatch. According to IAS 39 derivatives have to be measured at fair value through P&L, while e.g. loans are measured at amortized cost. Only in case the requirements concerning fair value hedge accounting under IAS 39 are met, loans can be measured at fair value related to interest rate risk so that the fair value changes effectively offset[1] the fair value changes of the hedging interest rate derivatives in P&L.[2]

---

1 Ineffectiveness may arise from e.g. counterparty risk on the hedging instrument.

2 In the following, for the sake of brevity, only fair value hedging is discussed in detail, since the underlying financial economics of cash flow hedging models are similar.

SECTION TWO

2

Interest Rate Hedge Accounting according to IAS 39

2.1 Main Requirements of Hedge Accounting – the Issue

According to IAS 39.AG99F the application requirements for hedge accounting are as follows:

a) The **designated risks** and **portions** must be **separately identifiable** components of the financial instruments;
b) changes in cash flow or the fair value of the entire financial instrument arising from changes in the designated risks and portions must be **reliably measurable.**

These definitions are also included in the Exposure Draft "Hedge Accounting" (ED 2010/13, §18), so the following analysis also holds for the new rules discussed for IFRS. Furthermore IAS 39.AG110 states that "… the hedge must relate to a specific and designated risk (hedged risk) […] and must ultimately affect the entity's P&L."

With respect to interest rate risk, the conditions above are assumed to be fulfilled for hedge accounting purposes; IAS 39 explicitly permits the designation of the LIBOR or EURIBOR component as a portion of the hedged item (e.g. bond / loan) (IAS 39.81, IAS 39.AG99C) and the utilization of a benchmark curve (e.g. "swap" curve) to determine the fair value of interest rate risk (e.g. discounting future cash flows) (IAS 39.86 (a), IAS 39.78, IAS 39.AG82 (a), IAS 39.AG102).[3]

This is illustrated by the following example:

FIGURE 1 **Decomposition of the Cash Flows according to IAS 39**

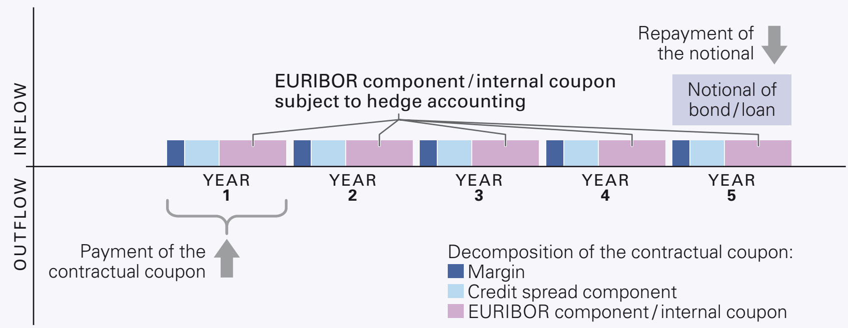


Figure 1 above shows the cash flows of a 5 year fixed coupon bullet bond / loan with annual payments. For the purpose of interest rate hedge accounting according to IAS 39, the contractual cash flows of the bond / loan can be decomposed into a margin, a credit spread and an EURIBOR component, although these components are not contractually specified. The EURIBOR component is subsequently utilized to define the "hedged risk" as the interest rate risk portion of the bond / loan as well as to determine the fair values for the postings and effectiveness test according to IAS 39.

---

3 According to IAS 39.AG107, IAS 39.AG108, IAS 39.IG F.4.3 the "benchmark curve" is supposed to reflect spreads resulting from counterparty and own credit spreads. This can be achieved also by applying the Absence of Arbitrage Principle. For the sake of simplicity we neglect the issue of "counterparty fair value (CVA)" adjustments.

It has to be acknowledged that IAS 39 does not provide explanations with respect to the economic reasoning of its hedge accounting model, especially there is no framework which justifies that hedged risks like interest rate risk, credit risk or foreign exchange risk fulfill the requirements of IAS 39 without being contractually specified. Like in the example, the EURIBOR component is not contractually specified in a fixed coupon bond / loan. How can it be justified that the EURIBOR component is a "portion" of the coupon bond / loan (hedged item), separately identifiable and reliably measurable? This is not only a theoretical issue but is of immediate relevance in practice if hedge accounting is implemented. The implementation of hedge accounting implies mathematical modeling with respect to the construction of benchmark curves (discount curves) and fair valuing. IAS 39 implicitly assumes that the required parameters are "observable" and taken from liquid markets; does IAS 39 provide any guidance in this respect? Do any further conditions (qualitative or quantitative) have to be fulfilled in order to meet the requirements of IAS 39 above?

The following paper provides a framework for hedge accounting and answers the questions raised.

The paper is structured as follows: Firstly we consider a simple hedge of a bond / loan using a coupon swap. This example is utilized to portray the application of IAS 39 and to derive a first set of results for a hedge accounting model. In *Section 3* it will be shown that the construction of discount curves involves modeling and explains the current issues in practice. The model setup will then be extended to stochastic term structure models in *Section 4* and hedges of portfolios of interest rate risk in *Section 5.*

2.2 Simple Hedges: Hedging of a Coupon Bond / Loan Using a Coupon Swap

2.2.1 Replication Strategy and Economic Hedging

In order to perform economic hedging with respect to interest rate risk, the cash flow component in the fixed coupon bond / loan, which is exposed to interest rate risk, has to be determined. For this purpose and with regard to the cash flow structure of the bond / loan the interest cash flow component termed "EURIBOR component (internal coupon)" is defined in the following way:

EQUATION 1 Definition of a EURIBOR Component (Internal Coupon)

$$\text{Notional} \overset{!}{=} \sum_{t=1}^{5} \frac{\text{EURIBOR Component (Internal Coupon)}}{(1+\text{Zero EURIBOR Swap Rate}(t))^{t}} + \frac{\text{Notional}}{(1+\text{Zero EURIBOR Swap Rate}(t))^{5}}$$

Please note that the equation above defines an equilibrium condition: Assuming that the market prices the bond / loan interest rate risk-neutral at par (notional l.h.s.), this constructs an interest-rate-risk-equivalent, synthetic bond / loan to a corresponding contractually specified bond / loan for a given maturity. Similar to the definition of "cost" in a funds transfer pricing method, the "EURIBOR component (internal coupon)" represents the cost of hedging against interest rate risk and is defined by means of the (derivative) Zero EURIBOR Swap Rates. Accordingly the "EURIBOR component (internal coupon)" coincides with the fixed rate payment of the coupon swap, and the hedging costs equal the price of the hedging instrument. Consequently for economic hedging only this synthetic bond / loan is considered, which is shown in the following *Figure 2*.

A motivation of this definition can be provided by the following economic rationale. Utilizing the example of the synthetic bond / loan in *Figure 2*, a portfolio consisting of a 5 year coupon bond / loan and the corresponding short-term cash funding on a six-month EURIBOR basis is considered. In this example the funding costs based on

FIGURE 2 **Construction of a Risk-Equivalent, Synthetic Bond / Loan to a Corresponding Contractually Specified Bond / Loan for a Given Maturity**

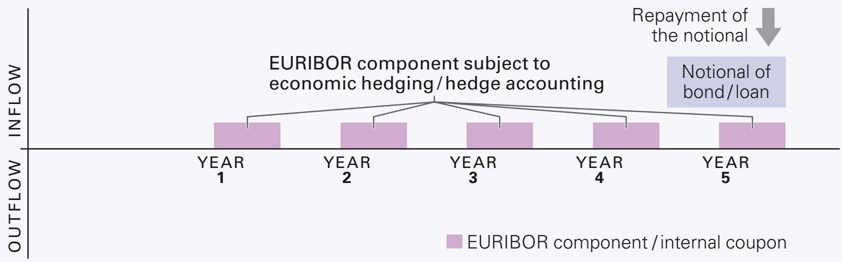
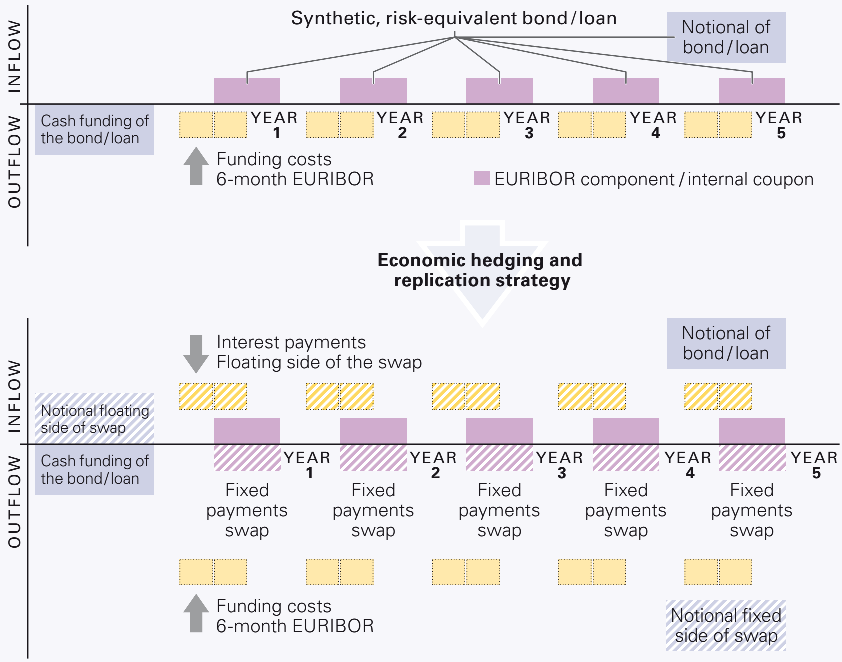


FIGURE 3 **Economic Hedging and Replication Strategy of an Interest Rate Hedge**[4]



4 The cash flow representation is adapted to that of an interest rate sensitivity gap (IRSG), i.e. arranged according to the next fixing date. The unfixed interest payments (usually not shown in an IRSG) are additionally indicated in broken frames.

EURIBOR are deterministic, since these are fixed for six months at the beginning of each period. This portfolio is exposed to interest rate risk, since the coupon payments of the synthetic bond/loan are fixed, and thus the fair value will change under changing market conditions, whereas the short-term funding is adapted to market conditions in short periods resulting in a fair value near 100% (being exact on fixing dates). Consequently the hedger enters into a six-month EURIBOR fixed coupon swap with a notional equal to the funding and the notional of the bond/loan and corresponding maturity schedule, which perfectly replicates the payoff of the portfolio.

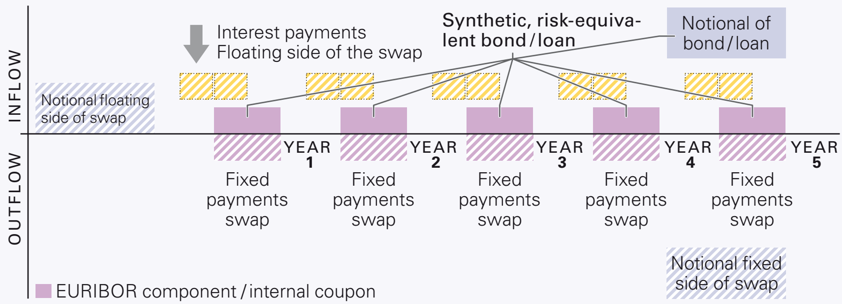
This is shown in *Figure 3*. This replication strategy corresponds to the absence of arbitrage in this example, which means that there is no risk-free revenue, neither by entering into the portfolio nor by entering into the swap; it is just the flip side of the same coin. In *Figure 3* the coupon swap is decomposed into a floating side and a fixed side in order to demonstrate that the portfolio is free of interest rate risk. This needs to be distinguished from the liquidity considerations.

2.2.2 Hedge Accounting according to IAS 39 Using a Coupon Swap

Continuing the example above, hedge accounting with respect to interest rate risk is performed neglecting the cash funding component, and it is assumed that all the requirements for fair value hedge accounting are met (IAS 39.78, IAS 39.86 (a) and IAS 39.88). Then the designation of a portion of cash flows using *Equation 1* is permitted according to IAS 39.81, IAS 39.AG99C (refer also to *Insights into IFRS,* KPMG, 7th Edition 2010/11, 3.7.180.10 and 3.7.180.12), provided that the internal coupon does not exceed the contractual coupon (counterexample sub-EURIBOR bond/loan). The hedge accounting example is portrayed in the following *Figure 4*.

The effectiveness testing is performed by evaluating the fair value changes of the coupon swap and the fair value changes of the interest-rate-risk-equivalent, synthetic bond/loan to a corresponding contrac-

FIGURE 4 **Hedge Accounting of Interest Rate Risk according to IAS 39**



tually specified bond/loan for a given maturity. This concept of a synthetic bond must not be mixed up with a "short cut" method, since here the inherent internal component attributable to the benchmark curve is derived and the measurement of effectiveness is retained in calculating the (hedge) fair value of the hedged item and hedging instrument. The hedge is termed to be effective according to IAS 39 if the ratio of both fair value changes is within the range of 80% to 125%. Due to the construction of the hedge in this "plain vanilla" case[5] the effectiveness will be close to 100%, only the fair value changes of the floating side of the coupon swap may cause ineffectiveness, e.g. deviation from the 100% ratio. Both the hedged item and the hedging instrument are "fair valued" using the same discount curve derived from the derivative market: (derivative) Zero EURIBOR Swap Rates.

2.2.3 First Results

Above a simple hedging relationship with deterministic cash flows has been considered, as a result the "portion" hedging approach advocated by IAS 39 can be justified by the Absence of Arbitrage Principle.

5 Additional features like prepayment options will be discussed below in *Section 2.2.3.* To simplify issues, timing differences of hedged item and hedging instrument or counterparty risk of the hedging derivative are neglected. In some cases, these issues may only influence hedge effectiveness, in other cases hedge accounting may not be achieved at all (cf. also *Section 3.4.2*).

This justification constitutes an equivalence using a replication argument, which implies the absence of arbitrage. For deterministic cash flows the replication strategy is model-independent, which means there is no further valuation model required. This will be much different if optionalities and stochastic cash flows are considered.

Using such a replication strategy implicitly assumes that there is only one "security market" for the derivative and the bond / loan. IAS 39 explicitly states in IAS 39.AG99C that a benchmark curve e.g. LIBOR can be applied. This (theoretical) "security market" has important properties: it assumes an integrated market for bond / loans and derivatives (no basis risk) as well as completeness. Accordingly the "security market" consists of interest rate swaps and zero coupon bonds derived from interest rate swaps so that any payoff of a bond / loan can be represented by (a combination of) payoffs of derivatives. Moreover the completeness of market is an important feature for financial economics, which will be addressed in connection with stochastic term structure models. The remaining requirements for hedge accounting such as separate identifiability and reliable measurability are tied to the existence and to the **liquidity** of the "benchmark curve". Interest rates or the benchmark curve are not directly observable in the market, but have to be derived from "liquid" financial instruments, like e.g. interest rate swaps. The benchmark curve derived from interest rate swaps defines the hedging costs which are equal to the price of the hedging instrument. This identity can be justified by the Absence of Arbitrage Principle and is summarized in the following *Table 1.*

Table 1 reveals that hedge accounting according to IAS 39 is tied to modern financial economics: the "benchmark curve" utilized as a discount curve for all financial instruments (hedging instrument and hedged item) introduces the "Law of One Price" and a "theoretical" security market, where all financial instruments are traded and the absence of arbitrage is assumed. A description of this theory and its mathematical theorems can be found in Cochrane (2001), Lengwiler (2004), Duffie (2001) and Dothan (2006).

TABLE 1 **The Requirements of IAS 39.AG99F and the Corresponding Concepts of Financial Economics**

Requirement IAS 39.AG99F	Financial economics
Portion	▸ Absence of arbitrage ▸ Completeness ▸ Integrated market for hedged item and hedging instrument through the common benchmark curve (derived from liquid market of hedging instruments) leading to the elimination of basis risk ▸ Determination of a (cash flow) component attributable to the designated risk by the hedging instrument
Separately identifiable	Identification by the hedging instruments and derivation of the "benchmark curve" – (derivative) Zero EURIBOR/LIBOR/OIS Swap Rates utilized for discounting.
Reliably measureable	Existence of a **liquid market** for the hedging instrument to derive the "benchmark curve", e.g. interest rate swaps (derivatives) based on EURIBOR/LIBOR/OIS, that covers all relevant market data to evaluate the portion of the hedged item attributable to the designated risk.

What kind of practical implications can be derived?

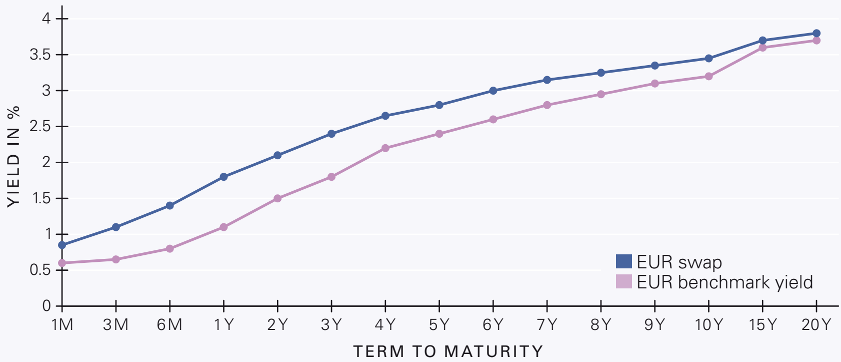
- In case of interest rate hedge accounting there is no contractual relationship between the hedged item and the hedging instrument; economic reasoning constitutes a basis for defining the hedged risk. This does not hold for every hedge accounting model. In case of credit risk hedge accounting there is, in addition to a similar economic reasoning, a contractual relationship between the hedging instrument and the hedged item. Utilizing the ISDA Matrix Approach for CDS markets, credit risk and economic risk transfer can be defined.
- The construction of a risk-equivalent, synthetic bond/loan to a corresponding contractually specified bond/loan for a given maturity is a well known and applied technique for interest rate treasury departments in context with term transformation and funds transfer pricing.[6] This can also be observed for credit

6 In general the real hedging instrument might differ slightly from the synthetic one due e.g. to counterparty risk which is not attributable to the hedged risk neither in accounting nor in the term transformation for the "profit center" treasury.

treasury departments for which the CDS spreads are used instead of the (derivative) Zero EURIBOR Swap Rates.

- In practice the derivative market is different from the cash market for bonds or loans. The following graph shows the EUR benchmark curves of the bond market (composed of government deposits and bond quotes) and the derivative market (composed of EURIBOR and swap quotes):

FIGURE 5 **EUR Benchmark Curves as of 03/15/2011** (Source: Reuters)



- As shown in the example, due to the fact that the derivative as well as the bond/loan are discounted using the (derivative) Zero EURIBOR Swap Rates, all differences with respect to liquidity in different markets are eliminated. Only by the elimination of basis risk ("difference between cash and derivative market") the effectiveness of the hedge can be proven according to IAS 39. Additionally the (derivative) Zero EURIBOR Swap Rates for a certain number of maturities represent quotes of unfunded financial instruments which are used to discount (funded) financial instruments like bonds/loans. Consequently the determination of the EURIBOR component according to IAS 39 is not only dependent on the (derivative) Zero EURIBOR Swap Rates but also on econometric modeling or interpolation. This will be discussed in the next section.

- The elimination of the basis risk has more severe impacts on hedge accounting if optionalities are involved. In practice market quotes of e.g. bonds with embedded options like prepayments include the value of these optionalities. For these instruments the changes in the fair values of the contracted cash flows as well as in the fair value of the prepayment option are dependent on interest rate moves.[7] Prepayment options are bond options – put option in the case of prepayable customer loans and call option in the case of issued bonds – and may be modeled as swaption since the synthetic and risk-equivalent bond equals the fixed leg of the swap, and the swap rate triggers the execution of the swaption. Pricing models in the bond market are calibrated on bond-specific discount curves (like government bond curves) and rather use historic volatilities. If hedge accounting is applied for such hedged items, modeling as swaption may be used, and parameters like the strike price of the prepayment option must be determined in accordance with the identified EURIBOR component. Since optionalities have to be evaluated by valuation models, the EURIBOR component becomes stochastic and model dependent! This will be discussed in context with the term structure models. From the effectiveness perspective we see: If the hedging instrument mirrors the embedded option, high effectiveness will be expected, otherwise ineffectiveness will occur depending on the extent the option is "in the money".

---

7 See also IAS 39.AG121 and IAS 39.BC178 where these circumstances are mentioned.

SECTION THREE

3

The Benchmark Curve according to IAS 39 for Hedges with Deterministic Cash Flow Profiles

3.1 Data and Notation

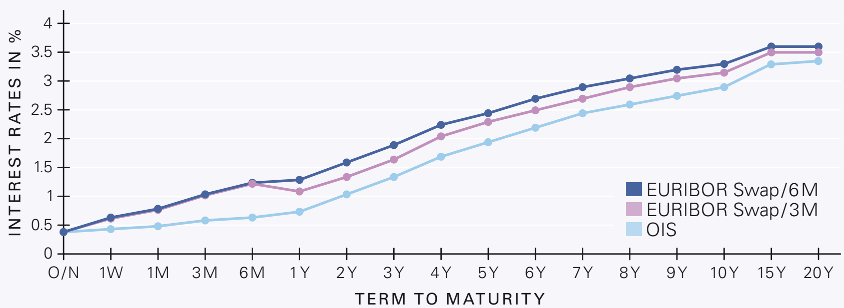
So far the concept of the "benchmark curve" has not been specified. The following table gives an example.

TABLE 2 Input Data for the Construction of a Benchmark Curve Dated 12/30/2010

Input data	Term	Maturity	Quotes	Input data	Term	Maturity	Quotes
EONIA	O/N	12/31/2010	0.393%	Swap 5Y/6M	5 Year	12/30/2015	2.466%
EURIBOR 1W	1 Week	01/06/2011	0.628%	Swap 6Y/6M	6 Year	12/30/2016	2.689%
EURIBOR 1M	1 Month	01/30/2011	0.790%	Swap 7Y/6M	7 Year	12/30/2017	2.891%
EURIBOR 3M	3 Month	03/30/2011	1.010%	Swap 8Y/6M	8 Year	12/30/2018	3.042%
EURIBOR 6M	6 Month	06/30/2011	1.234%	Swap 9Y/6M	9 Year	12/30/2019	3.170%
Swap 1Y/6M	1 Year	12/30/2011	1.303%	Swap 10Y/6M	10 Year	12/30/2020	3.281%
Swap 2Y/6M	2 Year	12/30/2012	1.566%	Swap 15Y/6M	15 Year	12/30/2025	3.632%
Swap 3Y/6M	3 Year	12/30/2013	1.898%	Swap 20Y/6M	20 Year	12/30/2030	3.691%
Swap 4Y/6M	4 Year	12/30/2014	2.214%				

The main data stem from swap rates with 6 month floating EURIBOR. Firstly it must be stated that there is no convention in the market as to what kind of input data, i.e. which maturities or instruments, are to be used in order to construct the "benchmark curve". Typically for the short term (less than one year), EONIA or EURIBOR are used, which can certainly be augmented by using future prices to increase the number of input data. Consequently a "benchmark curve" may consist of a composition of several financial instruments rather than one. Recently market conventions have changed, and e.g. OIS (Overnight Interest Rate Swap) quotes are used to value collateralized derivatives with Credit Support Annex.[8] Furthermore the example above uses EURIBOR swap/6 month floating in order to derive the term structure and the discount rates. Theoretically in this one-curve approach (i.e. discount factors and forward rates are calculated from the same curve), the swap rate should be independent of the tenor of the floating leg and coincide with the rate of the fixed rate bond given by the fixed leg. But market data show that this is not the case in practice as can be seen in *Figure 6*, which results in arbitrage possibilities.

FIGURE 6 **Comparison of Different Interest Swap Rate Curves**[9]



8 See e.g. "The price is wrong", in: *Risk Magazine,* 05/03/2010 or "LCH. Clearnet revalues $218 trillion swap portfolio using OIS", 17/06/2010 by C. Whittall and related articles.

9 OIS quotes are liquid up to 2 or at most 3 years, but single brokers treat also longer term OIS that also become more liquid due to the current development in the markets.

So the choice of the benchmark curve is not unique but depends on the specific data and the hedging instrument. Consequently changes in market conventions and different representations of discount curves may affect valuation models, since these are used to calibrate stochastic term structure models. In the deterministic case we have the choice between e.g. 25 ways to define the portion. Sole utilization of the benchmark curves (= 5 possibilities according to the possible choices of LIBOR 3M/6M, EURIBOR 3M/6M or OIS 1Day) or assignment of one curve as discounting curve and adjustment of the forward rates (= 4 times 5 possibilities for combinations of the choices mentioned above) is described below.

The differences in interest rates due to different tenors (repricing frequency on the float leg) of the corresponding swaps, also termed "basis" [10], result in different forward rates (i.e. future repricing rate of the float leg). When assigning "one" discount curve across different financial instruments (FRA, interest rate swaps, etc.), a multiple curve setup for pricing should be utilized. For a description refer e.g. to Bianchetti, M. (2010), Mercurio, F. (2009) or Fujii (2010). With respect to hedge accounting, if only one discount curve is assigned across different hedges and hedging instruments, ineffectiveness may occur. This ineffectiveness equals the fair value of the basis. But with the argument of portion hedging, the definition of the benchmark curve reduces this to a technical issue in connection with the implementation of hedge accounting on a large scale. The definition of the portion, the identification of the internal coupon and calculations of the hedge relationship will be done in the same way as shown for the example of an InArrears swap in *Section 4.2.8*. For the sake of simplicity we assume throughout the paper that discounting and forwarding is performed using the same interest rate curve ("benchmark curve").

10 Throughout the paper "basis" denotes the differences in cash and derivatives prices; at this point we adapt the concept to differences in the derivative market in the multiple curve context.

As a market convention the swap curve is defined on AA^- rating, so it is considered as a "credit risk free" curve by definition, but it certainly is not.

For the purpose of the paper the following notation is used:

$B(t_0, t)$ for $t \geq t_0$ denotes the value at time t_0 of one currency unit paid at time t. The function

$$B(t_0, \cdot): R_{\geq 0} \rightarrow R$$

is termed discount function or term structure curve at time t_0. $B(t_0, t)$ is the price of a zero coupon bond with face value 1 and maturity t (time to maturity $t - t_0$) at t_0.

3.2 Construction of the Benchmark Curve

Table 2 above shows the quotes of swaps with annual interest payments of the fixed leg (theoretically identical to annually paying fixed rate bonds); in order to derive zero coupon bond prices the "bootstrapping method" is applied, as is also mentioned in IAS 39.IG F.5.5. This method also relies on the Absence of Arbitrage Principle, assuming the same price for the zero and the coupon bearing bonds including compounding or equivalently assuming the same return when investing the same amount in a zero or a coupon bearing bond respectively, which is illustrated by an example.

Noting that the price of a 1 year zero coupon bond can be directly calculated from the quote by reverting the compounding method

$$B(t_0, t_0 + 1) = \frac{1}{1 + quote(Swap1Y\ /\ 6M)} = \frac{1}{1.01303} = 0.9871$$

the price for a 2 year zero coupon bond (notional EUR 1) is derived by interpreting the corresponding 2Y/6M swap as a portfolio of $quote(Swap2Y\ /\ 6M)$ pieces of a 1 year zero coupon bond for the first interest payment and $quote(Swap2Y\ /\ 6M)+1$ pieces of a 2 year zero coupon bonds for the interest and repayment at maturity. With $t_0 = 12/30/2010$ (assuming the same investment in zero and coupon bearing bond), this yields

$$1 = B(t_0, t_0+1) \cdot 0.01566 + B(t_0, t_0+2) \cdot (0.01566+1).$$

Rearranging and inserting the quotes for the 2Y/6M swap and $B(t_0, t_0+1)$ from above:

$$\left(\frac{1 - 0.9871 \cdot 0.01566}{(0.01566+1)}\right) = B(t_0, t_0+2) = 0.9694.$$

The entire zero coupon swap curve can successively be constructed in this way.

Accordingly the continuously compounded annual interest rates can be defined by the following equation (applying day count convention Act/365):

$$r(t_0, t) := \frac{-\ln\left(B(t_0, t)\right)}{(t - t_0)/365}.$$

The results of the "bootstrapping" and the evaluation of continuously compounded annual interest rates are shown in the following *Table 3*.

The "discounted cash flow method" uses the prices of the zero bonds to determine the price of a financial instrument by evaluating the cash flows. To determine the cash flow rollout for an instrument it might be necessary to predict also interest rates of future time periods leading to the concept of forward rates.

The Absence of Arbitrage Principle is also applied to derive the forward price and the forward rate of a zero coupon bond by defining

$$B(t_0, t, t+1) := \frac{B(t_0, t+1)}{B(t_0, t)} =: \left(\frac{1}{f_r(t_0, t, t+1)}\right).$$

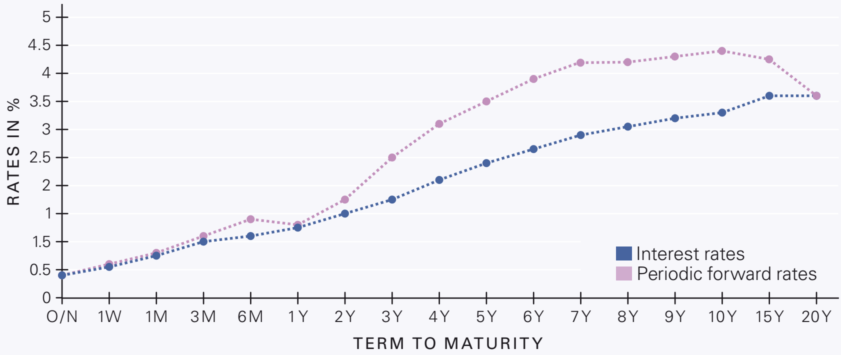
TABLE 3	Result of the "Bootstrapping" Zero Coupon Bonds Derived from Swap Rates				
Input data	**Term**	**Maturity**	**Quotes**	**Price of the "boot-strapped" zero coupon bonds**	**Continuously com-pounded annual interest rates**
EONIA	T/N	12/31/2010	0.393%	1.0000	0.40%
EURIBOR 1W	1 Week	01/06/2011	0.628%	0.9999	0.64%
EURIBOR 1M	1 Month	01/30/2011	0.790%	0.9993	0.80%
EURIBOR 3M	3 Month	03/30/2011	1.010%	0.9975	1.02%
EURIBOR 6M	6 Month	06/30/2011	1.234%	0.9938	1.25%
Swap 1Y/6M	1 Year	12/30/2011	1.303%	0.9871	1.29%
Swap 2Y/6M	2 Year	12/30/2012	1.566%	0.9694	1.55%
Swap 3Y/6M	3 Year	12/30/2013	1.898%	0.9449	1.89%
Swap 4Y/6M	4 Year	12/30/2014	2.214%	0.9155	2.21%
Swap 5Y/6M	5 Year	12/30/2015	2.466%	0.8841	2.46%
Swap 6Y/6M	6 Year	12/30/2016	2.689%	0.8507	2.69%
Swap 7Y/6M	7 Year	12/30/2017	2.891%	0.8159	2.90%
Swap 8Y/6M	8 Year	12/30/2018	3.042%	0.7825	3.06%
Swap 9Y/6M	9 Year	12/30/2019	3.170%	0.7496	3.20%
Swap 10Y/6M	10 Year	12/30/2020	3.281%	0.7173	3.32%
Swap 15Y/6M	15 Year	12/30/2025	3.632%	0.5814	3.61%
Swap 20Y/6M	20 Year	12/30/2030	3.691%	0.4851	3.61%

Assume buying a portfolio of $B(t_0,t,t+1)$ zero coupon bonds with a notional of EUR 1 and maturity t and selling one zero coupon bond with maturity $t+1$ and of notional EUR 1. The value of the portfolio is determined at time t_0, and the absence of arbitrage implies that the value is zero:

$$B(t_0,t,t+1)\cdot B(t_0,t)-B(t_0,t+1)=0.$$

In the following *Figure 7* the periodic forward rates and interest rates are compared. Observe that for the construction of these curves the Absence of Arbitrage Principle is used.

FIGURE 7 **Continuously Compounded Annual Interest and Periodic Forward Rates**



If the bootstrapping procedure is successful, i.e. there are no negative forward rates, the resulting curve on its own is arbitrage free. But as mentioned above, the benchmark curve depends on the choice of quoted products.

Continuing the example of the internal coupon and using the notation above, the internal coupon for a five year fixed rate bond/loan can be evaluated as follows (see *Equation 1*):

EURIBOR Component (Internal Coupon)

$$=\text{Notional}\cdot\left(\frac{1-\dfrac{1}{(1+\text{Zero EURIBOR Swap Rate}(t))^5}}{\sum_{t=1}^{5}\dfrac{1}{(1+\text{Zero EURIBOR Swap Rate}(t))^t}}\right)$$

$$=\text{Notional}\cdot\left(\frac{1-B(t_0,t_0+5)}{\sum_{t=1}^{5}B(t_0,t_0+t)}\right)=100\cdot\frac{0.1159}{4.7010}=2.46.$$

As outlined above, the internal coupon is by definition the 5 year quote of the swap rate (compare *Table 2*) and equals the cost of hedging for the particular 5 year bond/loan.

The following table gives a real example.

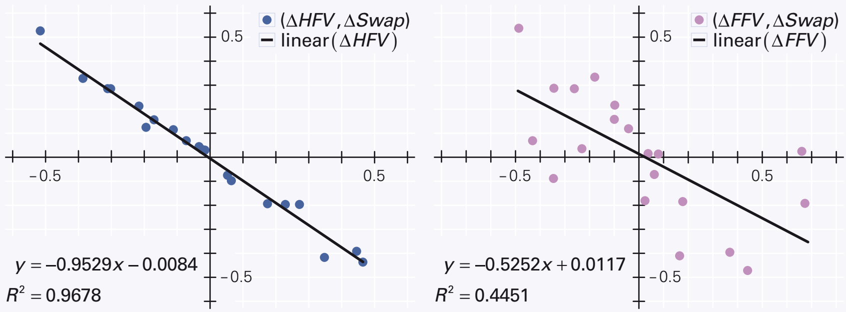
TABLE 4 **Terms and Conditions of the Example Bond (Deterministic Cash Flow Profile)**

Specifications	Example bond
Coupon:	4%
Frequency:	Annually
Notional:	100
Currency:	EUR
Maturity:	06/15/2012
Day count convention:	Act/Act

Fair value changes of an interest rate swap and fair value changes of the example (illustrated in *Table 4*) are compared: for the bond the full fair value change (ΔFFV, "changes in market quotes") and the fair value changes due to interest rate risk (ΔHFV) are shown. ΔHFV is evaluated using the discounted cash flow method based on the "benchmark curve". According to IAS 39 the effectiveness testing is performed by forming the ratio of the fair value change of the interest rate swap and the fair value changes of the bond ("hedged item") due to interest rate risk (ΔHFV). The effectiveness testing is successful if the absolute value of the ratio (b) lies between 80% and 125%.

The example above reveals that the effectiveness is only successful if the swap curve is used as "benchmark curve" ($b = -0.9529; R^2 = 0.9678$); when comparing market quotes for the bond with changes of the interest rate swap no effectiveness is achieved ($b = -0.5252; R^2 = 0.4451$). This example shows also the impact of the basis risk, i.e. the influence of other types of inherent risks in the market value and demonstrates the application of the "Law of One Price" using the benchmark curve. Furthermore it shows that fair value changes of interest rate swaps are of low explanatory power to describe changes in market quotes of the underlying cash product and are therefore of low performance with respect to real market quotes (full fair value) as shown in *Figure 8*. This performance issue will be discussed in *Section 4.2.7* below. Furthermore please note that the hedge fair value (as derived above) represents the hedging costs with respect to interest rate risk, whereas the full fair value (market quotes)

FIGURE 8 **Comparison of Fair Value Changes for Effectiveness Testing according to IAS 39**



corresponds to the liquidation costs of the entire hedge. Consequently these represent different valuation (measurement) approaches.

Hedge accounting of interest rate risk using the "benchmark curve" approach is also permitted under US GAAP (FAS 133). Although the portion hedging approach (FAS 133[11]) is permitted, according to FAS 133.21 f., the contractual cash flows have to be used in order to evaluate the fair value changes of the bond ("hedged item"). This only affects the result of the effectiveness test, but not the economic reasoning outlined above. Accordingly these results also apply to US GAAP and do not mean that US GAAP provides a hedge accounting model superior to that of IAS 39.

3.3 Impact on the Requirements of Hedge Accounting under IAS 39

The discussion above shows that the requirements "reliably measurable" and "separately identifiable" of IAS 39 are linked to a "liquid"

11 Refer to *Derivatives and Hedging Accounting Handbook,* KPMG, January 2011, 18.03.

benchmark curve. So far we have introduced a notion of the concept "liquid". Obviously a benchmark curve cannot be constructed if there is no observable data. Thus – as a practical consequence – hedge accounting is not possible for currencies where no data are available.

Before turning to a definition of "liquid" in terms of hedge accounting according to IAS 39, we address an additional feature of modeling benchmark curves. As shown in *Figure 7*, there are dotted lines between the various data points, so modeling the entire shape of the benchmark curve requires additional modeling. In the deterministic case we can distinguish between:

- "econometric" modeling and
- interpolation.

Modeling is required when the term of the EURIBOR component (cost of hedging, internal coupon) of bonds/loans does not coincide with the observable swap quotes. If e.g. the maturity of the bond/loan equals 4.5 years, econometric modeling or interpolation is required to determine the swap quote in terms of the internal coupon. In practice a number of models are applied to address this issue:

- parameter-based approximation models ("econometric model") incorporating general assumptions on basic features of the term structure, e.g. the Nelson-Siegel (ns) Representation

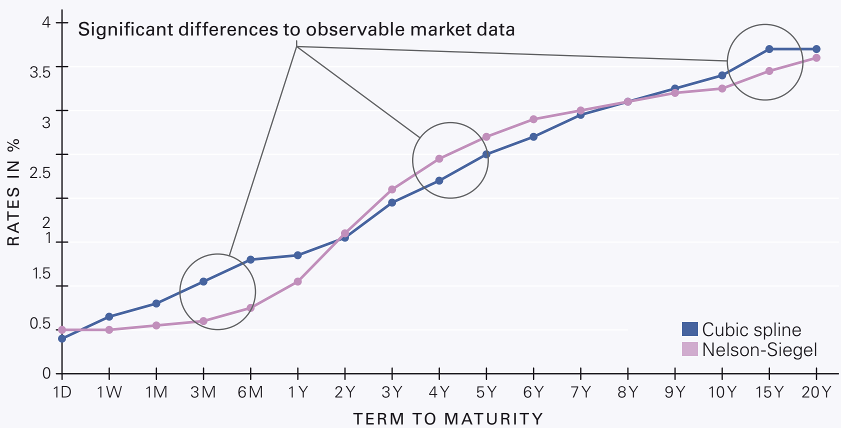
$$ns(T) = b_0 + (b_1 + b_2) \cdot \frac{\left[1 - \exp\left(-\frac{T}{\gamma}\right)\right]}{\frac{T}{\gamma}} - b_2 \exp\left(-\frac{T}{\gamma}\right),$$

  where b_0, b_1, b_2, γ represent shape parameters and T is maturity,
- (exact) interpolation methods that join the given nodes, e.g. linear interpolation, where only the nearest neighbours have influence, or cubic splines, polynomials of the 3rd order defined from node to node so that the catenation is continuous

and once differentiable in each node, resulting in a function defined piecewise that meets all given nodes. For cubic splines the conditions mentioned determine the polynomials uniquely, leaving some degree of freedom at the end points. These have to be fixed carefully since the effect can propagate through the entire structure of the curve. Theoretical details can be found in Stoer (1983) and details on the implementation in Press et al. (2002).

From the two examples above an important result can be derived. As becomes apparent from *Figure 9*, the Nelson-Siegel Representation does not necessarily recover the initial term structure, whereas the cubic spline interpolation does. So using an econometric model might lead to situations where there are "two" quotes for one term. This is inconsistent with the definition of fair value in IAS 39, since the term structure has to be uniquely defined in order to obtain unique portions and (hedge) fair values. Furthermore, this will result in arbitrage opportunities. Consequently "econometric modeling" is in general not appropriate for applying IAS 39!

FIGURE 9 Nelson-Siegel and Cubic Splines



This reveals that hedging according to IAS 39 depends on additional modeling so "liquid" can be defined in a statistical way:

- Use as many observable and liquid data (quoted financial instruments and parameters) as possible[12] for the construction of the "benchmark curve"; in practice the derivative market is considered as the most relevant source.
- The modeling required to set up the entire term structure is restricted by the fact that the defined benchmark curve has to be recovered after modeling. Therefore, in case of deterministic cash flows only exact interpolation is permitted.

At first glance this list of requirements appears to be weak, but it is not. As will be shown in connection with stochastic term structure modeling to derive prices for derivatives, mathematical modeling and hedge accounting are facing limitations. "Liquidity" implies **calibration** requirements, since IAS 39 requires benchmark curves and volatility term structures to be unique and recovered after modeling. Econometric modeling approaches are therefore not consistent with IAS 39.

3.4 Overview of the Basics of Interest Rate Hedge Accounting

In the following subsections basic steps and results of hedge accounting in the case of deterministic cash flows are summarized. As will be shown in the next chapter, the steps and results remain basically the same in the case of stochastic cash flows.

---

12 This is in line with the principle of "maximising the use of relevant observable inputs and minimising the use of unobservable inputs" as stated in recently issued IFRS 13 "Fair Value Measurement".

3.4.1 Steps in Practical Implementation

For hedge accounting of interest rate risk in general the following steps are taken:

1. Interest rate risk is an unobservable risk: Interest rates have to be derived from interest bearing financial instruments (interest rate swaps, government bonds). These derived rates may be considered as an observable input into a valuation model under IAS 39. You cannot go out in the market and buy 8% interest rate, you have to buy an interest-rate-bearing financial instrument (bond, loan, interest rate swap).
2. Pricing in cash markets (e.g. bonds) differs from that in derivative markets. This results from different liquidity, market conventions etc. This is termed basis risk.
3. The derivative market for interest rate derivatives (unfunded) is considered the most reliable and liquid source of prices for interest rates. Treasury departments use the derivative market to price interest bearing financial instruments (transfer pricing). There are different interest rate swap markets (EUR quotations):
   - LIBOR 3M/6M
   - EURIBOR 3M/6M
   - OIS 1 Day (for short-term maturities)
4. Define a "benchmark curve" (hedged risk), e.g. LIBOR 6M and construct the curve (bootstrapping): "Strip Zero Coupon Bonds" out of the interest rate swap market (LIBOR 6M).
5. Decide to "decompose" the contractual cash flows of the hedged item (bond/loan) or not: internal coupon vs. contractual ("portion" of cash flows).
6. Use the derivative zero coupon curve as discounting curve for the hedged item (bond/loan) and the derivative.
7. Evaluate the "fair values" and perform effectiveness testing.

3.4.2 Overview of Key Results

In the following the results and arguments of the present chapter for the case of deterministic cash flows are summarized.

- As soon as one benchmark curve is defined, all cash flows (derivative or bond/loan) are priced by the quotes from the derivative market (interest rate swap)! An integrated market for cash and derivative products is created.
- This is termed the "Law of One Price" for all financial instruments involved in hedge accounting.
- Basis risk is eliminated, i.e. all pricing differences resulting from different market data of cash market and derivative market are eliminated.
- In case of interest rate risk hedging there is no contractual relationship between the hedged item and the hedging instrument.
- Economics: The hedged item is priced according to its hedging costs!
- Provided similar "critical terms" of derivative and hedged item (same notional, maturity schedule etc.) and in the simplest case (no additional features like InArrears fixing or prepayment options) the only source of ineffectiveness is the floating side of the interest rate swap or fair value changes due to the credit spread or margin of the hedged item.
- In case the internal coupon method is used, the cash flows of the hedged item exactly match (in the simple case) the fix side of the swap. Furthermore, since discounting is performed by the same discount curve, these values offset each other perfectly. The portion equals the swap rate at inception.
- In case the contractual coupon method is applied, fair value changes of the margin and credit spread of the hedged item cause some ineffectiveness.
- In reality there is rarely, if ever, perfect effectiveness due to the floating leg, maturity mismatches, prepayment options, incongruities in payment frequencies, counterparty credit risk etc.

SECTION FOUR

4

The Benchmark Curve according to IAS 39 for Hedges with Stochastic Cash Flow Profiles

4.1 The Task

As shown above, in case of deterministic cash flow profiles replication arguments can be utilized to justify the hedge accounting approach. As soon as stochastic cash flow profiles depending on the evolution of interest rates are involved, the described setup cannot be applied. This becomes apparent if optionalities, caps, floors, Bermudan Options, InArrear features are involved in hedge accounting (eligible e.g. according to IAS 39.AG94), since the replication by zero bonds is no longer possible and the assumption of deterministic interest rates would lead to degeneration towards forward-type products and false evaluations.

Under these circumstances stochastic modeling of interest rates is required. These models are termed "term structure models of interest rates" and have been subject to academic research and development in recent years.

Additionally they represent a very complex topic, since sophisticated mathematical theory is required. In the following we only present the basic modeling approaches and ideas in order to show the connection to hedge accounting according to IAS 39; a full description of this topic is beyond scope. For a detailed description of this topic please refer e.g. to Sondermann (2006), Sandmann (2010), Björk (2004), Jarrow (1996), Jarrow / Turnbull (1996), Brigo / Mercurio (2006), Fries (2007), Musiela, M. / Rutkowski, M. (2010) or Rebonato (2004).

Term structure models can be classified into two main modeling approaches: short rate models and market models. In case of short rate models the unobservable instantaneous short rate is modeled so as to provide for each time t the basis to construct the entire term structure, whereas market models directly model observable "quantities" like (nominal) LIBOR or swap rates.

Irrespectively of the modeling approaches, the basic idea of such models is to use the absence of arbitrage to model the term structure relative to the current term structure. Term structure models describe the entire economy, so the quantities are modeled simultaneously for all different maturities (zero bond prices, LIBOR rates). So these modeled quantities cannot take any form even though they are modeled by random variables. In order to achieve this coherence result under the absence of arbitrage, a stochastic model and the calibration to observable market data (current term structure and term structure of volatilities) are required.

4.2 Term Structure Modeling – the Main Modeling Ideas

4.2.1 Definitions of Interest Rates and Descriptions of Term Structures

With respect to the term structure of interest rates there are different descriptions. In the previous section, zero coupon bonds and forward rates were introduced in a discrete setup. In the following definitions in continuous time $\{0 \le t \le T\}$ are provided:

(1) Zero coupon bonds:

$B(t,T)$ denotes the price of one monetary unit at time t deliverable at time T (maturity)

(2) Yields (continuously compounded):

$$y(t,T) := -\frac{1}{(T-t)} \log\left(B(t,T)\right)$$

(3) Instantaneous forward rates (continuously compounded):

$$f(t,T) := -\frac{\partial}{\partial T} \log\left(B(t,T)\right)$$

(4) Instantaneous spot rate, also termed "short rate":

$$r(t) = f(t,t) = -\frac{\partial}{\partial T} \log\left(B(t,T)\right)\Big|_{T=t}$$

It is important to note that (1) – (3) are equivalent descriptions of the interest rate term structure, whereas the description utilizing short rates requires an equivalent martingale measure. In order to portray the concept of equivalent martingale measures and term structure modeling, a simple one-factor model is presented. This example assumes a (discrete) evolution of spot rates modeled by a binomial (normally distributed) tree over three periods.

4.2.2 A One-Factor Model Example

In order to portray the concept of equivalent martingale measures and demonstrate the Absence of Arbitrage Principle in a term structure of interest rates setup, a simple one-factor model with three time periods is presented. According to *Table 3* we consider the following initial data (see *Table 5*).

The initial starting point in a term structure model is the definition of the modeled quantity and an assumption concerning its stochastic dynamic ("uncertainty", "term structure evolution"). In the following,

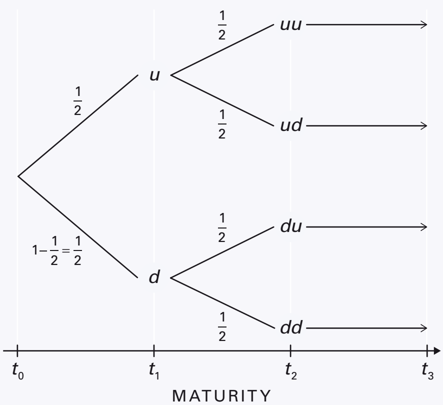
TABLE 5 **Initial Data for the One-Factor Model Example with Normally Distributed Interest Rates**

Input data	Term	Maturity	Price of zero coupon bonds	Continuously compounded annual interest rates	Volatility of spot rates
Swap 1Y/6M	1Y	12/30/2011	0.9871376	1.29%	0.02
Swap 2Y/6M	2Y	12/30/2012	0.9693612	1.55%	0.01
Swap 3Y/6M	3Y	12/30/2013	0.9449309	1.89%	

a one-factor time discrete model for the "short rate" is considered for the modeling of the term structure evolution. *Figure 10* shows the binomial state space tree diagram of such a model: At each node in the tree two possibilities u (for "up") and d (for "down") can occur; for each branch a positive probability ("real world probability") of $p = 0.5$ is assumed. The tree diagram is the same as for a coin tossing experiment.

Given the initial data in *Table 5* for $t_0 = 12/30/2010$ and the model in *Figure 10*, we construct the evolution of the term structure with $t_1 = 12/30/2011$, $t_2 = 12/30/2012$ and $t_3 = 12/30/2013$. For reasons of simplicity we write $B(i,j) = B(t_i, t_j), i, j = 0,\ldots,3$, and a sub index denotes the respective branch node at the starting point, i.e. $B_u(1,3)$ starts at time t_1 from state u.

FIGURE 10 **One-Factor State Space Tree Diagram**



Provided the stochastic dynamics of the short rates are given and thus the values of $r(t)$ for each node of the tree, *Figure 10*, the question is how to determine the evolution of the term structure, i.e. the values of $B(i,j), i, j = 1,\ldots,3$, especially the forward zero coupon bonds $B_u(1,3)$ and $B_d(1,3)$ that are not immediately derivable from the given data.

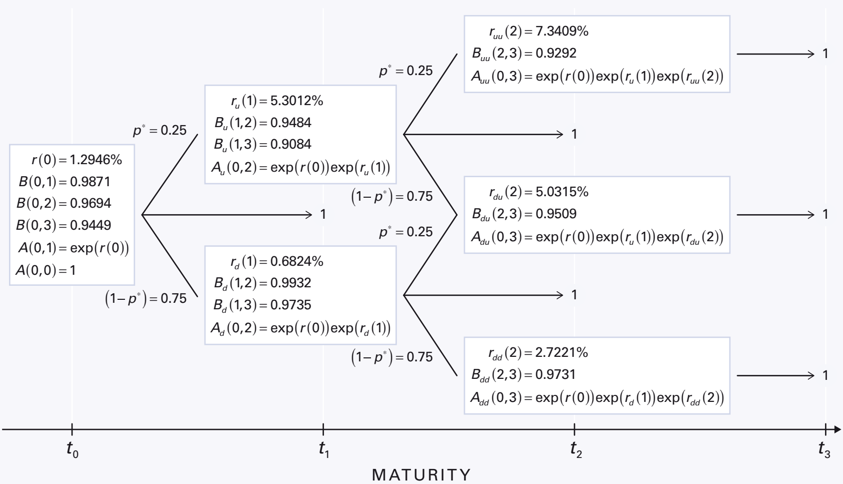
We start with the determination of the spot rate at time t_0, which is given by the following equation using the data in *Table 5*:

$$0.9871376 = \exp(-r(0)), \text{ resulting in } r(0) = 1.2946\%.$$

Using this and an assumed volatility of the periodic interest rate $r(t)$ of 2%, the values of $r(i), i = 1,2,3$ in each node of the tree are chosen in consistence with the given discount factors $B(0,i), i = 1,2,3$ as shown in *Figure 11*. From the periodic interest rate the one-period discount factors $B_u(1,2)$, $B_d(1,2)$, $B_{uu}(2,3)$, $B_{du}(2,3)$, $B_{dd}(2,3)$ and all values of the money market account $A_s(0,j), j = 1,2,3; s \in \{u,d,uu,du,dd\}$ are derived. Thus it just remains to determine $B_u(1,3)$ and $B_d(1,3)$, which will be done in the following using no arbitrage arguments and the derivation of a risk-neutral probability measure under which all discounted zero bonds fulfill the martingale property.

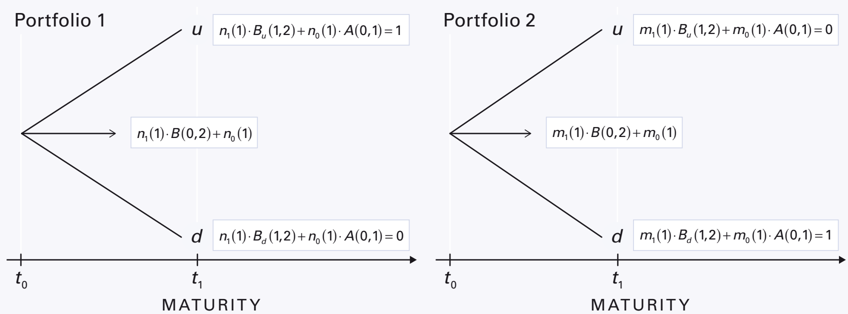
For the derivation of the risk-neutral measure in t_1 we construct two portfolios consisting of $B_u(1,2)$, $B_d(1,2)$ and the money market account $A(0,1)$. One portfolio consists of $B_u(1,2)$ and the money market account

FIGURE 11 Evolution of the Term Structure in the One-Factor Model (Results)



and pays off 1 at time t_1 if state u, and 0 if state d occurs. The other portfolio uses $B_d(1,2)$ instead and pays 1 at time t_1 if state d, and 0 if state u occurs. $n_0(1), n_1(1)$ and $m_0(1), m_1(1)$, respectively, denote the number of units invested in the money market account and the forward zero bonds in each portfolio.[13] Portfolio 1 and portfolio 2 are assets with state-contingent cash flows (Arrow-Debreu Securities) as shown in the following *Figure 12*.

FIGURE 12 Trading Strategies for State Contingent Portfolio with Payoff Profile 1



Solving for $n_0(1), n_1(1)$ yields the following:

$$n_1(1) = \frac{1}{B_u(1,2) - B_d(1,2)} = \frac{1}{0.9484 - 0.9932} = -23.31,$$

$$n_0(1) = -\frac{1}{A(0,1)} \cdot \left(\frac{B_d(1,2)}{B_u(1,2) - B_d(1,2)} \right) = 21.87\ .$$

Under the absence of arbitrage assumption, denoted by $\overset{a}{=}$, the price in t_0 of portfolio 1, denoted by $\varpi_{1,1}$, is:

$$\varpi_{1,1} \overset{a}{=} n_1(1) \cdot B(0,2) + n_0(1) = \frac{1}{A(0,1)} \cdot \left[\frac{A(0,1) \cdot B(0,2) - B_d(1,2)}{B_u(1,2) - B_d(1,2)} \right]$$

$$= 0.2468$$

13 Such a construction of a portfolio is an example of a "self-financing" strategy, since the portfolio weights are determined at the beginning and do not involve additional payments within a given period.

Similar calculations for the second portfolio, with a price in t_0 denoted by $\varpi_{1,0}$, yield:

$$m_1(1) = \frac{1}{B_d(1,2) - B_u(1,2)} = \frac{1}{0.9932 - 0.9484} = 23.31,$$

$$m_0(1) = -\frac{1}{A(0,1)} \cdot \left(\frac{B_u(1,2)}{B_d(1,2) - B_u(1,2)} \right) = -20.88,$$

$$\varpi_{1,0} \stackrel{a}{=} m_1(1) \cdot B(0,2) + m_0(1) = \frac{1}{A(0,1)} \cdot \left[\frac{A(0,1) \cdot B(0,2) - B_u(1,2)}{B_d(1,2) - B_u(1,2)} \right]$$
$$= 0.7404.$$

Since the sum of the two state-contingent portfolios pays 1 monetary unit with certainty in t_1, the price equals:

$$\frac{1}{A(0,1)} = B(0,1) \stackrel{a}{=} \varpi_{1,1} + \varpi_{1,0}.$$

Furthermore, a portfolio consisting of $\varpi_{1,1}$ pieces of $B_u(1,2)$ and $\varpi_{1,0}$ pieces of $B_d(1,2)$ is self-financing by the definition of the Arrow-Debreu Securities ($\varpi_{1,1}$ and $\varpi_{1,0}$ or portfolio 1 and 2) from above and pays 1 monetary unit in t_2 with certainty. Thus the price in t_0 is:

$$B(0,2) \stackrel{a}{=} \varpi_{1,1} B_u(1,2) + \varpi_{1,0} B_d(1,2).$$

If we define a so-called pseudo probability (that has nothing to do with the "real world"):

$$p^* = A(0,1) \cdot \varpi_{1,1} \text{ and } q^* := A(0,1) \cdot \varpi_{1,0} = A(0,1) \left(\frac{1}{A(0,1)} - \varpi_{1,1} \right) = 1 - p^*,$$

then we get, with $A(0,0) := 1$:

EQUATION 2 Martingale Property of Discounted Forward Price in t_1

$$\frac{B(0,2)}{A(0,0)} \stackrel{a}{=} \frac{p^*}{A(0,1)} B_u(1,2) + \frac{(1-p^*)}{A(0,1)} B_d(1,2) = \frac{1}{A(0,1)} E^{p^*}\left[B(1,2)\right],$$

where E^{P^*} denotes the expectation with respect to the probability p^*.

We observe that under p^* the discounted (stochastic) price process of a two year zero coupon bond is a martingale, the pseudo probability p^* is therefore termed martingale measure or risk-neutral probability.

Furthermore in *Equation 2* it can be seen that the risk-neutral measure is linked to the term structure in t_0, since the risk-neutral measure is derived from the assumed term structure evolution (stochastic process) and the current zero coupon bond prices. So, essentially the market chooses the risk-neutral measure (martingale measure). The determination of the parameters and risk-neutral measure using market quotes is termed "calibration" of the model.

Since we are interested in the evaluation of the fair value of the forward bonds $B_u(1,3), B_d(1,3)$ it is necessary to ask for the conditions to apply the martingale measure derived for t_1 for all states and times $t = 2,3$.

For this purpose we introduce the "excess return per unit risk":

$$\vartheta(t,T,p_t) := \frac{E^{p_t}\left[\frac{B(t+1,T)}{B(t,T)}\right] - (1+r(t))}{\sqrt{VAR^{p_t}\left[\frac{B(t+1,T)}{B(t,T)}\right]}},$$

where VAR^{p_t} denotes the variance with respect to the probability p_t.

For $t = 0, T = 2$, we have:

$$\begin{aligned}\vartheta(0,2,p_0) &= \frac{\left[\frac{p_0 \cdot B_u(1,2) + (1-p_0)\cdot B_d(1,2)}{B(0,2)}\right] - (1+r(0))}{\frac{1}{B(0,2)}\sqrt{p_0\cdot(1-p_0)}\cdot\left[B_u(1,2) - B_d(1,2)\right]} \\ &= \frac{\left[p_0\cdot B_u(1,2) + (1-p_0)\cdot B_d(1,2)\right] - B(0,2)(1+r(0))}{\sqrt{p_0\cdot(1-p_0)}\cdot\left[B_u(1,2)-B_d(1,2)\right]} \\ &= \frac{\sqrt{p_0}}{\sqrt{(1-p_0)}} + \left(\frac{B_d(1,2) - (1+r(0))B(0,2)}{B_u(1,2) - B_d(1,2)}\right)\cdot\frac{1}{\sqrt{p_0\cdot(1-p_0)}}.\end{aligned}$$

Using the definition of p_0^* we have:

$$\vartheta(0,2,p_0) = \frac{\sqrt{p_0}}{\sqrt{(1-p_0)}} - \frac{p_0^*}{\sqrt{p_0 \cdot (1-p_0)}} = \frac{p_0 - p_0^*}{\sqrt{p_0 \cdot (1-p_0)}}.$$

There are important results:

- Given the objective probability of p_0, the "excess return per unit risk" depends linearly on the risk-neutral probability p_0^*.
- "Excess return per unit risk" equals zero under risk-neutral probability ($p_0 = p_0^*$).
- Additionally it can be proven for the more general case that assuming the objective probabilities p_t the "excess return per unit risk" depends linearly on the risk-neutral probability p_t^* and thus p_t^* is the same for all maturities T, if and only if $\vartheta(t,T,p_t)$ is the same for all T (for a proof refer to Jarrow (1996), p. 102).

In order to determine the remaining $B_u(1,3), B_d(1,3)$, it is assumed that the "excess return per unit risk" $\vartheta(t,T,p_t)$ is equal for all maturities T, so $p^* := p_0^* = p_1^* = p_2^* = 0.25$. Consequently all discounted prices of securities are martingales under the corresponding risk-neutral measure P^*. Using this result we can evaluate:

$$\frac{B_u(1,3)}{A(0,1)} = E^{p^*}\left[\frac{B(2,3)}{A_u(0,2)}\right] = B_u(1,2) \cdot \left[p^* B_{uu}(2,3) + (1-p^*) B_{du}(2,3)\right],$$

using $A(0,1) = \exp(r(0))$ and $A_u(0,2) = \exp(r(0)) \cdot \exp(r_u(1))$ so that

$$\frac{A(0,1)}{A_u(0,2)} = \frac{1}{\exp(r_u(1))} = B_u(1,2).$$

This yields $B_u(1,3) = 0.9084$, and analogously from

$$\frac{B_d(1,3)}{A(0,1)} = E^{p^*}\left[\frac{B(2,3)}{A_d(0,2)}\right] = B_d(1,2) \cdot \left[p^* B_{du}(2,3) + (1-p^*) B_{dd}(2,3)\right]$$

follows $B_d(1,3) = 0.9753$. The results are given in *Figure 11.*

It also holds:

$$\frac{B(0,3)}{A(0,0)} = E^{P^*}\left[\frac{B(1,3)}{A(0,1)}\right] = E^{P^*}\left[E^{P^*}\left[\frac{B(2,3)}{A(0,2)}\right]\right].$$

Thus under the risk-neutral measure P^* any discounted state contingent cash flow (security) is a martingale and therefore all the Arrow-Debreu Securities can be evaluated. When denoting the security with a payout of one monetary unit at time $t = 2$ in the state $\{u,u\}$ by $\varpi_{2,2}$, the following results:

$$\frac{\varpi_{2,2}}{A(0,0)} = E^{P^*}\left[E^{P^*}\left[\frac{1_{\{u,u\}}}{A(0,2)}\right]\right] = \frac{1}{A_u(0,2)} P^*(\{u,u\})$$

$$= \frac{1}{A_u(0,2)} p^* \cdot p^* = \frac{1}{\exp(r(0)) \cdot \exp(r_u(1))} \cdot (p^*)^2.$$

Analogously $\varpi_{2,1}$ with payout 1 in the state $\{d,u\}$ and $\varpi_{2,0}$ with payout 1 in the state $\{d,d\}$ at time $t = 2$, we receive:

$$\frac{\varpi_{2,1}}{A(0,0)} = E^{P^*}\left[E^{P^*}\left[\frac{1_{\{d,u\}}}{A(0,2)}\right]\right] = \left[\frac{P^*(\{u,d\})}{A_u(0,2)} + \frac{P^*(\{d,u\})}{A_d(0,2)}\right]$$

$$= \left[\frac{1}{A_u(0,2)} + \frac{1}{A_d(0,2)}\right] \cdot (1-p^*)p^*$$

$$= \frac{1}{\exp(r(0))} \cdot p^*(1-p^*)\left[\frac{1}{\exp(r_u(1))} + \frac{1}{\exp(r_d(1))}\right],$$

$$\frac{\varpi_{2,0}}{A(0,0)} = E^{P^*}\left[E^{P^*}\left[\frac{1_{\{d,d\}}}{A(0,2)}\right]\right] = \frac{1}{A_d(0,2)} P^*(\{d,d\})$$

$$= \frac{1}{\exp(r(0)) \cdot \exp(r_d(1))} \cdot (1-p^*)^2$$

The utilization of discounting each payoff results in path dependence, since each branch of the tree has to be considered separately. In order to circumvent this, a new probability measure is introduced. A portfolio consisting of all securities $\varpi_{2,2}, \varpi_{2,1}, \varpi_{2,0}$ results in a payoff of 1 with certainty at time $t = 2$, therefore under the absence of arbitrage

$$B(0,2) \overset{a}{=} \varpi_{2,2} + \varpi_{2,1} + \varpi_{2,0} \Leftrightarrow 1 \overset{a}{=} \frac{\varpi_{2,2}}{B(0,2)} + \frac{\varpi_{2,1}}{B(0,2)} + \frac{\varpi_{2,0}}{B(0,2)}.$$

A probability distribution $Q^{t_2} = Q^{t_2}\left(q_{2,2}, q_{2,1}, q_{2,0}\right)$ can be defined by:

$$q_{2,2} := \frac{\varpi_{2,2}}{B(0,2)}, q_{2,1} := \frac{\varpi_{2,1}}{B(0,2)}, q_{2,0} := \frac{\varpi_{2,0}}{B(0,2)} = 1 - q_{2,2} - q_{2,1}.$$

The probability distribution Q^{t_2} depends on time $t_2 = 2$, but any payoff C_{t_2} at this time can be written as the sum of state contingent securities:

$$C_{t_2} \overset{a}{=} \alpha(\{u,u\})\varpi_{2,2} + \alpha(\{u,d\})\varpi_{2,1} + \alpha(\{d,d\})\varpi_{2,0} \overset{a}{=} \frac{1}{B(0,2)} \mathrm{E}^{Q^{t_2}}[\alpha(.)],$$

where $\alpha(.)$ denotes the corresponding function for the coefficients. The basic idea is to "change the numéraire" from the money market account $A(.,.)$ in case of measure P^* to the prices of zero coupon bonds $B(.,T)$. The new probability measure is of great importance in connection with term structure modeling, since it is used to evaluate the price (hedging costs) of interest rate derivatives. $Q^{t_2} = Q^{t_2}\left(q_{2,2}, q_{2,1}, q_{2,0}\right)$ is termed "forward risk adjusted measure at time t_2", and this change in measure represents the transition from the spot to the forward market.

4.2.3 The Role of Martingales in Financial Modeling

The description above reveals the importance of martingales in connection with asset pricing. In the following we briefly sketch some major properties and applications of the martingale theory.

Let $(\Omega, \mathcal{F}, P)$ denote a probability space, $\{\mathcal{F}_t, t = 0,1,\ldots,n\}$ is a filtration, i.e. an increasing family of sub-σ algebras of $\mathcal{F}_t, \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$. Such filtrations are used to describe the information structure in financial markets, where $\mathcal{F}_t$ denotes the available information in a financial market up to time t. A process $X = (X_t, t = 0,1,\ldots)$ is called martingale (relative to $(\{\mathcal{F}_t\}, P)$) if

(1) X is adapted to $\{\mathcal{F}_t\}$ (i.e. $\mathcal{F}_t$ includes all the information on X up to time t), i.e. $X = (X_t, t = 0,1,\ldots)$ is $\{\mathcal{F}_t\}$ measurable for each $t = 0,1,\ldots$,

(2) $E\left[|X_t|\right] < \infty$,

(3) $E\left[X_t \middle| \mathcal{F}_{t-1}\right] = X_{t-1}$, almost surely, $t \geq 1$. Intuitively this means that for given information $\{\mathcal{F}_{t-1}\}$ at $t-1$, the process X in average neither increases nor decreases.

A process $X = (X_t, t = 0, 1, \ldots)$ is called supermartingale (relative to $(\{\mathcal{F}_t\}, P)$) if (1), (2) hold and $E\left[X_t \middle| \mathcal{F}_{t-1}\right] \leq X_{t-1}$, almost surely, $t \geq 1$, and submartingale if $E\left[X_t \middle| \mathcal{F}_{t-1}\right] \geq X_{t-1}$, almost surely, $t \geq 1$.

According to the definitions above, a supermartingale decreases on the average, whereas a submartingale increases on average. This property can be utilized to describe fair and unfair games. If $X_t - X_{t-1}$ denotes the net winnings per unit stake at time $t \geq 1$, then:

- $E\left[X_t - X_{t-1} \middle| \mathcal{F}_{t-1}\right] = 0$ denotes a fair game, since on average you win and loose with the same probability (martingale case),
- $E\left[X_t - X_{t-1} \middle| \mathcal{F}_{t-1}\right] \leq 0$ denotes an unfair game, since on average there are more losses than wins (supermartingale case).

In asset pricing theory these concepts play an important role, since asset pricing can be modeled stochastically. Stock prices for example may be modeled as submartingale, since it is assumed that on average stock prices increase. The same holds for zero coupon bonds, whereas European Options e.g. decrease over time (according to the time value) and therefore may be considered as supermartingales.

A martingale is defined according to a certain probability distribution. In the previous section it was shown that asset prices (bond prices) can be converted into martingales if the probability distribution is suitably changed. Assuming the absence of arbitrage, it is possible to find such a "synthetic" probability distribution that discounted bond or stock prices behave like martingales. In mathematical finance there are two techniques available to convert submartingales into martingales: the Girsanov Theorem[14] and the Doob-Meyer Decomposition[15].

But martingales are also connected with another important topic in financial markets theory: efficient market hypothesis. According to this theory all relevant information is reflected in current prices in such a way that future price movements are "random". There is vast literature on this important topic so that it is impossible to describe it in full[16].

A process $C = (C_t, t = 0,1,2...)$ is called previsible if C_t is $\mathcal{F}_{t-1}$ measurable for all $t \geq 1$. This definition can be associated with (self-financing) trading strategies, since at time $t-1$ the investment strategy for time t is defined. In the section above such a self-financing trading strategy was utilized in order construct the Arrow-Debreu Securities.

In connection with trading or gambling strategies the winnings at time t can be additionally considered by means of $C_t (X_t - X_{t-1})$ and the total winnings using

$$\Psi_t = \sum_{1 \leq k \leq t} C_k (X_k - X_{k-1}) =: C \bullet X.$$

$C \bullet X$ is the martingale transform of X by C and represents the discrete analogue of the stochastic integral (Itô Integral) $\int C \bullet X$.

These stochastic integrals are of great importance, since they are utilized to define stochastic processes so that e.g. the stochastic evolution of a short rate, LIBOR rate, etc. is represented by a stochastic differential equation.

4.2.4 Stochastic Calculus and Term Structure Modeling – an Introduction

Let $W_t = (W_t^1, W_t^2, ...W_t^d)$ on $(\Omega, \{\mathcal{F}_t\}, P)$ denote a d-dimensional Brownian motion, which means that the changes of the stochastic process

14 For applications of the Girsanov Theorem refer e.g. to Neftci (2000), p. 322; Sondermann, D. (2006), p. 55; Björk, T. (2004), p. 154.

15 For the Doob-Meyer Decomposition refer to Doob, J.L.(1984), pp. 495 or Karatzas, I., Shreve, S. (1994), pp. 24

16 For an introduction to this topic refer to Copeland, T.E., Weston, F.J., Shastri, K. (2005), pp. 353– 370, for a critique of this concept refer to Haugen, R.A.(1995).

(increments) are described by a d-dimensional normally-distributed random variable. The stochastic evolution of zero coupon bonds is modeled by the following stochastic differential equation for $0 \le t \le T$:

$$\frac{dB(t,T)}{B(t,T)} = \mu(t,T)dt + \sigma(t,T) \circ dW_t,$$

where $\sigma(t,T) = \left(\sigma^1(t,T), \sigma^2(t,T), \dots, \sigma^d(t,T)\right)$ is the d-dimensional volatility and $\circ$ denotes the d-dimensional scalar product. According to the properties of the Brownian motion, zero coupon bonds as described above are lognormally distributed.

Under the condition of no arbitrage and several technical conditions it can be shown that there exists an adapted previsible d-dimensional process $\vartheta(t) = \left(\vartheta^1(t), \vartheta^2(t), \dots, \vartheta^d(t)\right)$, $0 \le t \le T^* \le T$ so that for all $0 \le t \le T^* \le T, r(t)$ the instantaneous short rate is:

$$\mu(t,T) = r(t) + \vartheta(t) \circ \sigma(t,T).$$

Therefore, under the absence of arbitrage, we can write:

$$\begin{aligned}\frac{dB(t,T)}{B(t,T)} &= \mu(t,T)dt + \sigma(t,T) \circ dW_t \\ &= r(t)dt + \sigma(t,T) \circ \left[dW_t + \vartheta(t)dt\right].\end{aligned}$$

The basic idea is to define an equivalent probability measure P^*, so that the drift term $\vartheta(t)$ can be eliminated. This is achieved by using the Girsanov Transformation: Let W_t^* denote the Brownian motion under the new probability measure P^*, then for all $0 \le t \le T^* \le T$:

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + \sigma(t,T) \circ dW_t^*.$$

This leads to the following results:

- $\vartheta(t), 0 \le t \le T^* \le T$ can be interpreted as the “market price of risk” and has already been discussed in the discrete example in *Section 3* above.

- With respect to the probabilities P and P^* we can provide the following economic interpretation. Under the real world probability P we have:

$$E^P\left[\frac{dB(t,T)}{B(t,T)}\middle|\mathcal{F}_t\right]=r(t)dt+\sigma(t,T)\vartheta(t)dt.$$

Therefore for the equilibrium price process of the zero coupon bonds under P the individual (subjective) expectations are taken into account and are represented by the "market price of risk" $\vartheta(t)$.

Under the probability measure P^* we have:

$$E^{P^*}\left[\frac{dB(t,T)}{B(t,T)}\middle|\mathcal{F}_t\right]=r(t)dt+0.$$

The "market price of risk" is zero under the probability measure P^*, the relation above is also termed "local expectation hypothesis". Since the market price of risk equals zero, the expectation can be interpreted as the expectation of a risk-neutral investor. But the equilibrium does not require the existence of representative investors; it just states that under the absence of arbitrage we do not need to know anything about individual attitudes towards risk (preferences).

- It is important to observe that, using the Girsanov Transformation, the price process of the zero coupon bonds has not changed, only the probability measure. So we are still considering the same zero bonds as before.
- Given the absence of arbitrage it is also clear that hedging and the price of the hedging instruments is just the flip side of the same coin. Under the absence of arbitrage one can form riskless portfolios which "grow" (locally) with the same "growth rate" as the risk-free asset $r(t)$.
- Under the absence of arbitrage condition, we can therefore work with a probability measure P^*, which facilitates the calculations: Particularly, the absence of arbitrage implies restrictions

on the drift term $\mu(t,T)$, whereas the application of the Girsanov Transformation eliminates the drift term $\mu(t,T)$ but "preserves" the volatility $\sigma(t,T)$.

- It also follows that the process $B(t,T)$ discounted with the money market account

$$A(t) = \exp\left(\int_0^t r(s)ds\right)$$

is a martingale under P^* with the dynamics $\sigma(t,T) \circ dW_t^*$.

- As an immediate consequence $B(T,T) = 1$ yields:

$$B(t,T) = E^{P^*}\left[\exp\left(-\int_t^T r(s)ds\right)B(T,T)\middle|\mathcal{F}_t\right]$$
$$= A(t)E^{P^*}\left[\frac{B(T,T)}{A(T)}\middle|\mathcal{F}_t\right] = A(t)E^{P^*}\left[\frac{1}{A(T)}\middle|\mathcal{F}_t\right].$$

The equation above is a special case of the "fundamental pricing rule".

4.2.5 The Heath-Jarrow-Morton Framework

Instead of assuming a stochastic process for zero coupon bonds like in the example above, forward rates can be modeled directly.

Within the Heath-Jarrow-Morton[17] Framework using the notation above, it is assumed that the forward rate $f(.,T)$ follows a stochastic differential equation

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T) \circ dW_t, T > 0,$$

assuming the forward rate $f(0,T)$ to be equal to the observed forward rate $f^*(0,T), f(0,T) = f^*(0,T)$.

Heath-Jarrow-Morton have proven that under the absence of arbitrage and the martingale measure the drift term $\alpha(t,T)$ only depends on the volatility $\sigma(t,T)$:

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds \quad T \geq t.$$

Therefore it has been shown that only volatility "matters" and the framework can be used to analyze term structure models under the absence of arbitrage. By imposing special forms of the volatility $\sigma(t,T)$, term structure models of Ho-Lee (constant volatility $\sigma(t,T)=\sigma$) or Hull-White (exponential volatility function $\sigma(t,T)=\sigma^{-a(T-t)}, a>0$) can be derived.

4.2.6 The LIBOR Market Model

One of the most popular term structure models is the LIBOR Market Model[18]. As shown above, instantaneous spot and forward rates cannot be directly observed in the market, additionally calibration for cap or swaption data can be complicated in these models. The LIBOR Market Model models discrete LIBOR rates and recovers the Black (1976) Volatilities quoted in the market.

We define the tenor τ_i for a fixed set of increasing maturities $T_0, T_1, T_2, ..., T_n$ by $\tau_i = T_i - T_{i-1}$, which typically equals a quarter of a year, so $\tau_i = 0.25$. The LIBOR forward rate contracted at time t for the period $[T_{i-1}, T_i]$ is defined by

$$L_i(t) = \frac{1}{\tau_i} \cdot \frac{B(t,T_{i-1}) - B(t,T_i)}{B(t,T_i)}, i = 1,...,n.$$

A cap with the cap rate R and the resettlement dates $T_0, T_1, T_2, ..., T_n$ is a contract which at time T_i provides the holder with the cap amount for each $i = 1,...,n$:

$$Y_i = \tau_i \max\left[\left(L_i(T_{i-1}) - R\right), 0\right].$$

A cap Y is just a portfolio of caplets Y_i, and the forward rate $L_i(T_{i-1})$ is the spot rate at time T_{i-1} paid at time T_i.

The Black (1976) Formula for caplets $Y_i = \tau_i \max\left[\left(L_i(T_{i-1}) - R\right), 0\right]$ is given by the expression:

---

17 Refer to Heath, D., Jarrow, R.A., Morton, A. (1992).

18 Refer e.g. to Miltersen, Sandmann, Sondermann (1997); Brace, Gatarek, Musiela (1997), Jamshidian, F (1997), Björk, T. (2004), p. 368.

$$Caplet_i(t) = \tau_i \cdot B(t,T_i)\left[L_i(t)N(d_1) - RN(d_2)\right], i = 1,\ldots,n,$$

where N denotes the cumulative normal distribution and

$$d_1 = \frac{1}{\sigma_i\sqrt{T_i - t}}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2(T_i - t)\right],$$

$$d_2 = d_1 - \sigma_i\sqrt{T_i - t}.$$

The constant σ_i is termed the Black Volatility. Caps are not quoted in monetary units but in implied Black Volatilities. A series of implied volatilities is termed volatility term structure and utilized for the calibration of the model.

The basic idea of the LIBOR Market Models is as follows: According to the fundamental pricing rule (see above), the (fair) value of the caplet can be written as:

$$Caplet_i(t) =$$

$$\tau_i \cdot E^{P^*}\left[\exp\left(-\int_0^{T_i} r(s)ds\right) \cdot \max\left((L_i(t) - R), 0\right) \Big| \mathcal{F}_t\right], i = 1,\ldots,n.$$

But is is also possible to write the (fair) value of the caplet in terms of the forward T_i measure denoted by P^{T_i}:

$$Caplet_i(t) = \tau_i \cdot B(0,T_i)E^{T_i}\left[\max\left((L_i(t) - R), 0\right) \Big| \mathcal{F}_t\right], i = 1,\ldots,n,$$

E^{T_i} denotes the expectation under the forward T_i measure denoted by P^{T_i}.

It can be shown that the LIBOR process $L_i(t)$ follows a martingale under the forward measure P^{T_i} on the interval $[0, T_{i-1}]$. It is therefore natural to define the LIBOR Market Model as follows:

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i(t), i = 1,\ldots,n,$$

where W^i denotes the Brownian motion under the forward T_i measure.

With respect to the LIBOR Market Model there are additional features like existence, the extension to volatility smiles etc. to be considered, but these are beyond scope[19]. In connection with IAS 39 it is important to observe that this term structure model recovers the input data: initial term structure and volatility term structure. As outlined in the previous section the requirement of "reliable measurability" is tied to the calibration of term structure models. Consequently the calibration and the recovery of input data drive the selection of term structure models under IAS 39.

4.2.7 Term Structure Models and Their Compliance with IAS 39

In the following *Table 6* major typical representatives of term structure models are listed (incorporating and summarizing results from several sources, e.g. Rebonato (2004)). This list is not complete, judging by the various models applied in practice. But the requirements of valuation model selection, which is driven by the fair value measurement rules under IAS 39, are shown and can be applied similarly to other term structure models.

With respect to the determination of fair values IAS 39 permits the usage of valuation techniques (refer to IAS 39.AG74; IAS 39.AG75; IAS 39.AG76, IAS 39.AG81 and IAS 39.AG82). The main requirements are as follows[20]:

- Valuation techniques should be used which are most common to the market;
- valuation technique should be consistent with economic methodologies;
- the inputs to the valuation technique should consider market information whenever possible.

---

19 Refer e.g. to Brigo, Mercurio (2006); Rebonato, R. (2008); Rebonato, R., McKay, K., White, R. (2009)

20 These requirements go in line with those of recently issued IFRS 13 „Fair Value Measurement".

As becomes apparent, the major difference between the determination of fair value and the evaluation of fair value in connection with hedge accounting is that hedge accounting requires the term structure model to be calibrated to reproduce current market data, whereas for fair value measurement (approximate) econometric approaches can also be used.

In connection with valuation models it is important to observe that IAS 39 does not impose performance requirements on the valuation

TABLE 6 **Major Types of Term Structure Models and Their Compliance with Hedge Accounting**

Model type	Model	Diffusion process
Short rate model	Vasicek	$dr = (b - ar)dt + \sigma dW$
	Cox-Ingersoll-Ross (CIR)	$dr = a(b - r)dt + \sigma\sqrt{r}dW$
	Ho-Lee	$dr = \Theta(t)dt + \sigma dW$
	Hull-White	$dr = (\Theta(t) - ar)rdt + \sigma(t)dW, a > 0$
	Hull-White (extended CIR)	$dr = (\Theta(t) - a(t)r)rdt + \sigma(t)\sqrt{r}dW, a(t) > 0$
	Black-Derman-Toy	$d\ln(r) = \left(\frac{\partial w(t)}{\partial t} - \frac{\partial \ln\sigma(t)}{\partial t}(w(t) - \ln(r))\right)dt + \sigma(t)dW$
	Black-Karasinski	$d\ln(r) = (\Theta(t) - a(t)\ln(r))dt + \sigma(t)dW$
Market model	LIBOR Market Model (consistent caplet pricing)	$dL_i(t) = \mu_i(t)L_i dt + L_i(t)\sigma_i(t)dW_i(t), i = 1,\ldots,n$ in forward adjusted measure with joint numéraire $B(t,T_j)$ for all i: $\mu_i(t) = 0$ for $i = j$, $\mu_i(t) = \mu_i(\sigma_k, L_k), i \neq j, k = \min(i,j)+1,\ldots,\max(i,j)$
	Swap rate market model (consistent swaption pricing)	$dS_{i,k}(t) = \mu_{i,k}(t)S_{i,k}(t) + \sigma_{i,k}(t)S_{i,k}(t)dW_{i,k}(t), i = 1,\ldots,n; k > i$ in forward swap measure with joint numéraire $C_{j,m}(t) = \sum_{l=j+1}^{k}(T_{l+1} - T_l)B(t,T_l)$ for all i,k: $\mu_{i,k}(t) = 0$ for $i = j, k = m$

model. It only requires that the model used should be one that market participants would use. So the explanatory power of cash prices using derivative prices or the predictive power of valuation models (stochastic models) is not addressed in IAS 39. For example the Black-Scholes Option pricing formula does not take smile effects into account, but it is still used to price options. Some short rate models like the Ho-Lee Model may result in negative interest rates, but despite this undesirable property they are utilized to determine fair values. Issues with respect

Input parameters (dependent on the financial instrument and calibration "strategy")	Number of fitted parameters (dependent on the financial instrument and calibration "strategy")	Properties ▸ Recovery of initial term structure ▸ Recovery of volatilities or cap or swaption prices
Initial term structure	4: $a, b, \sigma, r(0)$ or λ (market price of risk)	No
Initial term structure	4: $a, b, \sigma, r(0)$ or λ (market price of risk)	No
Initial term structure	$n+2$: $\sigma, r(0), \Theta(t)$ for given terms	Only the initial term structure, not necessarily the volatility
Initial term structure	$2n+3$: $a, r(0), \sigma(t), \Theta(t)$ for given terms	Yes, but calibration to caplet prices difficult
Initial term structure	$3n+2$: $r(0), \sigma(t), \Theta(t), a(t)$ for given terms strongly non-stationary	Yes, but calibration to caplet prices difficult
Initial term structure	$2n+2$: $r(0), \sigma(t), w(t)$ for given terms time-decaying short-rate volatility	Yes, but calibration to caplet prices difficult
Initial term structure	$3n+2$: $r(0), \sigma(t), \Theta(t), a(t)$ for given terms	Yes, but calibration to caplet prices difficult
Initial term structure, caplet prices (volatilities)	Calibration to caplet prices: $2n-1$, $L_i(0), i = 0,\ldots,n-1$, $\sigma_i, i = 1,\ldots,n-1$	Yes (no smile, i.e. only one caplet per maturity)
Initial term structure, swaption prices (volatilities)	$S_{i,k,0}, i = 0,\ldots,n-1; k > i$ $\sigma_{i,k}, i = 1,\ldots,n-1; k > i$	Yes

to the performance or stability of numerical procedures do not play any role for IAS 39, either.

As described in the previous sections, hedge accounting under IAS 39 requires the unique definition of a benchmark risk curve (i.e. the values of the benchmark curve are uniquely defined) and the portion at inception of the hedge. This definition of the portion cannot be changed until a de-designation of the hedge takes place. Additionally the reliable measurability requires the utilization of all market prices which are available (observable derivative prices).

In contrast to the case of deterministic cash flows there are infinite possibilities to define a portion in the stochastic case due to the choice of term structure models and parameters. However, the number of definitions of a portion may be infinite but they are not arbitrary. This results from the absence of arbitrage property (martingale property) and must be satisfied by the term structure model.

Like with deterministic cash flows it is important to note that in general econometric approaches to estimate the model parameters outlined in *Table 6* above cannot be applied to hedge accounting since such techniques cannot ensure the uniqueness of the definition of the benchmark curve or the portion. Therefore there is only the calibration technique to evaluate the parameters needed for the term structure models. But the calibration technique in connection with the "uniqueness" requirement implies that not all types of term structure models can be applied to hedge accounting purposes under IAS 39.

In *Table 6* above (last column) the properties of term structure models are portrayed. If term structure models cannot recover the initial input data like the initial term structure or the term structure of volatilities, they are precluded from hedge accounting under IAS 39. This shows that IAS 39 implicitly requires model selection based on a market participant view excluding e.g. the econometric model as carried out in *Section 3.3*. *Table 6* must be used carefully, since calibration itself relies on decisions to be made before the application. For example the LIBOR Market Model stated is calibrated with respect to caplet prices (volatilities) intending an exact pricing of caps by Black's Formula. It could also be calibrated to swaption prices (volatilities) with the intention of exact swaption pricing using Black's Formula (see e.g. Brigo/Mercurio (2004)). A simultaneous recovery of Black Prices of both plain vanilla options is not possible but in practice the arising discrepancies are small (for details see Rebonato (2004) and references therein). There are further calibration approaches that provide the recovery of caplet and swaption prices (cf. e.g. Schoenmakers & Coffey (1999) or Schoenmakers (2002)). More advanced market models deal with e.g. stochastic volatility profiles (cf. Brigo/Mercurio (2001) or Rebonato (2004) and references therein), but this is beyond scope since the performance of a model is not a direct issue under IAS 39.

Irrespective of the implicit model selection under IAS 39, the prices of derivatives evaluated under the absence of arbitrage assumption (equivalent martingale measure) are identical to the hedging costs. Consequently, like in the deterministic case, the hedged items are "priced" (fair value according to the hedged risk) according to their hedging costs. Ineffectiveness only occurs if the cash flow profile does not match that of the derivative, e.g. if there are different maturities or prepayment features in the hedged item, which is not reflected in the hedging instrument (derivative), or due to counterparty credit risk. The "portion" can now be defined according to financial economics: The portion equals its hedging costs evaluated under equivalent martingale measure. Accordingly the portion is clearly defined, but depends on the model chosen and the hedging instrument used, e.g. prepayment options are evaluated by a swap rate model, whereas the LIBOR Market Model is used to evaluate caps etc. Since financial modeling is involved to determine the benchmark curve as well as term structure models, the notions of "portion", "reliably measurable" and "separately identifiable" correspond with equilibrium conditions used in financial modeling.

There are two additional properties (implicit assumptions) in connection with term structure modeling and hedge accounting which are of importance:

- efficient market hypotheses;
- replication / completeness.

But according to the requirement of IAS 39 to use observable market prices, it is implied that IAS 39 relies on the efficient market hypotheses. This means that it is assumed that all relevant information to determine the fair value is reflected by market prices. The assumed markets are incomplete, if short rate models are used, especially in dimensions >1, e.g. the number of risk factors (short rate) is larger than that of traded zero bonds.[21]

TABLE 7 The Requirements of IAS 39.AG99F and the Corresponding Concepts of Financial Economics Using Term Structure Models

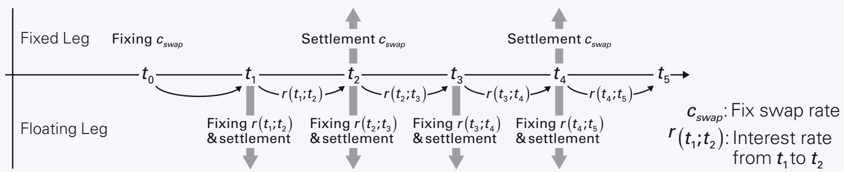
Requirement IAS 39.AG99F	Financial economics
Portion	▸ Absence of arbitrage ▸ Integrated market for hedged item and hedging instrument through the common benchmark curve (derived from liquid market of hedging instruments) leading to the elimination of basis risk. ▸ Determination of a (cash flow) component attributable to the designated risk by the hedging instrument. ▸ Completeness / replication depends on the selected term structure model and / or efficient market hypotheses being applied
Separately identifiable	Identification by the hedging instruments and derivation of the "benchmark curve" – (derivative) Zero EURIBOR / LIBOR / OIS Swap Rates utilized for discounting. Completeness / replication depends on the selected term structure model and / or efficient market hypotheses being applied.
Reliably measurable	Existence of a **liquid market** for the hedging instrument to derive the "benchmark curve", e.g. interest rate swaps, options (caps, floors, prepayment) based on EURIBOR / LIBOR / OIS and terms structure of volatilities that covers all relevant market data to evaluate the portion of the hedged item attributable to the designated risk. Only calibration can be applied and only models which recover initial data can be used.

Other models like LIBOR Market Models assume market completeness and therefore imply that any payoff profile of the hedged item can be replicated by a derivative. This is a critical assumption since e.g. prepayments are not always driven by interest rates but also by other factors like GDP (gross domestic product), etc. As a consequence, the assumed completeness may not be given. *Table 7* summarizes the requirements under IAS 39 and its corresponding concepts in financial economics; it is similar to *Table 1* contemplated by the results from term structure modeling.

4.2.8 Example: Interest Rate Hedge Accounting Using an InArrears Swap

In the following an example of an interest rate fair value hedge is considered comprising a hedging instrument with stochastic cash flows due to its InArrears property. The hedge consists of a fixed rate bond and an InArrears swap with a fixed leg having the same cash flow structure as the hedged bond and a floating leg that is fixed InArrears, i.e. the fixing rate is determined at the end of the fixing period (not in advance like for a plain vanilla interest rate swap). The cash flow structure of an InArrears swap is given in *Figure 13*.

FIGURE 13 Cash Flow Structure of an InArrears Swap



InArrears swaps are eligible for hedge accounting – following US GAAP even the short-cut method is not precluded for InArrears swaps[22].

21 See Björk, T. (2004), p. 319.

22 Cf. *Derivatives and Hedging Accounting Handbook 2011,* KPMG, A5.54, A5.89 Q 7, A6.53 and references therein

In the case of an InArrears swap the floating rate is fixed and settled at the same time. Thus a replication by suitable zero bonds as in the case with deterministic cash flows is not applicable, so an appropriate term structure model (valuation model) is needed. For the given example three valuation methods are considered:

- approximate valuation by the discounted cash flow method with convexity adjustment[23] (DCF);
- short-term model: Hull-White (HW);
- (lognormal) LIBOR Market Model with calibration to cap volatilities (LMM).

The data of the bond and the swap are shown in *Table 8*.

TABLE 8 Example: Bond and InArrears Swap – Data

Specifications	Bond	Swap fixed leg	Swap floating leg
Coupon / swap rate:	3.500%	3.3461%; 3.3418%; 3.3438%	–
Margin (incl. credit spread):	0.1539%; 0.1582%; 0.1562%24	–	–
Reference rate:	–	–	6M EURIBOR
Type of fixing:	–	–	InArrears
Frequency:	Annually	Annually	Semiannually
Notional:	100	100	100
Currency:	EUR	EUR	EUR
Issue / value date:	10/09/2009	10/09/2009	10/09/2009
First coupon date:	10/17/2010	10/17/2010	04/17/2010
Maturity:	10/17/2016	10/17/2016	10/17/2016
Day count convention:	Act/Act	Act/Act	Act/360

23 Evaluation following Pelsser (2000)

24 The margin/swap rate is model dependent: discounted cash flow (DCF), Hull-White (HW), LIBOR Market Model (LMM)

TABLE 9 **Fair Value Calculations for the InArrears Swap Using Different Valuation Models**

Date	PV_ LMM	PV_HW	PV_DCF _Conv	Date	PV_ LMM	PV_HW	PV_DCF _Conv
10/09/2009	0.0008	0.0002	0.0000	11/01/2010	−4.9532	−4.9487	−4.9946
05/01/2010	−4.2402	−4.2436	−4.2603	12/01/2010	−3.1115	−3.1017	−3.1436
06/01/2010	−6.1287	−6.1263	−6.1614	01/01/2011	−3.0538	−3.0438	−3.0791
07/01/2010	−6.7377	−6.7191	−6.7552	02/01/2011	−1.0865	−1.0658	−1.0860
08/01/2010	−6.4161	−6.4031	−6.4539	03/01/2011	−1.5766	−1.5456	−1.5597
09/01/2010	−9.6392	−9.6251	−9.6747	04/01/2011	−0.3175	−0.3152	−0.3182
10/01/2010	−8.6681	−8.6617	−8.7040				

Following the requirement that the value of a derivative should be zero initially, the application of different evaluation models may lead to differences in the choice of the characteristic parameter such as the swap rate, as in the present example of an InArrears swap. Following the hedging costs arguments above, this consequently results in different definitions of the portion (represented by different internal rates and margins respectively), depending on the valuation model used. However the hedged risk remains economically identical. Thus, as already mentioned in the previous section, there are many possibilities to define the portion but they are not arbitrary.

In T*able 9* the values of the InArrears swap are calculated using the three methods listed above. Furthermore, the fair values are portrayed for a series of times (one year monthly) which are of relevance for effectiveness calculations.

There are actual differences between the valuations as portrayed in the following *Figure 14*.

FIGURE 14 **Differences of Swap Fair Values Calculated by Different Valuation Models**

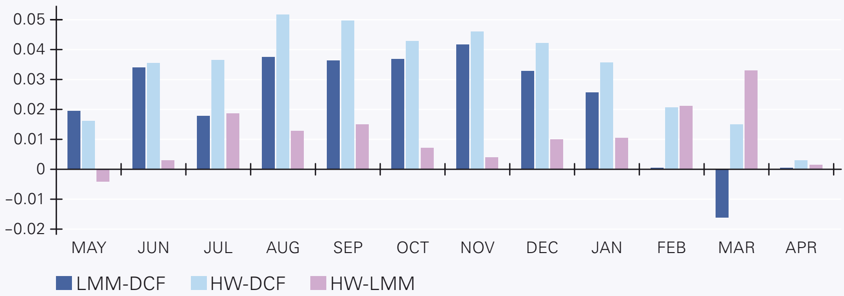
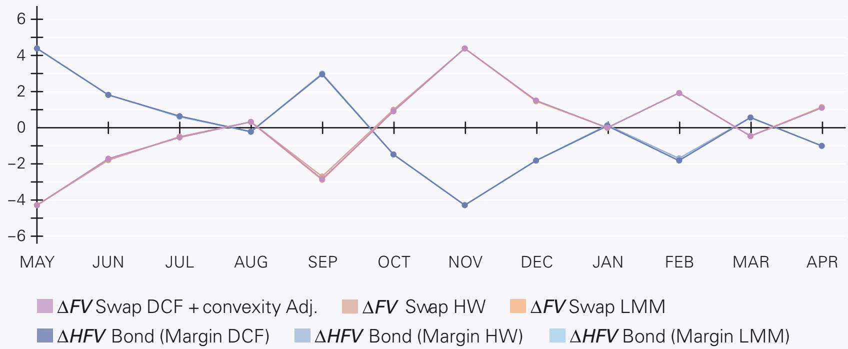


FIGURE 15 **Changes in Hedge Fair Value and Swap Fair Value Calculated with Different Valuation Models**



For the calculation the following market data were used at each point of measurement:

- initial term structure: EONIA, EURIBOR (SW, 1 to 12M), Swap/6M EURIBOR (18M, 1 to 7Y);
- short rate: volatilities, mean reversion (calibrated to caps);
- LMM/convexity adjustment: cap volatility term structure derived from flat cap volatilities from 1 to 7Y per 09/10/2009.

TABLE 10 Effectiveness by Valuation Methods

Effectiveness measurement regression analysis	R^2	Slope	Axis intercept	Swap rate
DCF method w. convexity adjustment	0.986311	–0.984767	0.039767	3.3461%
Short rate model – Hull-White	0.986648	–0.982553	0.040286	3.3418%
(lognormal) LIBOR Market Model	0.986585	–0.983169	0.039885	3.3438%

Actually, the effect of the valuation differences on the fair value differences over time relevant for hedge accounting is small with respect to the fair value differences over time, as can be seen in *Figure 15*.

The regression analysis *(Table 10)* shows that the hedge is effective for all three types of valuation methods.

In this illustrative example the different acceptable valuation methods for pricing the hedging swap with InArrears fixing show differences in the value of the swap. Due to the simple character of the instrument and simplifying valuation assumptions (e.g. same volatilities for all times) these differences are small with respect to the fair value changes over time, and effectiveness can be shown for all cases. This can be different when more complex products with stochastic cash flows (e.g. options) and more sophisticated valuation models are used.

Remark:
The hedge accounting with swaps priced by a multiple-curve approach as mentioned in *Section 3.1* can be treated similarly.

SECTION FIVE

5

Portfolio Hedging of Interest Rate Risk under IAS 39

5.1 Preliminaries

In the following we consider portfolio hedge accounting according to IAS 39. Under IAS 39 two types of "portfolio" hedges must be distinguished:

- Hedges of a "group of items": Assets, liabilities etc. can be summarized into one portfolio and designated as "hedged item". Forming such a portfolio implies that the requirement of "similarity" under IAS 39 is met. This type of hedge is also permitted under US-GAAP/FAS 133 (refer to KPMG, *Derivatives and Hedging Accounting Handbook,* January 2011, A5.78). This hedge accounting model will be analyzed in more detail below.
- Hedges of "portfolio fair value hedges of interest rate risk"[25] represent an approach applicable to IAS 39 only (IAS 39.78,

25 According to IAS 39.84, IAS 39.AG101 / 114 (c) the designation of net amount is precluded, but for a fair value hedge the changes in fair value due to the hedged risk must be allocated to either assets or liabilities being hedged (IAS 39.IG F.6.2).

IAS 39.81A, IAS 39.AG114 (c), IAS 39.AG116 and IAS 39.AG 118), which offers the possibility of forming time buckets ("repricing dates") and allocating "cash flows" to these buckets. Despite the approach differing from that for "hedges of a group of items", the Absence of Arbitrage Principle, the term structure modeling, etc. are identical, therefore this hedge accounting model is not analyzed in detail.

5.2 IAS 39 Requirements for a Group of Items

According to IAS 39.78 (b) it is generally eligible to designate a group of assets, liabilities, firm commitments, highly probable forecast transactions or net investments in foreign operations with similar risk characteristics as hedged item. IAS 39.83 and IAS 39.84 specify this requirement of similarity, which will be outlined in detail below. The similarity of the risk characteristics is defined with respect to the hedged risk.

According to IAS 39.83 two requirements are claimed for the designation of a group of items with similar risk characteristics (IAS 39.78 (b)) as hedged items:

- The individual assets, liabilities or future transactions in the portfolio share the same characteristics with respect to the hedged risk; and
- The change in fair value attributable to the hedged risk for each individual item in the portfolio is expected to be approximately proportional to the overall change in fair value attributable to the hedged risk of the group (test of homogeneity).

IAS 39 makes no detailed statements on the term "similar assets/ liabilities" in a portfolio. In practice the following facts are typically considered in case of interest rate risk hedge accounting:

- type of the assets/liabilities; and
- interest rate (fixed or floating) and the level; and
- currency; and
- scheduled maturity[26]; and
- cash flow structure.

For convenience it will be assumed in the following that the portfolio of hedged items consists of "plain vanilla" assets like bonds/loans with the same maturity, currency and cash flow structure (e.g. bonds with fixed coupon payments).

5.3 Test of Homogeneity and the Dependence Conception under IAS 39

The test of the homogeneity parameter represents a crucial step in hedge accounting for a group of items; it illustrates the proportion of hedge fair value changes of the entire portfolio and hedge fair value changes of each single item of the hedged portfolio. For each item *i* in the hedged portfolio, the homogeneity parameter is defined by

$$\beta_i = \frac{\Delta HFV_{\text{hedged item}} / HFV_{\text{hedged item}}}{\Delta HFV_{\text{item } i} / HFV_{\text{item } i}}.$$

The test of homogeneity has to be performed at the time of designation of the hedging relationship. ΔHFV indicates the difference between the actual hedge fair value and the scenario hedge fair value. Usually the scenario is defined by shifting the benchmark curve. If parameter β_i fulfills $90\% \leq \beta_i \leq 110\%$[27] for each *i*, the homogeneity test is considered passed.

The economic perception of the test of homogeneity is that it measures the elasticity of fair value changes common to each individual asset in

---

26 Cf. PriceWaterhouseCoopers (ed.): *IFRS Manual of Accounting – 2011,* chapter 10

27 This condition is adapted from US GAAP: FAS 133 paragraph 21 (a) (1) and ASC paragraph 815-20-55-14, respectively; see also Kuhn, Scharpf (2007) the portfolio with respect to the defined benchmark curve. It does not measure the "dependence" of the hedged risk. The test of homogeneity does not quantify the dependence (correlation) between two assets with respect to interest rate risk; the dependence with respect to interest rate risk is covered by the definition of the benchmark curve which is assumed to be common for all individual assets in the portfolio.[28] Hedge accounting for portfolios under IAS 39 uses a dependence concept which is similar to classical factor modeling in finance (Arbitrage Pricing Theory (APT) Model[29], Single Index Model (Capital Asset Pricing Model)[30] or Credit Risk+[31] ("sector analysis")). In such models e.g. the return is decomposed into common ("systematic") factors and other ("unsystematic") factors.

In *Figure 16* the underlying "dependence" conception of IAS 39 is portrayed.

The basic idea under IAS 39 is the definition of a benchmark curve which is assumed to be common to all individual fair value changes of each asset (e.g. bond) in the portfolio, whereas e.g. the margin or credit spread is individual to each asset (e.g. bond), which causes inhomogeneity as well as ineffectiveness. An additional important observation is that the test of homogeneity as well as the effectiveness test depend on the method chosen for the determination of the fair value changes. In case of

FIGURE 16 Dependence Model of IAS 39

$$\Delta HFV_1 = \Delta \sum_{t=1}^{n} \frac{\text{Coupon}_1}{(1-\text{Interest Rate}+\text{Credit Risk}_1+\text{Margin}_1)^t}$$

$$\Delta HFV_2 = \Delta \sum_{t=1}^{n} \frac{\text{Coupon}_2}{(1-\text{Interest Rate}+\text{Credit Risk}_2+\text{Margin}_2)^t}$$

$$\dots$$

$$\Delta HFV_n = \Delta \sum_{t=1}^{n} \frac{\text{Coupon}_n}{(1-\text{Interest Rate}+\text{Credit Risk}_n+\text{Margin}_n)^t}$$

Common risk factor interest risk, represented by the defined benchmark curve (hedged risk)

28 On the other hand if, as in *KPMG's Insights* 3.7.20\0, in an example for a portfolio hedge the fact of sharing the same (single) risk risk-free interest rate is defined as similarity, implying a certain correlation between the assets of the portfolio, the test of homogeneity will easily be passed.

29 Refer to Ross, S. (1976)

30 For a description of Factor Models and Empirical Tests refer to Gourieroux, C., Jasiak, J. (2001).

31 Credit Risk+ (1997)

portion hedging (internal coupon), a homogeneity of 1 can be expected on the basis of similar maturities, etc.

Example:

Using the EURIBOR/6M swap curve as benchmark curve, a portfolio of debt instruments can be designated as hedged item. The test of homogeneity

$$\beta_i = \frac{\Delta HFV_{\text{hedged item}} / HFV_{\text{hedged item}}}{\Delta HFV_{\text{item } i} / HFV_{\text{item } i}}, i = 1,\ldots,7$$

is easily calculated in such a case, as the *HFV* of the hedged item is simply the sum of the hedge fair values of each single debt instrument. *Table 11* shows an example of a seven-bond portfolio:

TABLE 11 **Example – Bond Specifications**

Specifications	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7
Coupon:	4.000%	2.375%	8.875%	5.000%	4.500%	2.600%	8.875%
Frequency:	Annually	Annually	Semi-annually	Annually	Annually	Annually	Semi-annually
Notional:	100	100	100	100	100	100	100
Currency:	EUR	EUR	EUR	EUR	EUR	EUR	EUR
Maturity:	06/15/2012	06/15/2012	06/15/2012	06/15/2012	06/15/2012	06/15/2012	06/15/2012
Day count convention:	Act/Act	Act/Act	Act/Act	Act/Act	Act/Act	Act/Act	Act/Act

The **test of homogeneity** must be performed at the time of designation of the hedging relationship. In this example the scenario used for the calculations equals a 10 bps shift of the benchmark curve, implying the following homogeneity parameters:

TABLE 12 **Example – Test of Homogeneity**

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7
β_i	99.72%	99.31%	100.44%	100.31%	100.45%	99.31%	100.44%

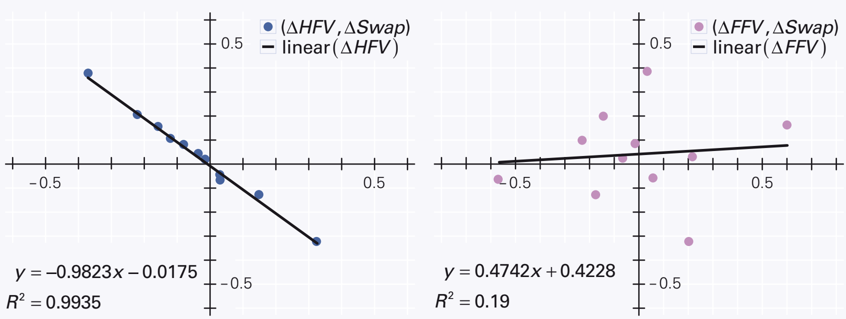
The homogeneity test is passed, since parameter β_i fulfills $90\% \leq \beta_i \leq 110\%$ for each $i = 1,\ldots,7$. As can be expected, the homogeneity is quite high.

5.4 Effectiveness Testing and the Performance of Hedges

In the following *Figure 17* the effectiveness testing for the portfolio and the comparison towards the full fair value is performed. The results are very similar to the single name hedge in the deterministic hedge accounting model (see *Figure 8*).

The impact of the utilization of the "swap curve" as benchmark curve becomes obvious. If the full fair value changes of the portfolio are compared with the changes in fair value of the interest rate swap portfolio, then the explanatory power is low ($R^2 = 0.19$). Only if the "benchmark curve" approach is used, the hedge becomes effective ($0.8 \leq b = 0.9823 \leq 1.25$) according to IAS 39 and attains $R^2 = 0.9935$. For the sake of completeness we also give the explanatory power (R^2) of the interest rate swap curve of all considered bonds in the paper in order to demonstrate its poor performance. Consider bond 3 and

FIGURE 17 Effectiveness Testing of a Portfolio Hedge and Comparison to Full Fair Value Approach



bond 7: The terms and conditions are identical but R^2 is quite different and low!

TABLE 13 **R^2 of All Considered Bonds on Full Fair Value and Hedge Fair Value Basis in the Paper**

R^2	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7	Portfolio
ΔFFV / $\Delta Swap$	0.4451	0.0332	0.0409	0.6550	0.5979	0.0338	0.0008	0.0061
ΔHFV / $\Delta Swap$	0.9666	0.9658	0.9657	0.9678	0.9680	0.9658	0.9657	0.9935

We omitted the optionalities, caps/floors and InArrear features in this example, so term structure modeling is not required to cover random (stochastic) future events. Consequently the justification with respect to the application of separately identifiable and reliable measurability is similar to single hedges and represented by the Absence of Arbitrage Principle.

But in connection with the hedging of portfolios, the “proof of existence” of the benchmark curve has additional aspects in comparison to single hedges. The applied Absence of Arbitrage Principle constitutes also the “common factor” approach used as a dependence concept by IAS 39. Under an IAS 39 portfolio model the derivatives (interest rate swaps) may be applied to define the common factor (“dependence”). A benchmark curve, derived from the derivative market, may be used to demonstrate that portfolio items share “similar risk” according to IAS 39. The required test of homogeneity measures the elasticity of fair value changes common to each individual asset in the portfolio with respect to the defined benchmark curve. To demonstrate the impact of such an approach, we consider the correlation matrix evaluated on full fair value basis (see *Table 14*). As becomes obvious, the correlations amongst the bonds are very different and, not surprising from to the poor explanatory power in *Table 14*, these seems to have nothing in common.

By construction of the portfolio hedge accounting model under IAS 39, the correlation matrix based on hedge fair values, evaluated by the swap curve, yields a correlation of 1 (see *Table 15*). The comparison reveals that this is a very strong assumption.

Putting it in terms of the discussion of risk-neutral probability measure, the real world correlations (real world probability measure) given in *Table 14* are not relevant for hedge accounting purposes, only

TABLE 14 **Correlation Matrix Evaluated on Full Fair Value Changes for the Example Bonds**

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7
Bond 1	1.000	0.481	0.105	0.718	0.497	0.022	0.465
Bond 2	0.481	1.000	–0.024	0.123	0.048	–0.051	–0.054
Bond 3	0.105	–0.024	1.000	0.187	–0.266	0.034	0.326
Bond 4	0.718	0.123	0.187	1.000	0.727	0.245	0.268
Bond 5	0.497	0.048	–0.266	0.727	1.000	0.387	0.034
Bond 6	0.022	–0.051	0.034	0.245	0.387	1.000	–0.012
Bond 7	0.465	–0.054	0.326	0.268	0.034	–0.012	1.000

TABLE 15 **Correlation Matrix Evaluated on Hedge Fair Value Changes for the Example Bonds**

	Bond 1	Bond 2	Bond 3	Bond 4	Bond 5	Bond 6	Bond 7
Bond 1	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Bond 2	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Bond 3	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Bond 4	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Bond 5	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Bond 6	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Bond 7	1.000	1.000	1.000	1.000	1.000	1.000	1.000

the correlations evaluated under the martingale measure (derived from the derivative market). This is consistent with common pricing approaches.

5.5 Dependence Conception and Absence of Arbitrage under IAS 39

As a consequence IAS 39 relies on the existence of such “common” risk factors, which needs to be explained.

TABLE 16 “Similar Risk Approach” and Common Risk Factors in IAS 39

“Similar risk” according to IAS 39.83 and IAS 39.84	Financial economics
Interest rate risk (e.g. swap curve as benchmark risk curve)	Quantitative reason: ▸ Absence of arbitrage (existence of liquid derivative contracts and pricing models in order to derive the benchmark risk curve) ▸ Factor model approach, dependence is modeled by one benchmark curve ▸ The application of the benchmark curve under IAS 39 results in perfect dependence (correlation of 1) No contractual relationship between the portfolio of hedging instruments and the portfolio of hedged items.

The factor model and the dependence concept are also utilized for the hedges of “portfolio fair value hedges of interest rate risk” (IAS 39.78, IAS 39.81A, IAS 39.AG114 (c), IAS 39.AG116 and IAS 39.AG118), so the economic perception (underlying assumptions) is identical, although the hedge accounting model looks different.

SECTION SIX

6

Hedge Accounting according to IAS 39: a Valuation Concept

6.1 Résumé of the Absence of Arbitrage Pricing Principle and Hedge Accounting

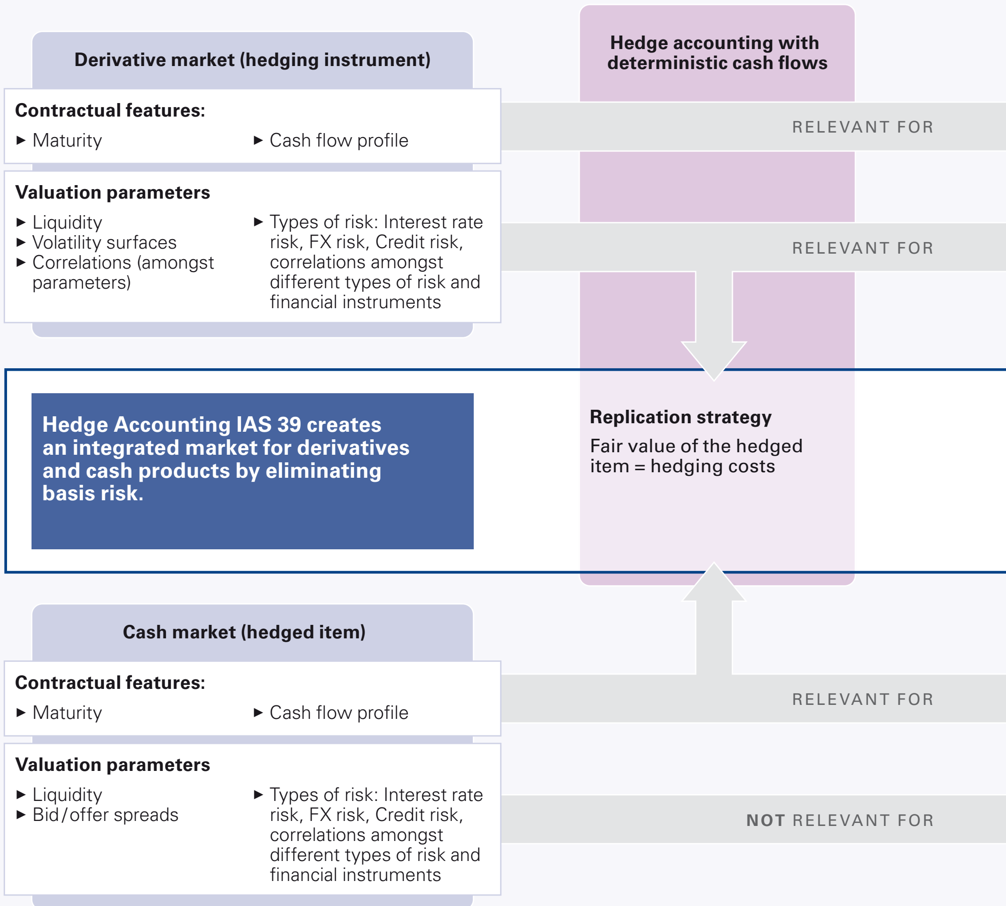
In the following the results of the previous sections are summarized. As demonstrated, the Absence of Arbitrage Principle is present in all types of hedge accounting models. We have distinguished between the following types of hedges:

- hedges with deterministic cash flows;
- hedges with stochastic cash flows; and
- hedges of portfolios for hedged items.

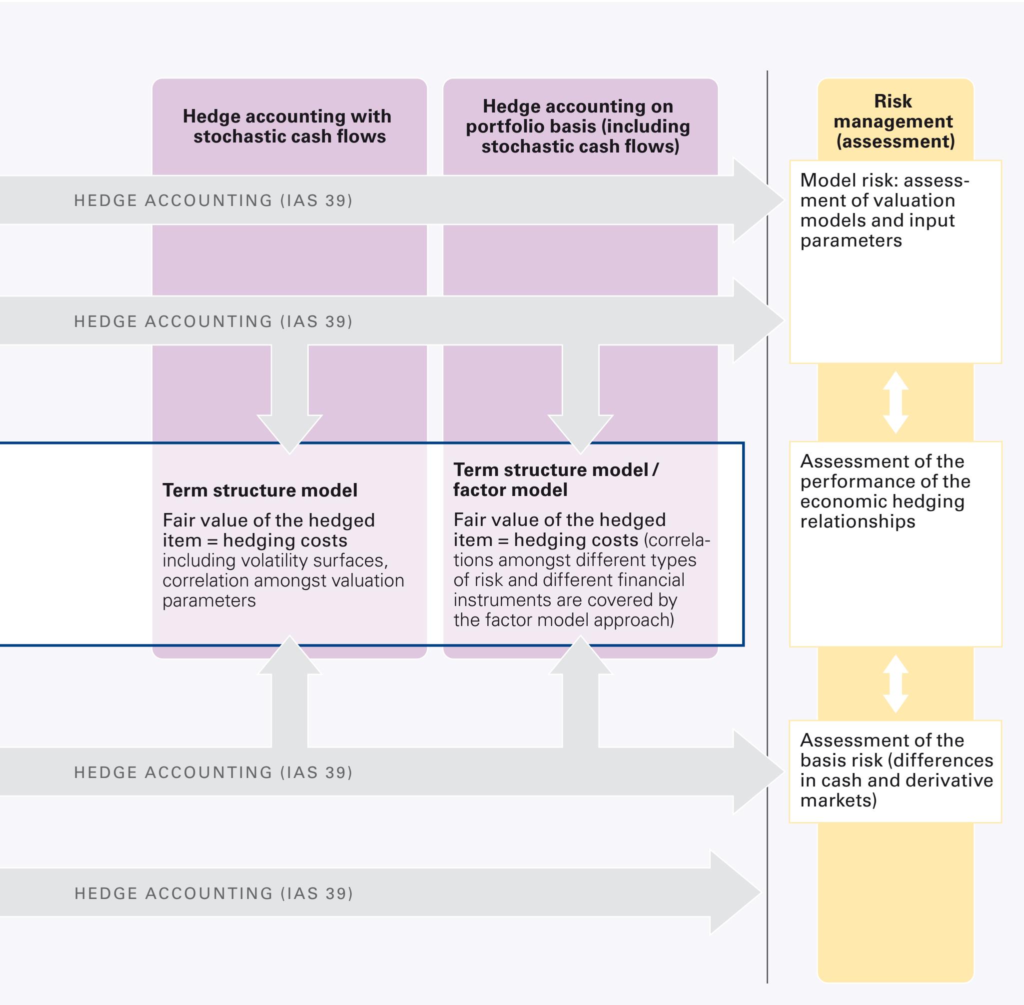
It is apparent that for each type of hedges the derivative market plays the pivotal role. The derivative market serves as "the source" for the derivation of the benchmark curve as well as of the parameters relevant for the valuation. Consequently any differences arising from cash and derivative markets are eliminated or substituted by the derivative market. The justification of such an approach is represented by the

Absence of Arbitrage Principle, which is common standard for valuation models. The reliance on the derivative market also affects hedges of portfolios. In this case the Absence of Arbitrage Principle is combined with the factor model approach, which is also a widely applied approach for dealing with dependence. Accordingly dependence (correlations) between different financial instruments in the cash market is not of explicit relevance for hedge accounting purposes, since it is

FIGURE 18 **Summary of the Absence of Arbitrage Principle and Hedge Accounting**



assumed to be reflected in the benchmark curve derived from the derivative market. In essence the analysis reveals that the hedge accounting rules create an integrated market for derivatives and cash products (hedged items). In this integrated market the derivative market determines the price for all financial instruments. Therefore the fair value of the hedged item with respect to the hedged risk (interest rate risk) is equal to its hedging costs. In combination with the portion hedging



permitted under IAS 39, provided the contractual features are similar, strong hedge effectiveness can be expected. *Figure 18* summarizes the results of the analysis.

In the light of the application of hedge accounting in practice, the concepts of reliably measurable and separately identifiable have nothing in common with arithmetical accuracy. It has been shown that these concepts coincide with the Absence of Arbitrage Principle and model calibration. The performance of the stochastic term structure models and factor model (dependence concept – "similar risks"), e.g. explanatory or predictive power of models, are not addressed in IAS 39. As a result hedge accounting incorporates a valuation model and is, like any valuation model, exposed to weaknesses. But this is of great importance, so as a consequence hedge accounting models need to be contemplated by a risk management framework.

6.2 Interaction between Hedge Accounting and Risk Management

As outlined in the previous section, hedge accounting according to IAS 39 is a specific valuation approach and uses various kinds of assumptions. Consequently there is "hedge accounting model risk" involved if hedge accounting is applied. Within banks economic hedging activities are monitored by risk management departments. As portrayed in *Figure 18* there are three main features of importance in connection with economic hedging and risk management: Assessment of model risk, economic hedging relationship and basis risk. Model risk arises if the model delivers unreasonable results for a set combination of input parameters and/or does not cover specific features of the derivative instrument. Furthermore risk management departments assess the performance of economic hedging relationships, e.g. by performing stress or simulation scenarios, in order to check if the hedging relationship delivers the desired results. This is closely related to analyses concerning the basis, especially if cash products (cash derivative products) are hedged by OTC derivatives. As was shown in the previous analysis, the performance of the hedge accounting model is rather poor. The explanatory power of market quotes for interest rate swaps was low compared with market quotes of bonds ("full fair value"). Similar results hold in connection with hedges of portfolios and the dependence concept of IAS 39. These properties do not represent an issue if hedge accounting is contemplated by sound risk management approaches.

It is easy to imagine various circumstances for which hedge accounting is applicable under IAS 39 but not leading to the desired results from the risk management perspective. This especially holds for hedges of interest rate and foreign exchange risk, but not for hedges of credit risk. In this hedge accounting approach there is a contractual relationship between the hedged item and the derivative, which ensures, if applied properly, a transfer of economic risk to the counterpart (refer also to credit risk mitigation rules under Basel II). This is very close to the assessment of (credit) risk management departments. But in general, a derivation of an integrated setup of hedge accounting and risk management activities is very complex and beyond the scope of the present paper. Nonetheless in such an integrated setup the Arbitrage Pricing Theory and its implications play an important role.

SECTION SEVEN

7

Conclusion

The result of the previous analysis is that hedge accounting under IAS 39 incorporates a valuation model which can be applied if the conditions according to IAS 39 are met. As shown, the valuation technique used under IAS 39 is consistent with the Absence of Arbitrage Theory, which is widely used in practice to determine “fair values” especially for the pricing of derivatives.

As generally accepted, the most reliable source of “risk free” prices (quotes) is represented by the derivative market. Economically the reliance on derivatives markets and the usage of the quotes of a derivative market for the pricing (“fair valuing”) of hedged items (like bonds/loans) constitutes the “Law of One Price” of all financial instruments (derivatives and cash instruments (hedged items)). Consequently an integrated market for both derivatives and cash instruments is created, which eliminates the basis risk – differences between the pricing of cash and derivative instruments due to different market data and conventions. This is a condition necessary for meeting the requirements imposed by the effectiveness test. Furthermore the reliance on the derivative market implies the notion of "liquidity", since only a "liquid" derivative market is a precondition to the application of hedge accounting under IAS 39. Accordingly the requirements of hedge accounting under IAS 39.AG99F concerning "reliable measurability" and "separate identifiability" include the Absence of Arbitrage Principle and liquidity considerations in the derivative market.

Liquidity considerations in connection with "reliable measurability" also entail calibration requirements due to the requirement of a "unique benchmark curve", which also drives the model selection with respect to term structure models that are used to price payoffs dependent on future (random) events. As a consequence not every term structure model can be used for hedge accounting. Since financial modeling is involved to determine the benchmark curve as well as term structure models, the "portion", "reliable measurability" and "separate identifiability" correspond with equilibrium conditions used in financial modeling. Consequently IAS 39 is principle-based and uses the Absence of Arbitrage Principle as the common and superior principle for hedge accounting purposes.

The consistency of the Absence of Arbitrage Principle and hedge accounting under IAS 39 does not only provide a framework for hedge accounting but also defines the interaction with risk management. Hedge accounting requirements according to IAS 39 also imply a "division of labor" between financial accounting and risk management. This stems from the fact that hedge accounting represents a valuation model, which naturally relies on assumptions and shortcomings of various kinds. These assumptions and shortcomings need to be assessed and monitored by risk management. Consequently integrated approaches for financial accounting and risk management can be derived. The idea of aligning hedge accounting with risk management is a principle advocated in ED 2010/13 and therefore the analysis above carries over to current discussions on hedge accounting.

8 References

M. Bianchetti (2010): "Two Curves, One Price: Pricing & Hedging Interest Rate Derivatives Decoupling Forwarding and Discounting Yield Curves", in: *Risk Magazine,* August 2010

M. Bianchetti, M. Carlicchi (2011): "Interest Rates after the Credit Crunch: Multiple Curve Vanilla Derivatives and SABR", working paper

T. Björk (2004): *Arbitrage Theory in Continuous Time,* 2nd Edition Oxford

Brace, Gatarek, Musiela (1997): "A Market Model of Interest Rate Dynamics", in: *Mathematical Finance,* 7, pp. 127 – 154

D. Brigo, F. Mercurio (2006): *Interest Rate Models – Theory and Practice, with Smile, Inflation and Credit,* 2nd Edition, Springer

D. Brigo, M. Morini (2009): "New developments on the analytical cascade swaption calibration of the LIBOR Market Model", working paper

J. H. Cochrane (2001): *Asset Pricing,* Princeton University Press, Princeton and Oxford

T. E. Copeland, F. J. Weston, K. Shastri (2005): *Financial Theory and Corporate Policy,* Addison Wesley

Credit Risk+ (1997): A Credit Risk Management Framework, Credit Suisse Financial Products, Technical Document

J. L. Doob (1984): *Classical Potential Theory and Its Probabilistic Counterpart,* Springer

M. U. Dothan (2006): *Prices in Financial Markets,* Oxford University Press

D. Duffie (2001): *Dynamic Asset Pricing Theory,* 3rd Edition, Princeton University Press

C. Fries (2007): *Mathematical Finance – Theory, Modeling, Implementation,* Wiley & Sons Ltd.

M. Fujii, Y. Shimada, A. Takahashi (2010): "A Note on Construction of Multiple Swap Curves with and without Collateral", in: *FSA Research Review Vol. 6,* pp.139 – 157

C. Gourieroux, J. Jasiak (2001): *Financial Econometrics,* Princeton Series in Finance

R. A. Haugen (1995): *The New Finance – the Case Against Efficient Markets,* Prentice-Hall Inc.

D. Heath, R. A. Jarrow, A. Morton (1992): “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation”, in: *Econometrica,* No. 1, Vol. 60, pp. 77 – 105

F. Jamshidian (1997): “LIBOR and swap market models and measures”, in: *Finance and Stochastics 1,* pp. 293 . 330

R. A. Jarrow (1996): *Modelling Fixed Income Securities and Interest Rate Options,* McGraw Hill Series in Finance

R. A. Jarrow, S. Turnbull (1996): *Derivative Securities,* International Thomson Publishing

I. Karatzas, S. Shreve (1994): *Brownian Motion and Stochastic Calculus,* Springer

Derivatives and Hedging Accounting Handbook, KPMG, January 2008

Insights into IFRS, 7th Edition, KPMG, 2010 / 11

S. Kuhn, P. Scharpf: *Rechnungslegung von Financial Instruments nach IFRS,* Schäfer Poeschel

Y. Lengwiler (2004): *Microfoundations of Financial Economics – An introduction to General Equilibrium Asset Pricing,* Princeton University Press

F. Mercurio (2009): “Interest Rates and the Credit Crunch: New Formulas and Market Models”, working paper

Miltersen, Sandmann, Sondermann (1997): “Closed Form Solutions for Term Structure Derivatives with Log-Normal Interest Rates”*,* in: *Journal of Finance,* 52, pp. 409 – 430

M. Musiela, M. Rutkowski (2010): *Martingale Methods in Financial Modelling,* Springer

S. N. Neftci (2000): *An Introduction to the Mathematics of Financial Derivatives,* 2nd Edition, Academic Press

C. R. Nelson, A. F. Siegel (1987): “Parsimonious Modeling of the Yield Curve”, in: *Journal of Business,* Volume 60, Issue 4 (Oct.), pp. 473 – 489

A. Pelsser (2000): *Efficient Methods for Valuing Interest Rate Derivatives,* Springer

W. H. Press et al. (2002): *Numerical Recipes in C++,* 2nd Edition

PriceWaterhouseCoopers (ed.): *IFRS Manual of Accounting – 2011*

R. Rebonato (2004): *Interest-Rate Term-Structure Pricing Models: A Review,* Proceedings of the Royal Society London, 460, pp. 667 – 728

R. Rebonato (2008): *Volatility and Correlation – the Perfect Hedger and the Fox,* 2nd edition, Wiley & Sons Ltd.

R. Rebonato, K. McKay, R. White (2009): *The SABR/LIBOR Market Model – Pricing, Calbration, Hedging for Complex Interest Rate Derivatives,* Wiley & Sons Ltd.

S. Ross (1976): “The Arbitrage Theory of Capital Asset Pricing”, in: *Journal of Economic Theory,* December, pp. 343 – 362

K. Sandmann (2010): *Einführung in die Stochastik der Finanzmärkte,* 3rd Edition, Springer

J. Schoenmakers, B. Coffey (1999): *LIBOR rate models, related derivatives and model calibration,* 1999

J. Schoenmakers (2002): “Calibration of LIBOR models to caps and swaptions: a way around intrinsic instabilities via parsimonious structures”, working paper no. 740, Weierstrass-Institut für Angewandte Analysis und Stochastik, Berlin

B. Snowdon, H. R. Vane: *Modern Macroeconomics,* Cheltenham 2005

D. Sondermann (2006): *Introduction to Stochastic Calculus for Finance – A New Didactic Approach,* Springer

J. Stoer (1983): *Einführung in die Numerische Mathematik I,* Springer

D. Williams (1994): *Probability with Martingales,* Cambridge

Contact

KPMG AG
Wirtschaftsprüfungsgesellschaft

THE SQUAIRE
Am Flughafen
60549 Frankfurt/Main
GERMANY

Dr. Dirk Schubert
Partner, Audit Financial Services
T +49 69 9587-3626
dschubert@kpmg.com

www.kpmg.de