Management of economic and demographic risk in life insurance and pensions

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1 Introduction

The purpose of life insurance is to relieve individuals of economic risk associated with accidents, sickness etc.... The insurer covers the risk against a fixed, risk-free premium. It works because in a sufficiently large portfolio of independent risks, the gains and losses on the individuals will balance on average and the premium is set according to principle of equivalence: expected discounted premium are equal to expected discounted benefits.

However, a presence of *collective risk* factors that effect all policies in the portfolio, the indepence assumption may hold true conditionally, given the outcome of these factors, but unconditionally the individual risks become dependent. Examples:

- catastrophes
- uncertain economic development
- demographic development

Increasing the size of the portfolio exacerbates rather the mitigates such fors of risk. How to manage the risk?

1. Internal risk management - design of individual contracts. Two cases: *with-profit insurance*, where the premium as set sufficiently high to be adequate under all likely economic-demographic scenarios, and systematic surpluses are redistributed to the policy-holders in arrears. *Index-linked insurance*, where contractual payments depend, not only on the

individual life events, but also on interest and mortality. Ideally equivalence is attained under any economic-demographic scenario.

2. Reinsurance. Today they cover short term catastophes, but they do not offer coverage of long term economic-demographic risk.

3. Securitization, tradeable derivative securities of with payoffs depending on indices related to collective risk factors. This open hedging opportunities for insurers. For 15 years catastophic derivaties with pay-offs linked to indices for natural catastrophes are traded. In 2003 Swiss Re issued a USD 250m 4 year mortality bond with coupons following a survival function based on population statistics. A 25 year bond was also launched with interest related to the mortality experience in pensions but it was no success.

Three problem with securitization.

1. The contractual liabilities of an insurer cannot be transferred to a third party that is not supervised by the insurance regulator.

2. Pension contracts are exceedingly longterm and any prediction of interest and mortality is highly uncertain.

3. A mortality derivative must be based on a mortality index that can be equally understood by market agents and the insurer.

Ragnar Norberg presented some different models under which he derived the prospected reserves (or the future liability the insurer has to the insured).

2 Life is a process

Let the state of life at time $t \in [0, T]$ be

$$Z(t) \in \mathcal{Z} = \{0, 1, 2, \ldots, J\}, \quad Z(0) = 0.$$

Introduce the indicator processes

$$I_g(t) = 1[Z(t) = g],$$

and the counting process

$$N_{gh}(t) = \#\{\tau; Z(\tau-) = g, Z(\tau) = h, \tau \in (0, t)\}$$

(N_{gh} is the number of jumps from state g to h in the timeinterval $(0, t]$). We will assumpe that Z is a Markov process and thus have transition probabilities

$$p_{j,k}(t, u) = \mathbb{P}[Z(u) = k | Z(t) = j]$$

and intensities

$$\mu_{jk}(t) = \lim_{h \downarrow 0} \frac{p_{jk}(t, t+h)}{h}.$$

The compensated counting processes are square integrable orthogonal martingales:

$$dM_{gh}(t) = dN_{gh}(t) - I_g(t)\mu_{gh}(t)dt,$$

with expectation zero

$$\mathbb{E}[dM_{gh}(t)|\mathcal{F}_{t-}] = 0$$

and covariance

$$\mathbb{E}[dM_{gh}(t)dM_{jk}(t)|\mathcal{F}_{t-}] = \delta_{gh,jk} I_g(t)\mu_{gh}(t)dt,$$

where δ is the Krockner-delta.

2.1 Two states

$$\begin{cases} 0, & \textit{alive} \\ 1, & \textit{dead} \end{cases}$$

The "usual" survival probability: $p_{00}(t, u) = e^{-\int_t^u \mu dt}$.

2.2 Multiple decrement model

$$\begin{cases} 0, & \text{alive} \\ j, & \text{dead cause j}, j = 1, 2, \ldots, J \end{cases}$$

Total mortality intensity

$$\mu(t) = \sum_{j=1}^{J} \mu_j(t).$$

Probabiliy of death from cause j:

$$p_{0j}(t, u) = \int_t^u \underbrace{e^{-\int_t^\tau \mu}}_{(i)} \mu_j(\tau)\, d\tau.$$

(i) probability of being alive at time τ.

2.3 Disability model

Three types of states: active, disabled, dead, where "dead" is the only absorbing state, in particular one can go from invalid to active with some intensity.

There is no closed form for this system, but one has to use Kolomogorov forward equation

$$\frac{\partial}{\partial t} p_{ij}(t) = \sum_{g;g\neq j} p_{ig}(s,t)\mu_{gj}(t) - p_{ij}(s,t) \sum_{g;g\neq j} \mu_{jg}(t),$$

or Kolmogorov backward differential equation

$$\frac{\partial}{\partial t} p_{jk}(t,u) = -\sum_{g;g\neq j} \mu_{jg}(t)p_{gk}(t,u) + \sum_{g;g\neq j} \mu_{jg}(t)p_{jk}(t,u),$$

both equipeed with zero-transition $p_{jk}(u,u) = \delta_{jk}$.

The idea is to look at a population and estimate the transitions and then solve the differential equation numerically. Lax-Wendroff?

3 Insurance in Life

Consider individual multi-state policy issued at time 0 expiring at time T. $B(t)$ is total payment of benefits less premiums in $[0,T]$.

$$dB(t) = \sum_{j} I_j(t)dB_j(t) + \sum_{j\neq k} b_{jk}(t)dN_{jk}(t),$$

where B_j is a general life annuity running during sojourns in state j and b_{jk} is a sum assured payable upon transition from state j to state k. The annuity decomposes into a continuous part and a jump part:

$$dB_j(t) = b_j(t)\,dt + \Delta B_j(t).$$

The life history is represeted as a filtration

$$\mathbf{H} = \{\mathcal{H}_t\}_{t\geq 0}; \mathcal{H}_t = \sigma\{Z(\tau); 0 \leq \tau \leq t\}.$$

Suppose payments are currently invested/withdrawn from an account that bears interest at deterministic rate $r(t)$ at any time t. The reserv (the sum of future discounted liabilities) is

$$V_{\mathbf{H}}(t) = \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r} dB(\tau) \Big| \mathcal{H}_t\right]$$

which in the Markov case reduces to

$$V_{Z(t)}(t) = \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r} dB(\tau)\middle| Z_t\right].$$

Thus, we need only the *state-wise prospective reserves*

$$\begin{aligned} V_j(t) &= \mathbb{E}\left[\int_t^T e^{-\int_t^\tau r} dB(\tau)\middle| Z(t) = j\right] = \\ &= \int_t^T e^{-\int_t^\tau r} \sum_g p_{jg}(t,\tau)\left(dB(\tau) + \sum_{h;h\neq g} b_{gh}(\tau)\mu_{gh}(\tau)\,d\tau\right). \end{aligned}$$

The integralexpression is simple: it is the sum of all expected discounted future payments. It is one part from going to state j to state g, and a second part for continuing to another state h, i.e. $j \to g \to h$ when lumpsum $b_{gh}(\tau)$ is paid.

Differentiating this we get

$$\begin{aligned} \frac{dV_j(t)}{dt} &= -e^{-\int_t^t} \sum_g p_{jg}(t,t)\left(b_j(t) + \sum_{h;h\neq g} b_{gh}(t)\mu_{gh}(t)\right) + r(t)V_j(t) + \\ &+ \int_t^T e^{-\int_t^\tau r(s)\,ds} \sum_g \left(-\sum_{k;k\neq j} \mu_{jk}(t)(p_{gk}(t,\tau) - p_{j,g}(t,\tau))\right)\left(b_j\,dt + \sum_{h;h\neq g} b_{gh}(\tau)\mu_{gh}(\tau)\,d\tau\right) = \\ &= -\left(b_j(t) + \sum_{k;k\neq j} b_{jk}(t)\mu_{jk}(t)\right) + r(t)V_j(t) + \\ &+ \int_t^T e^{-\int_t^\tau r(s)\,ds} \sum_g -(\sum_{k;k\neq j} \mu_{jk}(t)\,(p_{gk}(t,\tau) - p_{j,g}(t,\tau)))\left(b_j\,dt + \sum_{h;h\neq g} b_{gh}(\tau)\mu_{gh}(\tau)\,d\tau\right) = \\ &= r(t)V_j(t) - b_j(t) - \sum_{k\neq j} \mu_{jk}(t)\left(b_{jk}(t) + \int_t^T e^{-\int_t^\tau r(s)\,ds} \sum_g (p_{gk}(t,\tau) - p_{jk}(t,\tau))\right)(\dots) = \\ &= r(t)V_j(t) - b_j(t) - \sum_{k;k\neq j} \mu_{jk}(t)R_{jk}(t), \end{aligned}$$

where

$$R_{jk}(t) = b_{jk}(t) + V_k(t) - V_j(t) \quad \text{"sum at risk"}.$$

Thus the differential equation is

$$V'(t) = r(t)V_j(t) - b_j(t) - \sum_{k;k\neq j} \mu_{jk}(t)R_{jk}(t)$$

which is Thieles differential equation.

3.1 The principle of equivalence

The principle state that the discounted benefits and the discounted premiums should balance on average

$$\mathbb{E}[e^{-\int_0^\tau r(s)\,ds}\,dB(\tau)] = 0.$$

Recall that $dB(t)$ is the change in annuity

$$dB(t) = \sum_j I_j(t)dB_j(t) + \sum_{j\neq k} b_{jk}dN_{jk}(t).$$

Spelled out in integral form the principle of equivalence is

$$\int_{0-} e^{-\int_0^\tau r(s)ds} \sum_g p_{0g}(0,\tau)\left(dB_g(\tau) + \sum_{h;h\neq g} b_{gh}(\tau)\mu_{gh}(\tau)\,d\tau\right) = 0.$$

In terms of the reserv it reads

$$\Delta B_0(0) + V_0(0) = 0.$$

Thus if no initial payments are made then $V(0) = 0$.

3.2 Semi-Markov model and path-dependent payments

Norberg now makes the model more general, i.e. by letting the transition intensities ("death intensities") depend on the state duration $S(t)$ (which is time elapsed since entry in the current state). For instance if we make a model from being ill and be able to work probably those that have been ill for 5 years has less probability to recover then those who has been ill just a week. The model has the folloing setup

$$\mu_{jk}(t, S(t)), \text{intensity of transition}$$

$$b_j(t,S(t)), \text{rate of annuity payment}$$
$$b_{jk}(t,S(t-)), \text{sum assured}$$
$$\Delta B_j(S(T)), \text{terminal endowment}$$

(I think of T as fixed and non-stochastic). $V_j s, t$ reserve in state $j$at policvy duration t and state duration $S(t) = s$. Taking the conditional expected value of what happens in a small time interval $(t, t + dt)$, gives

$$\begin{aligned} V_j(s,t) = &\underbrace{(1-\mu_j.dt)}_{\text{stay in j}}\big(b_j(s,t)dt + e^{-r(t)dt}V_j(s+dt,t+dt)\big)+ \\ &+\sum_{k;k\neq j}\mu_{jk}(s,t)\,dt(b_{jk}(s,t)+V_k(0,t))+o(dt)\end{aligned}$$

This leads to the first order partial differential equations (have not checked details)

$$\begin{aligned}\frac{\partial}{\partial t}V_j(s,t) = r(t)V_j(s,t) - \frac{\partial}{\partial s}V_j(s,t) - b_j(s,t) \\ - \sum_{k;k\neq j}\mu_{jk}(s,t)(b_{jk}(s,t)+V_k(0,t)-V_j(s,t))\end{aligned}$$

With the terminal condition $V_j(s,T-) = \Delta B_j(s)$. This can be solved with numerical methods such as Lax-Wendroff.

4 Martingales in Life

Start with the martingale associated with the total discounted payments under the contract:

$$\begin{aligned} M(t) = \mathbb{E}[\int_{0-}^{T} e^{-\int_0^\tau r}dB(\tau)|\mathcal{H}_t] = \\ = [past is known] = \int_{0^-}^{t} e^{-\int_0^\tau r}dB(\tau) + e^{-\int_0^t r}\mathbb{E}[\int_t^T e^{-\int_t^\tau r}dB(\tau)|\mathcal{H}_t] = \\ = \int_{0^-}^{t} e^{-\int_0^\tau r}dB(\tau) + e^{-\int_0^t r}V_{Z(t)}(t).\end{aligned}$$

Taking the differential using the Itô formula we get

$$
\begin{aligned}
dM(t) &= e^{-\int_0^t r} dB(t) + e^{-\int_0^t r}(-r(t)\,dt) \sum_j I_j(t) V_j(t) + \\
&\quad + e^{-\int_0^t r} \sum_j I_j(t) dV_j(t) + e^{-\int_0^t r} \sum_{j \neq k} dN_{jk}(t)(V_k(t) - V_j(t-)) = \\
&= e^{-\int_0^t r} \sum_j I_j(t) \underbrace{\left(dB_j(t) - r(t)V_j(t)dt + dV_j(t) + \sum_{k;\neq j} \mu_{jk} dt R_{jk}(t) \right)}_{=0} + \\
&\quad + e^{-\int_0^t r} \sum_{j \neq k} R_{jk}(t) dM_{jk}(t).
\end{aligned}
$$

The drift of a martingale should have expectation zero! We obtained the ODE again, but with another method.

5 The markov chain market and demographic hedging

The idea is to build a financial market model under the same framework as the life history analysis.

The continuous time Markov chain. Let $\{Y(t)\}_{t\geq 0}$ be a Markov chain on $\mathcal{Y} = \{1, \dots, n\}$. Transition probabilities

$$p_{ef}(t) = \mathbb{P}[Y(\tau + t) = f | Y(\tau) = e]$$

and intensities

$$\lambda_{ef} = \lim_{to\downarrow 0} \frac{p_{ef}(t)}{t}, e \neq f.$$

$$\lambda_{ee} = -\lambda_{e\cdot} = -\sum_{f;f\in\mathcal{Y}_e} \lambda_{ef}.$$

States directly accessible from state e:

$$\mathcal{Y}_e = \{f; \lambda_{ef} > 0\}, n_e = |\mathcal{Y}_e|.$$

In this framework Norberg explaines optimal quadratic hedging, and hedging mortality in particular.