Risk capital allocation for RORAC optimization*

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Abstract

This paper considers the financial optimization problem of a firm with several sub-businesses striving for its optimal RORAC. An insightful example shows that the implementation of classical gradient capital allocation can be suboptimal if division managers are allowed to venture into all business whose marginal RORAC exceeds the firm's RORAC. The marginal RORAC requirements are refined by adding a risk correction term that takes into account the interdependencies of the risks of different lines of business. It is shown that under certain stationarity conditions this approach can guarantee that the optimal RORAC will eventually be achieved.

Keywords: Risk capital, Economic capital, Capital allocation, Gradient allocation, Euler allocation, RORAC

JEL Classification: C61, D81, D82, G21, G22

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1. Introduction

The allocation of risk capital in financial firms for the purpose of performance measurement and risk-return optimization is well established in theory as well as in practice. Throughout this paper, we will use the terms risk capital and economic capital synonymously for an estimate of the amount of equity a firm needs to cover potential losses generated by its business. In contrast to regulatory capital, which is calculated using externally given rules and methodologies, economic capital represents an internal estimate of the risks. While the use of economic capital and its decomposition into a sum of single contributions of sub-businesses has become a standard approach in many banks (see Rosen and Saunders, 2010) and insurance companies (see Myers and Read, 2001), the academic world is still discussing methodological aspects and, to an extent, even the very significance of this concept.

There are several strands of literature which deal with risk capital allocation from various points of view. Most articles can be attributed to the mathematical finance context, in which rigorous arguments and axiomatics form the main focus (e.g. Denault, 2001; Kalkbrener, 2005; Tasche, 2004; Buch and Dorfleitner, 2008). Another strand of literature has a definite insurance-linked perspective (e.g. Dhaene et al., 2003; Furman and Zitikis, 2008; Gatzert and Schmeiser, 2008) and seeks to explore the advantages of risk capital allocation for insurance companies. A third strand looks at risk capital allocation from a more financial economics point of view (e.g. Merton and Perold, 1993; Stoughton and Zechner, 2007) and is therefore more closely related to the question concerning why capital allocation is a sensible procedure from an economic perspective.

In any case, a sound risk capital allocation framework requires at least two theoretical fundaments, namely a proper definition of a risk measure and an allocation principle. The combination of these two items yields a concrete allocation rule. In addition, several ad hoc allocation rules, like e.g. the covariance allocation rule[1], exist without explicit reference to the combination of a risk measure and an allocation principle. Much attention has recently been given to coherent risk measures (Artzner et al., 1999), which have several economically favorable properties, and to the gradient allocation principle (Tasche, 2008; Rosen and Saunders, 2010), also sometimes called the Euler allocation principle. The gradient allocation principle is well-suited to firms with homogeneous sub-businesses consisting of a continuum of single contracts, whereas in the case of few large single contracts an incremental allocation (Merton and Perold, 1993) seems to be more appropriate, where the risk capital allocated to sub-businesses is derived from looking at the firm with and without the sub-business under consideration and allocating economic capital proportional to the difference in overall risk capital.

While many contributions examine technical aspects of risk capital allocation rigorously and in

[1]See e.g. Kalkbrener (2005), who also points out the shortcomings of this allocation rule. Urban et al. (2004) use the covariance principle for calculating relative weights of each segment independently of the overall portfolio risk measure.

great detail, the actual economic justification remains mostly verbal. Typically, it is stated that the allocation is necessary to control risks ex ante by assigning limits to individual business units and its necessity for performance measurement is emphasized. On the other hand, risk capital allocation is also subject to criticism. In fact, Gründl and Schmeiser (2007) argue that capital allocation is completely senseless and that firms should rather refrain from using it. Even if one does not wish to follow this argument, the question emerges concerning why the optimum amounts of every line of business are not more adequately directly optimized by the headquarters.

The contribution of Stoughton and Zechner (2007) is the first to actually consider an economic optimization problem. The authors show that if the firm as a whole pursues maximization of the economic value added it is consistent with allocating capital to the sub-businesses, which are characterized by private information of managers, and allowing them to maximize the economic value added, based on the allocated capital. However, due to the restriction to normally distributed risks and a very specific incremental Value-at-Risk allocation rule, which is largely identical to the covariance allocation, their results are only of limited usefulness in terms of practical application.

This paper focuses on financial firms with different lines of business, for which the managerial decision concerns whether to expand or reduce rather than to create newly or abandon completely. We do not restrict ourselves to certain specific risk measures or distributional assumptions. Our approach comprises banks and insurance companies, both of which are subject to risk capital allocation. In banks the economic capital to be allocated could cover market, credit, and operational risk (Alessandri and Drehmann, 2010; Breuer et al., 2010; Embrechts et al., 2003) or classically credit risk in a portfolio context (Rosen and Saunders, 2010), while in insurance companies risk capital could be allocated for different lines of insurance contracts (Urban et al., 2004).

This paper contributes to the literature by developing a justification of risk capital allocation with a rather mathematical finance argumentation, which is well suited to the many extensively axiomatic contributions made on risk measures and economic capital found in the literature. To our knowledge there is no contribution, which argues without restricting the probability distribution of losses and the risk measure chosen that capital allocation could be reasonable when pursuing a maximization problem. In this paper we fill the gap by developing a procedure concerning capital allocation that is designed to maximize the RORAC of a company. Our analysis is based on the work of Tasche (2004) who, however, is not able to state a maximization problem due to assumptions, which are too simplistic. We assume that the segment managers have superior knowledge concerning the possible profits induced by segment reductions or expansion, while the risk of the portfolio is calculated centrally by the headquarters. Based on this we question RORAC maximization utilizing naive risk capital allocation and develop a more sophisticated rule for RORAC maximization.

The remainder of this paper is structured as follows: Firstly, we present the general organizational framework for capital allocation and return maximization in Section 2. Afterwards, we introduce our specific model in Section 3 and derive certain requirements for the existence of a company-wide optimal RORAC. Moreover, we propose an explicit control strategy that directs a firm to the optimal RORAC. We then give a numerical example in Section 4, showing how a classical risk allocation rule to sub-businesses can impede a company in attaining its optimal RORAC and elaborating the success of the afore-mentioned control strategy. Practical aspects are discussed in Section 5 and Section 6 concludes our paper.

Notice that the proofs of all lemmas, theorems, and corollaries can be found in the appendix.

2. General organizational framework

We consider a firm with n lines of business (subsequently called segments), in which each segment conducts a certain amount of business. Let the vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ symbolize the amount of business of each segment. We call u the *portfolio* and $U \subseteq \mathbb{R}^n$ the set of all portfolios. Let the future profits of segment $k = 1, \ldots, n$ stemming from an amount of business u_k be represented by the discrete-time process $(Y_k^t(u_k))_{t=1,2,\ldots}$ of integrable random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $(\mathcal{F}^t)_{t\geq 0}$. To be precise, $Y_k^t(u_k)$ denotes the profits generated by segment k in the period $(t-1, t)$. Note that the portfolio u can change over time. Since the firm's overall profit consists of the profits of each segment, we can add up $Y^t(u) := \sum_{k=1}^n Y_k^t(u_k)$, so that Y^t denotes the total profit of the firm in the corresponding period.

The risk capital of a portfolio u for time interval (s, h) is assumed to be $\varrho((Y^t(u))_{t=s+1,\ldots,h})$. Formally, ϱ is a multi-period risk measure, i.e., a mapping of the set of all tuples of random variables $(Y^{s+1}(u), \ldots, Y^h(u))$ into the real numbers $\mathbb{R}$. For further details on the construction of such a multi-period risk measure, see e.g. Artzner et al. (2007); Frittelli and Scandolo (2006). If ϱ is sub-additive, the firm's overall risk $\varrho((Y^t(u))_{t=s+1,\ldots,h})$ will be lower than the sum of the segments' stand-alone risks $\sum_{k=1}^n \varrho((Y_k^t(u_k))_{t=s+1,\ldots,h})$. This motivates the definition of an allocation principle, which fully proportions the firm's overall risk to the individual segments.

Definition 2.1. *Let $\mathcal{P}$ be defined as a set of tuples $(Y^{s+1}, \ldots, Y^h, u)$, with $u \in U$, where U is a non-empty set in $\mathbb{R}^n$ and $s < h$. Given a risk measure ϱ, an* allocation principle *on $\mathcal{P}$ is defined as a mapping $A^\varrho : \mathcal{P} \to \mathbb{R}^n$ with*

$$A^\varrho : (Y^{s+1}, \ldots, Y^h, u) \mapsto \begin{bmatrix} A_1^\varrho(Y^{s+1}, \ldots, Y^h, u) \\ \vdots \\ A_n^\varrho(Y^{s+1}, \ldots, Y^h, u) \end{bmatrix}$$

such that

$$\sum_{k=1}^n A_k^\varrho(Y^{s+1}, \ldots, Y^h, u) = \varrho((Y^t(u))_{t=s+1,\ldots,h}) \, .$$

The expression $A_k^{\varrho}(Y^{s+1}, \ldots, Y^h, u)/u_k =: a_k^{(s,h)}(u)$ *is referred to as the* per-unit risk contribution *of segment* k *for time interval* (s, h).

The capital allocation can be seen as a means to split up the diversification benefits stemming from the pooling of the segments' risks. It should be noted that the risk capital allocated does not coincide with real capital invested to fund the business in the segments. Since we consider financial firms, we can assume that the investments are financed to a large extend through debt, while equity in the form of economic capital is only essential to cover the risks of the investments. Therefore, risk capital allocation in a financial institution requires a different approach than classical capital budgeting in non-financial firms (Saita, 2007). The economic capital is merely allocated virtually to express each segment's contribution to the overall risk and to provide a benchmark for the profitability of each segment's business.

Next, we introduce a return function for a certain time horizon h (possibly infinity), linking the yield and the risk dimension of the firm. Formally, given the information set $\mathcal{F}^s$, let $r_{Y,\varrho}^{(s,h)} : U \to \mathbb{R}$ denote the return of portfolio u for the time interval (s, h) with $s < h$. In general, the return depends on the expected profits $\mathbb{E}(Y^{s+1}(u) \,|\, \mathcal{F}^s), \ldots, \mathbb{E}(Y^h(u) \,|\, \mathcal{F}^s)$ and the risk arising from the profits $Y^{s+1}(u), \ldots, Y^h(u)$ for all periods up to time h. Notice, however, that given the information set $\mathcal{F}^s$ the return function $r_{Y,\varrho}^{(s,h)}$ is deterministic.

The natural managerial control problem is to maximize the return $r_{Y,\varrho}^{(s,h)}$ at time s for a given time horizon h. We embed this maximization problem into a general systems framework. Therefore, we consider the profits $Y^{s+1}(u^{s+1}), \ldots, Y^h(u^{s+1})$ as the state variables, which can be controlled by the choice of the portfolio u^{s+1} at time s. If the return function is known to the headquarters, it can directly optimize the return by solving the above-mentioned control problem using e.g. a stochastic control theory approach. In this case, division managers and capital allocation become superfluous therewith.

Therefore, in the remainder of this paper we examine decentralized firms. That is, we assume that only the headquarters can evaluate the risk $\varrho((Y^t(u^s))_{t=s+1,\ldots,h})$ of the whole portfolio at any time s, i.e., risk modeling and risk calculations take place at the headquarters. On the other hand, the expected profits $\mathbb{E}(Y^{s+1}(u) \,|\, \mathcal{F}^s), \ldots, \mathbb{E}(Y^h(u) \,|\, \mathcal{F}^s)$ are unknown at the corporate level except at the present portfolio extent u^s. We can assume, however, that the expected profits $\mathbb{E}(Y_k^{s+1}(u_k) \,|\, \mathcal{F}^s), \ldots, \mathbb{E}(Y_k^h(u_k) \,|\, \mathcal{F}^s)$ are known to the kth segment for u_k in some neighborhood of u_k^s, i.e., the amount of business currently undertaken. This conveys the idea that the segments have more extensive knowledge than the headquarters about which profit margins can be generated by signing additional business of the same kind due to their direct negotiations with their business partners.

Since the segment managers have only partial information available, it is generally unlikely that they will be able to achieve the optimal portfolios directly in advance. In theory, the search for the optimal portfolio could be intermediated by a Walrasian auctioneer who sets and adjusts risk prices until an equilibrium is obtained. Nevertheless, we pursue the approach of economic capital allocation, which is by far more common in financial institutions due to its straightforwardness in terms of implementation.

At each instant of time s, the headquarters calculates the future return $r^{(s,h)}_{Y,\varrho}$ and the corresponding company-wide risk of the current portfolio u^s and allocates each segment its per-unit risk contribution $a^{(s,h)}_k$. Afterwards, each segment's manager can control the kth amount of business by setting up a new value of u^{s+1}_k. The main input he or she has available at time s are the per-unit risk contribution $a^{(s,h)}_k$, the company-wide return $r^{(s,h)}_{Y,\varrho}(u^s)$ as communicated by the headquarters, and the segment's expected profits for the future periods $\mathbb{E}(Y^{s+1}_k(u^{s+1}_k) \mid \mathcal{F}^s), \ldots, \mathbb{E}(Y^h_k(u^{s+1}_k) \mid \mathcal{F}^s)$ for u^{s+1}_k in some neighborhood of u^s_k. It is now desirable to control the next portfolio u^{s+1} in such a way that the performance judged by the criterion $r^{(s,h)}_{Y,\varrho}$ becomes as large as possible.

Within the general framework, there are many possible control strategies, which can be used to search the optimal portfolio, depending inter alia on the profit process Y^t and the specification of the return function. Without further assumptions, it is clear that the convergence to the optimal portfolio of a strategy cannot be guaranteed, as the optimal return may change arbitrarily.

3. RORAC optimization using gradient allocation in a stationary setting

In this section we first specify the general model and show that gradient allocation can be linked to the search for the optimal portfolio. Finally we derive a converging control strategy and incorporate risk limits.

3.1. Model specification

We commence specifying the model with a restriction of the profit process:

Assumption 3.1. *The profit process $(Y^t(u))_{t=1,2,\ldots}$ is strictly stationary for each $u \in U$.*

Notice that due to the stationarity of $(Y^t(u))_{t=1,2,\ldots}$, the company-wide expected profit $M(u) := \mathbb{E}(Y^t(u) \mid \mathcal{F}^s)$ and each segment's expected profit $M_k(u_k) := \mathbb{E}(Y^t_k(u_k) \mid \mathcal{F}^s)$ become independent of the time t and the information set $\mathcal{F}^s$, $s < t$. Therefore, we drop the time index on these terms. By the same token, when specifying the return function $r^{(s,h)}_{Y,\rho}$, it suffices to consider the expected profit of a single period only. Moreover, the stationarity condition of the profit process also motivates using a one-period risk measure.

Assumption 3.2. *There exists a coherent one-period risk measure ρ such that $\rho(Y^{s+1}(u)) = \varrho((Y^t(u)_{t=s+1,\ldots,h})$.*

Notice that $\rho(Y^{s+1}(u))$ is also time-independent. Therefore, we drop the time index in the following and define $\rho_Y(u) := \rho(Y^{s+1}(u))$. Next, we specify the return function by using the *return on risk adjusted capital (RORAC)* linking the yield and the risk dimension of the firm.

Assumption 3.3. *The return function $r_{Y,\rho} : U \to \mathbb{R}$ is defined as $r_{Y,\rho} : u \mapsto \frac{\mathbb{E}(Y(u))}{\rho(Y(u))}$.*

The RORAC represents the ratio of the expected profit margin due to business to the extent given by u and the economic capital due to the corresponding risk of the portfolio u. Furthermore, we have to specify a capital allocation rule according to Definition 2.1. Using RORAC as the return function, it is illustrated below in Subsection 3.2 that it is natural to employ gradient allocation, sometimes also called Euler allocation, which is defined as follows.

Definition 3.1. *Let $X(u)$ be an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ depending on the portfolio u. Let $\mathcal{P}_{X,\rho}$ be defined as a set of tuples (X, u), with $\rho(X(u))$ being homogeneous and differentiable in $u \in U$. The mapping $A^{\rho}_{\nabla} : \mathcal{P}_{X,\rho} \to \mathbb{R}^n$ defined as*

$$A^{\rho}_{\nabla} : (X, u) \mapsto u * \nabla\rho(X(u))$$

is called the gradient allocation principle *associated with ρ. Here, $*$ denotes the Hadamard or component-wise product. Thus, $A^{\rho}_{\nabla}(X, u)$ is a vector.*

Notice that as gradient allocation is well-defined for homogeneous functions only, we first need to further restrict the process $(Y^t(u))_{t=1,2,\ldots}$ by splitting it up into the (deterministic) expected profit and a homogeneous risk-bearing part.

Assumption 3.4. *The (random) function $X^t_k(u_k) := Y^t_k(u_k) - M_k(u_k)$ is homogeneous with respect to u_k for all $k = 1, \ldots, n$ and all $t = 1, 2, \ldots$.*

We call $X^t(u) := \sum_{k=1}^{n} X^t_k(u_k)$ the *profit fluctuation* of portfolio u. Notice that due to the stationarity of the profit process, the risk $\rho_X(u) := \rho(X^t(u))$ is independent of the time t. Using gradient allocation, we can only allocate the risk capital of the profit fluctuations $\rho_X(u)$ instead of the total risk capital $\rho_Y(u)$. However, due to the translation invariance of ρ we have $\rho_Y(u) = \rho_X(u) - M(u)$. Thus, an allocation of the profits $A^{\rho}(Y, u)$ can be naturally constructed as the expected profits $M_k(u_k)$ bear no risk:

Assumption 3.5. *The allocation principle of the firm is defined as*

$$(Y, u) \mapsto A^{\rho}_{\nabla}(X, u) - \boldsymbol{M}(u),$$

where $\boldsymbol{M}(u) := (M_1(u_1), \ldots, M_n(u_n))'$.

Notice that Assumptions 3.4 and 3.5 imply that at the headquarters level, only the risk of the profit fluctuations is distributed among the segments. That is, the segments receive their per-unit risk contribution related to their fraction of the profit fluctuation X^t. Once these contributions are calculated, the actual required and allocated economic capital is determined by subtracting each segment's expected profit.

3.2. Linking gradient allocation and the control problem

In this subsection, we link the concept of gradient capital allocation to the control problem of maximizing the return. Notice that the RORAC can be written as

$$r_{Y,\rho}(u) = \frac{\mathbb{E}(Y(u))}{\rho(Y(u))} = \frac{M(u)}{\rho_X(u) - M(u)}.$$

Given the notion of a per-unit risk contribution of the profit fluctuation $a_k(u) := \left(A^{\rho}_{\nabla}(X,u)\right)_k / u_k$ one can define the marginal RORAC of segment k.

Definition 3.2. *Let M be partially differentiable in U. The function $r_{Y,\rho}(u_k \,|\, u) : U \to \mathbb{R}$ defined as*

$$r_{Y,\rho}(u_k \,|\, u) : u \mapsto \frac{\frac{\partial M(u)}{\partial u_k}}{a_k(u) - \frac{\partial M(u)}{\partial u_k}} = \frac{M_k'(u_k)}{a_k(u) - M_k'(u_k)}$$

is called marginal RORAC *of segment k given portfolio u associated with Y and ρ.*

This ratio expresses the expected additional profits in relation to the additional risk capital for the additional business. It is natural to postulate that a suitable allocation principle should state that a business extension is useful for every segment k whenever its marginal RORAC is higher than the present RORAC. Based on this idea we follow Tasche (2008) and define the notion of RORAC compatibility of an allocation principle.

Definition 3.3. *An allocation principle A^{ρ} is called* RORAC compatible *if there holds:*

1. *For all portfolios $u \in U$ and $k = 1, \ldots, n$ there is some $\epsilon_k > 0$ such that*

$$r_{Y,\rho}(u_k \,|\, u) > r_{Y,\rho}(u) \quad \Rightarrow \quad r_{Y,\rho}(u + \tau e_k) > r_{Y,\rho}(u)$$

   holds for all $0 < \tau < \epsilon_k$, where e_k denotes the kth unit vector.

2. *For all portfolios $u \in U$ and $k = 1, \ldots, n$ there is some $\epsilon_k > 0$ such that*

$$r_{Y,\rho}(u_k \,|\, u) < r_{Y,\rho}(u) \quad \Rightarrow \quad r_{Y,\rho}(u - \tau e_k) > r_{Y,\rho}(u)$$

   holds for all $0 < \tau < \epsilon_k$, where e_k denotes the kth unit vector.

The following theorem, which has been proven for linear profit functions in Tasche (2004), links the concepts of RORAC compatibility to the gradient allocation principle:

Theorem 3.4. *Let $\rho_X : U \to \mathbb{R}$ be partially differentiable in U with continuous derivatives. Then A^ρ is RORAC compatible if and only if $A^\rho(Y, u) = A^\rho_\nabla(X, u) - \boldsymbol{M}(u)$.*

Note that Theorem 3.4 can be considered as an additional justification for the gradient allocation principle in the form of Assumption 3.5. Since firms strive to maximize their return as stated above, the next lemma provides sufficient conditions to ensure that the RORAC $r_{Y,\rho}$ has a finite global maximum.

Lemma 3.5. *Let $U \subseteq \mathbb{R}^n_{\geq 0}$ be a closed convex set. Assume that there holds:*

(a) The expected profit function M is differentiable, positive, and concave on U.

(b) The risk function ρ_X is differentiable and positive on U.

(c) For all $k = 1, \dots, n$ there holds $\lim_{u_k \to \infty} M'_k(u_k) = 0$.

Then, the RORAC $r_{Y,\rho}$ either attains its global maximum in the interior of U where $\nabla r_{Y,\rho} = 0$, or on some boundary point of U.

While the adherence to conditions (a) and (b) of Lemma 3.5 is straightforward in an economic context, it is advisable to briefly discuss condition (c). This assumption corresponds to the decreasing profitability of additional business of the same nature. Moreover, we assume that the marginal profits will eventually converge to zero. We regard this assumption as uncritical from a theoretical viewpoint as we assume an incomplete market, i.e., we implicitly assume that $X(u)$ cannot be traded on the capital market at a unique price. Such behavior also depicts the fact that one has to offer better conditions when expanding into a market, which is becoming increasingly satiated.

3.3. Second-order condition

Definition 3.3 merely states that if a segment's marginal RORAC exceeds the firm's RORAC, then there is an ϵ_k such that the expansion by ϵ_k units increases the firm's RORAC, i.e., that $r_{Y,\rho}(u + \epsilon_k e_k) > r_{Y,\rho}(u)$, but provides no conditions on the size of ϵ_k. In fact, this can lead to over-expanding a section's business and hence to a decline of the firm's RORAC, as the example in Section 4 demonstrates. The following theorem ensures that ceteris paribus a change of the portfolio cannot reduce the firm's RORAC.

Theorem 3.6. *Let $H(u) = \left[\frac{\partial^2 \rho_X(u)}{\partial u_i \partial u_j}\right]$ be the Hessian of $\rho_X(u)$. Assume that $\|H(u)\|$ is bounded on a convex set $U \subseteq \mathbb{R}^n_{\geq 0}$. Let $\Lambda \geq \max_{u \in U} \lambda_{\max}(H(u))$ be an upper bound for the largest eigenvalue of*

$H(u)$. If $u \in U$, $u + \epsilon \in U$, $M(u) > 0, r_{Y,\rho}(u) > 0$, and if for all $k = 1, \ldots, n$ there holds

$$r_{Y,\rho,\Lambda}(\epsilon_k \,|\, u) \geq r_{Y,\rho}(u), \tag{1}$$

where

$$r_{Y,\rho,\Lambda}(\epsilon_k \,|\, u) := \frac{M_k(u_k + \epsilon_k) - M_k(u_k)}{\epsilon_k a_k(u) + \frac{1}{2}\epsilon_k^2 \Lambda - (M_k(u_k + \epsilon_k) - M_k(u_k))}, \tag{2}$$

with strict inequality in (1) given for at least one $k = 1, \ldots, n$, then there also holds

$$r_{Y,\rho}(u + \epsilon) > r_{Y,\rho}(u). \tag{3}$$

There are two differences concerning equation (1) when compared to the requirement of Definition 3.3 which are worth mentioning. Notice that an expansion of the marginal RORAC yields

$$r_{Y,\rho}(u_k \,|\, u) = \frac{\epsilon_k M_k'(u_k)}{\epsilon_k a_k(u) - \epsilon_k M_k'(u_k)}. \tag{4}$$

The first change of equation (2) as compared to equation (4) concerns the valuation of the additional expected profit generated by segment k by an expansion of ϵ_k. While in equation (4) the additional expected profit is estimated with the first-order approximation $\epsilon_k M_k'(u_k)$, equation (2) uses the actual increase of the expected profit $M_k(u_k + \epsilon_k) - M_k(u_k)$. The second change consists of the additional term of $\frac{1}{2}\epsilon_k^2 \Lambda$ in equation (1). Notice that this term is quadratic on ϵ_k. While it disappears for small values of ϵ_k, it decreases the marginal RORAC with the size of the expansion.

Returning to the aim of maximizing the firm's RORAC, Theorem 3.6 gives rise to the following control strategy:

Control Strategy 3.7. *Let $\alpha \in (0, 0.5]$.*

1. *For each $k = 1, \ldots, n$ check if*

$$r_{Y,\rho}(u_k^{t-1} \,|\, u^{t-1}) \begin{array}{c} > \\ = \\ < \end{array} r_{Y,\rho}(u^{t-1}) :$$

   a) *If "$=$" holds, set $\epsilon_k^t = 0$.*

   b) *If "$>$" holds, calculate $\epsilon_{k,\max}^t = \max \epsilon_k^t > 0$ such that*

$$r_{Y,\rho,\Lambda}(\epsilon_k^t \,|\, u^{t-1}) \geq r_{Y,\rho}(u^{t-1}) \tag{5}$$

   is fulfilled and $u_k^{t-1} + \epsilon_{k,\max}^t \in U$. Choose $\epsilon_k^t \in [\alpha \epsilon_{k,\max}^t, (1-\alpha)\epsilon_{k,\max}^t]$.

c) If "<" holds, calculate $\epsilon^t_{k,\min} = \min \epsilon^t_k < 0$ *such that*

$$r_{Y,\rho,\Lambda}(\epsilon^t_k \,|\, u^{t-1}) \geq r_{Y,\rho}(u^{t-1}) \tag{6}$$

is fulfilled and $u^{t-1}_k + \epsilon^t_{k,\min} \in U$*. Choose* $\epsilon^t_k \in [(1-\alpha)\epsilon^t_{k,\min}, \alpha\epsilon^t_{k,\min}]$.

2. Set $u^t = u^{t-1} + \epsilon^t$.

It is noteworthy to show that this strategy guarantees convergence to the optimal RORAC, which is carried out in the following corollary.

Corollary 3.8. *Assume that the conditions (a)–(c) of Lemma 3.5 apply and* $u^{opt} := \arg\max_{u\in U} r_{Y,\rho}(u)$ *lies in the interior of the convex set* $U \subseteq \mathbb{R}^n_{\geq 0}$*. Then the sequential application of Control Strategy 3.7 leads to the optimal RORAC, i.e.,* $\lim_{t\to\infty} u^t = u^{opt}$.

3.4. Risk limiting

The application of Control Strategy 3.7 implicitly assumes that the available economic capital of the firm is unlimited. In fact, however, a natural limit for the economic capital is given by the amount of equity in each period. Therefore, we extend the strategy by including risk limits. Let $\rho^t_{\max}$ denote the maximal risk capital available for period $(t-1,t)$. First, we have to allocate the maximal risk capital $\rho^t_{\max}$ to the individual segments such that $\sum_{k=1}^n \rho^t_{\max,k} = \rho^t_{\max}$. While there are many possible rules for such a decomposition, we precisely propose the following:

$$\rho^t_{\max,k} = \frac{A^\rho_k(Y,u^{t-1})}{\rho_Y(u^{t-1})} \cdot \rho^t_{\max} = \frac{u^{t-1}_k a_k(u^{t-1}) - M_k(u^{t-1}_k)}{\rho_X(u^{t-1}) - M(u^{t-1})} \cdot \rho^t_{\max},$$

which implies that the maximal risk capital is allocated to the segments relatively to the portion of the overall risk capital currently used.

The following theorem places restrictions on the individual values of ϵ^t_k in the control strategy to ensure that the risk limit $\rho^t_{\max}$ is maintained in the period $(t-1,t)$.

Theorem 3.9. *If for all* $k = 1,\ldots,n$ *there holds*

$$u^{t-1}_k a_k(u^{t-1}) + \epsilon^t_k a_k(u^{t-1}) + \tfrac{1}{2}(\epsilon^t_k)^2\Lambda \leq \rho^t_{\max,k} + M_k(u^t_k), \tag{7}$$

where $u^t_k = u^{t-1}_k + \epsilon^t_k$*, then there also holds*

$$\rho_Y(u^t) = \rho_X(u^t) - M(u^t) \leq \rho^t_{\max}.$$

Extending Control Strategy 3.7 to include such risk limits is straightforward. Equation (7) is quadratic in ϵ_k^t and provides an additional bound for the maximal size $\epsilon_{k,\max}^t$ of an extension or reduction of a segment's business in steps 1(b) and 1(c) of Control Strategy 3.7.

Notice that for each single period the maximal risk capital is constant since it is governed by the amount of equity. However, this amount can in principle be adjusted by issuing or repurchasing shares. It is a task for the firm's headquarters to expand or reduce equity in the long run in such a way that the risk limit eventually nears the overall economic capital.

4. Numerical example

Consider a firm with two risky segments. Let the profit process of the firm be represented by

$$Y^t(u) = \mu(u) + A(u)W^t,$$

where

$$\mu(u) = \begin{bmatrix} \log(u_1 + \frac{1}{2}) \\ \log(u_2 + \frac{1}{2}) \end{bmatrix} \quad \text{and} \quad A(u) = \begin{bmatrix} u_1 & 0 \\ 0.5u_2 & 0.866u_2 \end{bmatrix}$$

are constant, and W^t is Gaussian 2-dimensional white noise for all $t = 1, 2, \ldots$. Notice that as $\mu(u)$ and $A(u)$ are time-independent, the process $(Y^t(u))_{t=1,2,\ldots}$ is strictly stationary. It is easy to verify that

$$\begin{aligned} M_1(u_1) &= \mu_1(u_1), \\ M_2(u_2) &= \mu_2(u_2), \\ X_1(u_1) &\sim \mathrm{N}(0, u_1^2) \text{ iid}, \\ X_2(u_2) &\sim \mathrm{N}(0, u_2^2) \text{ iid}, \\ \mathrm{Corr}(X_1, X_2) &= 0.5\,. \end{aligned}$$

Moreover, let the risk measure ρ be the Value-at-Risk (VaR) at the 99.97% level.[2]

4.1. First-order approach

We first show that a traditional marginal RORAC expansion or reduction strategy without using the insights of Theorem 3.6 can hinder the firm from being controlled optimally. Let the portfolio be $(u_1^1, u_2^1) = (1.5, 1.7)$ in the first period. Therewith,

$$\rho_X(u^1) = 3.43\sqrt{(u_1^1)^2 + 2\mathrm{Corr}(X_1^1, X_2^1)u_1^1 u_2^1 + (u_2^1)^2} = 9.2100.$$

[2]Note that the VaR is generally not a coherent risk measure since it fails to meet the sub-additivity property (see e.g. Tasche, 2002). However, when restricted to the multinormal distribution it is in fact sub-additive (Artzner et al., 1999).

The per-unit risk contributions of the profit fluctuation are then

$$a_1(u^1) = \frac{\partial \rho_X(u)}{\partial u_1}(u^1) = 2.9067 \quad \text{and} \quad a_2(u^1) = \frac{\partial \rho_X(u)}{\partial u_2}(u^1) = 3.0304\,.$$

The RORAC for the present portfolio is

$$r_{Y,\rho}(u^1) = 18.451\%\,.$$

Marginal RORAC analysis according to Definition 3.2 leads to

$$r_{Y,\rho}(u_1^1 \,|\, u^1) = 20.775\% \quad \text{and} \quad r_{Y,\rho}(u_2^1 \,|\, u^1) = 17.647\%\,.$$

We assume that the firm follows a marginal RORAC expansion or reduction strategy, respectively, i.e., that segments whose marginal RORAC exceeds the firm's overall RORAC will expand, while segments whose marginal RORAC falls below the firm's RORAC will reduce their business. Therefore, segment 1 would expand and segment 2 would reduce. However, they do not know to which extent.

We assume an extension and reduction, respectively, to the new values $(u_1^2, u_2^2) = (1.85, 1.55)$. The new RORAC is now

$$r_{Y,\rho}(u^2) = 18.410\%$$

and thus has decreased slightly despite the intention for it being to increase. In fact the real optimum of $r_{Y,\rho}(u^{\text{opt}}) = 18.508\%$ at $(u_1^{\text{opt}}, u_2^{\text{opt}}) = (1.6555, 1.6555)$ has been missed, as Figure 1(a) displays.

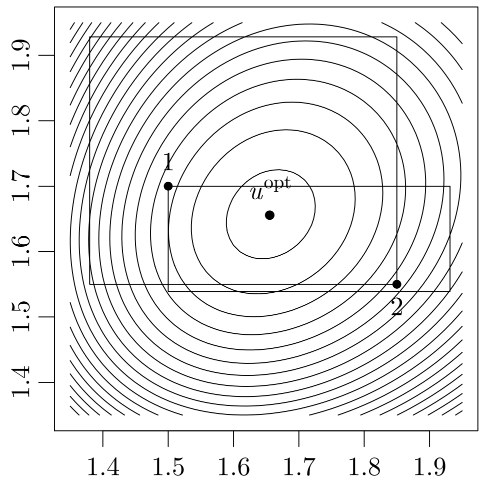
Now since the functions M_k are known to the segments one could consider a more sophisticated approach analogously to the first remark after Theorem 3.6 and replace the first-order approximation of the additional expected profit $\epsilon_k^{t+1} M_k'(u_k^t)$ with the actual additional expected profit $M_k(u_k^t + \epsilon_k^{t+1}) - M_k(u_k^t)$ both in the numerator and denominator of equation (4). In fact the values above of the new (u_1^2, u_2^2) were chosen in such a way that the expected profit-corrected marginal RORAC

$$r_{Y,\rho}^{\text{epc}}(\epsilon_k^{t+1} \,|\, u^t) := \frac{M_k(u_k^t + \epsilon_k^{t+1}) - M_k(u_k^t)}{\epsilon_k^{t+1} a_k(u^t) - (M_k(u_k^t + \epsilon_k^{t+1}) - M_k(u_k^t))}$$

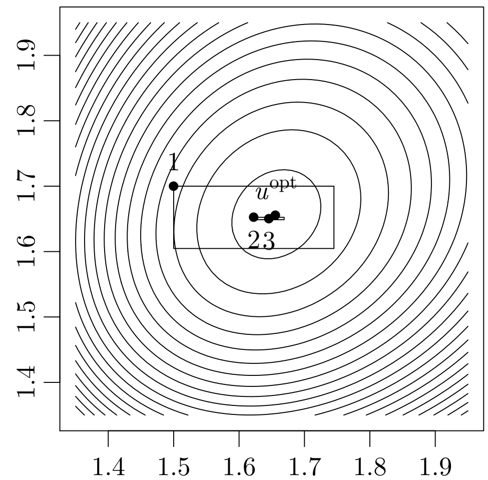
of the extension was still higher than the original RORAC (segment 1: 18.834%) and the expected profit-corrected marginal RORAC of the reduction was still lower than the original RORAC (segment 2: 18.393%).

In the next period, the per-unit risk contributions become

$$a_1(u^2) = \frac{\partial \rho_X^2(u)}{\partial u_1}(u^2) = 3.0539 \quad \text{and} \quad a_2(u^2) = \frac{\partial \rho_X^2(u)}{\partial u_2}(u^2) = 2.8794\,,$$



(a) Sequence of portfolios generated by a first-order marginal RORAC strategy. The boxes indicate the feasible regions where $r^{\text{epc}}_{Y,\rho}(\epsilon^t_k \mid u^{t-1}) > r_{Y,\rho}(u^{t-1})$. Notice that it is possible for a firm to swap between portfolios 1 and 2 all the time.



(b) Sequence of portfolios generated by Control Strategy 3.7. The boxes indicate the feasible regions according to equations (5) and (6). The sequence of portfolios converges to u^{opt}.

Figure 1: Contour plots of the return function $r_{Y,\rho}$ as defined in Section 4 with different control strategies.

and the marginal RORACs are

$$r_{Y,\rho}(u_1^2 \mid u^2) = 16.190\% \quad \text{and} \quad r_{Y,\rho}(u_1^2 \mid u^2) = 20.397\% \,.$$

This time, segment 1 would decline and segment 2 would expand. In fact, the new portfolio for period 3 could be $(u_1^3, u_2^3) = (1.5, 1.7)$, where we started in period 1. Again, in this case the expected profit-corrected marginal RORAC of the reduction was still lower than the RORAC (segment 1: 17.769%) and the expected profit-corrected marginal RORAC of the extension was still higher than the RORAC (segment 2: 19.546%).

Therefore, this example proves that a control strategy proceeding in the way described above does not necessarily lead the firm to the optimum even under the assumption of a stationary profit function.

4.2. Second-order approach

In order to implement Control Strategy 3.7, we first have to constrain the feasible region. Let $U = \{(u_1, u_2) : u_1 \geq 1, u_2 \geq 1\}$. Since the eigenvalues of the Hessian $H(u)$ of $\rho_X(u)$ are

$$\begin{aligned} \lambda_1 &= 0\,, \\ \lambda_2 &= 2.5725((u_1)^2 + (u_2)^2)/\left((u_1)^2 + (u_2)^2 + u_1 u_2\right)^{3/2}\,, \end{aligned}$$

we can use the upper bound $\Lambda = H(1,1) = 0.99016$.

Starting again at $(u_1^1, u_2^1) = (1.5, 1.7)$, we choose $\epsilon_k^t = 0.5\epsilon_{k,\max}^t$ or $\epsilon_k^t = 0.5\epsilon_{k,\min}^t$, respectively, for each $k = 1, 2$ at each time. In the first period, equations (5) and (6) yield

$$\epsilon_{1,\max}^2 = 0.24505 \quad \text{and} \quad \epsilon_{2,\min}^2 = -0.09530.$$

Thus, we can calculate

$$u_1^2 = u_1^1 + \tfrac{1}{2}\epsilon_{1,\max}^2 = 1.6225 \quad \text{and} \quad u_2^2 = u_2^1 + \tfrac{1}{2}\epsilon_{2,\min}^2 = 1.6523.$$

The new RORAC becomes $r_{Y,\rho}^2(u^2) = 18.506\%$. For the next period, the maximum step sizes become

$$\epsilon_{1,\max}^3 = 0.04645 \quad \text{and} \quad \epsilon_{2,\min}^3 = -0.00363,$$

which leads to new values of

$$u_1^3 = u_1^2 + \tfrac{1}{2}\epsilon_{1,\max}^3 = 1.6457 \quad \text{and} \quad u_2^3 = u_2^2 + \tfrac{1}{2}\epsilon_{2,\min}^3 = 1.6505,$$

and to a RORAC of $r_{Y,\rho}^3(u^3) = 18.508\%$. Figure 1(b) displays the progress of the portfolios u^t for $t = 1, 2, 3$.

5. Practical aspects

In this section we will discuss several issues that are of importance when putting the concepts of Section 3 into practice. Our paper focuses on financial firms and thereby comprises banks and insurance companies. The following remarks are mainly valid for both types of firms, however, with a certain emphasis on banks.

1. For practical applications it is realistic to use the sequential one-period model instead of the general approach of Section 2. It represents the status quo in financial firms, being that there is a RORAC to be optimized in each period, as banks tend to drive their business by sight in order to prevent misjudgments caused by model errors. However, as Artzner et al. (2007)

point out, the future beyond the current period can be accounted for by considering market prices, or where these are missing, fair values of the assets when calculating the expected profit and profit fluctuation.

2. The risk measure ρ is in practice often chosen to be the Value-at-Risk (VaR). The economic rationale behind this is that lodging economic capital of at least $\mathrm{VaR}_{1-\alpha}$ yields a probability of default of at most α. Hence a firm can control its probability of default and therefore its rating. However, as already mentioned above, it is a well-known fact that the VaR is not sub-additive. Nevertheless, there are many possible distributions besides the normal distribution for which the VaR is sub-additive at least in the tail region (see Daníelsson et al., 2010), which is the relevant case in the economic capital context, where the threshold α will generally be chosen at a very low level.

3. Many authors focus on the *risk adjusted return on capital* (RAROC), see e.g. Hallerbach (2004), or the *risk adjusted return on risk adjusted capital* (RARORAC) instead of on the RORAC. While in some cases this is simply another name for the same concept, there are other cases where indeed the interest rate on the economic capital is added or the product of the capital costs and the economic capital is subtracted in the numerator. In any case, such conceptualizations are essentially just an additive translation of the objective function focused upon here. Thus, our considerations also remain valid for these return functions.

4. From an economic viewpoint the restriction of assuming X to be linear with respect to u literally means that we assume perfectly dependent business within each segment. For many segments, like for instance a portfolio of various single loans, this restriction may appear to be unrealistic. However, in credit risk modeling the risk of a segment indeed scales (almost) linearly with the number of units held in well-diversified segments for many distributional assumptions (see McNeil et al., 2005, Sec. 8.4.3). Therefore, this restriction can be assumed to hold unless $X_k(u_k)$ is scaled down to values of u_k close to zero. For many applications, especially credit risk modeling, the latter assumption appears to be noncritical.

5. When implementing risk limits, it is necessary to monitor the compliance with risk limits preferably on a daily basis. If a segment's loss exceeds its allocated risk limit $\rho^t_{\max,k}$, there need to be safeguard procedures to be invoked immediately.

6. In practice, as the overall risk limit $\rho^t_{\max}$ depends on the available equity, it also depends on realized profits of the previous periods. Therefore, in the case of losses, the risk limit of the next period will decrease unless the firm is able to raise additional equity. Vice versa, retained profits yield an increase of the risk limits.

7. The assumption of stationarity in Section 3 sounds quite restricted. However, notice that in the end this restriction is only required for the risk limits and for the proof of the convergence of the control strategy. It is clear that convergence of any strategy cannot be guaranteed in an arbitrary non-stationary setting, as in such a setting the optimal return may also change arbitrarily. Nevertheless, if the change in the profit Y^t is slow relative to the convergence of the control strategy, the Control Strategy 3.7 still traces the moving optimal return.

8. Although not considered explicitly in our notation, it is clear that it allows for updates of the profit process and of the risk function. Correlation and risk regimes may change. Such changes could be included in the profit process itself. In practice, however, the model will be updated from time to time ad hoc by replacing the current process by a new one that suits empirical data and new insights more appropriately. Such an update is also necessary if a new line of business is established. It appears realistic that updates take place before the control strategy has led the firm to the optimal RORAC. Therefore in most cases the firm will never actually reach the optimum, even if it permanently is en route to it.

9. A limitation of the approach's implementation is the fact that in a gradient allocation framework there may be spillover effects of one's segment's actions to the other segments. Although this phenomenon could be interpreted as a kind of competitive setting, since the improvement in one segment implies a higher benchmark for the other ones, this effect, which is present all the time, can hamper the acceptance and viability of such a regime. Therefore, in practical applications the integration of the entire risks of different organizational divisions needs to be uncoupled to a certain extent.

10. Finally, it is helpful to put our approach into perspective to the financial crisis of 2007–10. In this light, a concern regarding one-period models generally lies in their myopic risk governance. In principle, a long-term perception of risks would have been appropriate without doubt. However, even the myopic approach presented here may possibly have helped to prevent large risk concentration in banks, e.g. by entering large quantities of CDO tranches on the asset side of the balance sheets, because it would have set higher requirements for the yields of the signed contracts due to the second-order correction term and the implementation of risk limits.

6. Summary and Conclusion

This paper contributes to the literature on risk capital allocation by considering conditions that are required for capital allocation to be a useful tool for obtaining the optimal value of a return function of a decentralized financial firm. We regard the maximization problem as a managerial control problem and embed it into a general systems framework. Contrary to the majority of the relevant literature, we do not restrict ourselves to considering a one-period model but rather employ a discrete multi-period model. However, we derive significant results only in a specification with a stationary profit process and a one-period return function, which is of the RORAC type.

Our results are as follows: In a classical capital allocation-based RORAC framework the headquarters calculates the firm's overall risk according to its risk model once every period. Simultaneously, the headquarters also determines the overall RORAC and each segment's per-unit risk contribution. Therewith each segment can decide individually whether an additional business is profitable by calculating the marginal RORAC of the business. Yet, traditional capital allocation with the gradient approach in the marginal RORAC framework linearizes the embedded risk function, which can lead to over-expansions and over-reductions of businesses and may even yield a reduction of the overall RORAC.

The approach suggested by Theorem 3.6 is similar. However, instead of using the linear approximation of the risk function for the calculation of the marginal RORAC, the risk is adjusted by the additional quadratic risk correction term (RCT). This requires additional business to be more profitable than in the naïve capital allocation procedure. The size of this effect grows along with the expansion, implying de facto limits for every segment's expansion or reduction at every period. Now, instead of venturing into all business whose marginal RORAC exceeds the overall RORAC, only those businesses are undertaken whose marginal RORAC with RCT exceeds the overall RORAC to prevent the overall RORAC from decreasing. Based on these results we develop a control strategy that enables each segment to decide on expansions or reductions once a period. We can show that the strategy does indeed lead to the optimal RORAC.

Finally, we also address risk limiting in a static form by the headquarters, which imposes additional bounds on the expansion or reduction the segments may perform.

In the future, and as a consequence of the financial crisis of 2007–10, it seems desirable to overcome one-periodic modeling. However, any type of consideration of a hyperopic optimal strategy requires a specification of the multi-period risk measure ϱ, the general return function and the non-stationary process $(Y^t)_{t>0}$. It remains a big but promising challenge for further research to proceed in this direction.

A. Proofs

Proof of Theorem 3.4. Sufficiency: We have

$$\frac{\partial r_{Y,\rho}(u)}{\partial u_k} = (\rho_X(u) - M_k(u_k))^{-2} \left(\rho_X(u) M_k'(u_k) - M(u) \frac{\partial \rho_X(u)}{\partial u_k} \right). \tag{8}$$

If $a_k(u) = \frac{\partial \rho_X(u)}{\partial u_k}$ then $\rho_X(u)M'_k(u_k) > M(u)a_k(u)$ implies that

$$\rho_X(u)M'_k(u_k) - M(u)a_k(u) > 0. \tag{9}$$

Equation (9) in conjunction with equation (8) implies that

$$\frac{\partial r_{Y,\rho}(u)}{\partial u_k} > 0\,.$$

Hence, there is an $\epsilon > 0$ such that for all $\tau \in (0, \epsilon)$ we have

$$r_{Y,\rho}(u) < r_{Y,\rho}(u + \tau e_k).$$

Analogously (by replacing "$>$" through "$<$"), part 2. in Definition 3.3 is proven.

Necessity: The proof of the necessity follows the steps of Tasche (2004) exactly with $M(u) = m'u$. $\square$

Proof of Lemma 3.5. Clearly, $r_{Y,\rho}$ is pseudoconcave, as its numerator is concave and its denominator is convex. Therefore, each local maximum of $r_{Y,\rho}$ is also a global maximum on U (Cambini and Martein, 2009). Since $r_{Y,\rho}$ is defined on the convex set U, we have to consider two cases:

Case 1: U is compact. In this case, $r_{Y,\rho}$ either attains its global maximum in the interior of U where $\nabla r_{Y,\rho} = 0$, or on some boundary point of U.

Case 2: U is not compact. Note that since

$$\begin{aligned}\frac{\partial r_{Y,\rho}(u)}{\partial u_k} &= M'_k(u_k)(\rho_X(u) - M_k(u_k))^{-1} \\ &\quad - M_k(u_k)\Big(\frac{\partial \rho_X(u)}{\partial u_k} - M'_k(u_k)\Big)(\rho_X(u) - M_k(u_k))^{-2},\end{aligned}$$

and by assumption (c)

$$\frac{M_k(u_k)\Big(\frac{\partial \rho_X(u)}{\partial u_k} - M'_k(u_k)\Big)(\rho_X(u) - M_k(u_k))^{-2}}{M'_k(u_k)(\rho_X(u) - M_k(u_k))^{-1}} > 1$$

for u_k sufficiently large, we have

$$\frac{\partial r_{Y,\rho}(u)}{\partial u_k} < 0$$

for u_k sufficiently large. Therefore, the return $r_{Y,\rho}$ is reduced when increasing u_k, if u_k is sufficiently large. Hence, we can safely ignore all portfolios exceeding a certain u_k and thus fall back upon Case 1. $\square$

$\square$

Proof of Theorem 3.6. Equation (1) is equivalent[3] to

$$M_k(u_k + \epsilon_k)\rho_X(u) - M_k(u_k)\rho_X(u) - M(u)\epsilon_k \frac{\partial \rho_X(u)}{\partial u_k} - \tfrac{1}{2}\epsilon_k^2 M(u)\Lambda \geq 0.$$

Summing up over all $k = 1, \ldots, n$, we derive

$$M(u+\epsilon)\rho_X(u) - M(u)\rho_X(u) - M(u)\epsilon'\nabla\rho_X(u) - \tfrac{1}{2}M(u)\epsilon'\epsilon\Lambda > 0, \tag{10}$$

since $M(u) = \sum_{k=1}^n M_k(u_k)$. Clearly, as ρ_X is convex, H is positive definite and therefore $\Lambda > 0$. Moreover, since by the Rayleigh-Ritz-Theorem (Horn and Johnson, 1990) $\Lambda \geq \max_{\epsilon \neq 0} \frac{\epsilon' H(\upsilon)\epsilon}{\epsilon'\epsilon}$, the inequality

$$\tfrac{1}{2}M(u)\epsilon'\epsilon\Lambda \geq \tfrac{1}{2}M(u)\epsilon' H(\upsilon)\epsilon$$

holds for all $\upsilon \in U$. Therefore, equation (10) yields

$$M(u+\epsilon)\rho_X(u) - M(u)\rho_X(u) - M(u)\epsilon'\nabla\rho_X(u) - \tfrac{1}{2}M(u)\epsilon' H(\upsilon)\epsilon > 0,$$

which is equivalent[4] to

$$\frac{M(u+\epsilon)}{\rho_X(u) + \epsilon'\nabla\rho_X(u) + \frac{1}{2}\epsilon' H(\upsilon)\epsilon - M(u+\epsilon)} > \frac{M(u)}{\rho_X(u) - M(u)}. \tag{11}$$

Note that according to the Mean Value Theorem (Nocedal and Wright, 2006) there is some $\upsilon^* \in [u, u+\epsilon]$ such that $\rho_X(u+\epsilon) = \rho_X(u) + \epsilon'\nabla\rho_X(u) + \frac{1}{2}\epsilon' H(\upsilon^*)\epsilon$. Letting $\upsilon = \upsilon^*$ in equation (11) yields $r_{Y,\rho}(u+\epsilon) > r_{Y,\rho}(u)$. $\square$

Proof of Corollary 3.8. We refer to the Convergence Theorem A from Zangwill (1969) to show that a sequential application of Control Strategy 3.7 leads to the optimum. Clearly, the portfolio remains constant at critical points of $r_{Y,\rho}(u)$. Since $r_{Y,\rho}$ is pseudoconcave, each critical point is a global maximum (Cambini and Martein, 2009). Hence it suffices to show:

1. *Compactness*: By assumption (c) of Lemma 3.5, $\epsilon_k < 0$ for u_k sufficiently large, since the sign of ϵ_k follows the sign of $\frac{\partial r_{Y,\rho}(u)}{\partial u_K}$ (cf. proof of Theorem 3.4). Therefore, to show that the sequence of portfolios generated by Control Strategy 3.7 is bounded, we only need to show

[3] Notice that the denominator of the LHS of equation (1) must be positive: If it were negative, the numerator would also have to be negative because the whole LHS needs to be positive since $r_{Y,\rho}(u) > 0$. However, if the numerator of the LHS is negative, the denominator of the LHS is clearly positive, which is a contradiction.

[4] A similar argument to that in the footnote above shows that the denominator of the LHS of equation (11) has to be positive.

that ϵ_k is bounded if $\frac{\partial r_{Y,\rho}(u)}{\partial u_k} > 0$: Due to the concavity of $M(u)$, there holds

$$M_k(u_k + \epsilon_k) - M_k(u_k) \leq \epsilon_k M_k'(u_k).$$

Therefore, equation (1) yields

$$\frac{M_k'(u_k)}{a_k(u) + \frac{1}{2}\epsilon_k \Lambda - M_k'(u_k)} \geq r_{Y,\rho}(u).$$

Clearly, the right hand side is bounded from below, since the strategy creates an increasing sequence of return function values, and $r_{Y,\rho}(u^1) > 0$ as $M(u^1) > 0$. On the other hand, $M_k'(u_k)$ is bounded from above, and therefore ϵ_k is bounded from above.

2. *Adaption*: Follows directly from the proposition of Theorem 3.6.

3. *Closedness*: Holds since ϵ_k is chosen from a closed set.

□

Proof of Theorem 3.9. Summing up over all $k = 1, \dots, n$, equation (7) yields

$$\rho_X(u^{t-1}) + (\epsilon^t)'\nabla\rho_X(u^{t-1}) + \tfrac{1}{2}(\epsilon^t)'\epsilon^t\Lambda \leq \rho^t_{\max} + M(u^t).$$

Since $\frac{1}{2}(\epsilon^t)'H(\upsilon)\epsilon^t \leq \frac{1}{2}(\epsilon^t)'\epsilon^t\Lambda$ for all $\upsilon \in U$, there holds

$$\rho_X(u^{t-1}) + (\epsilon^t)'\nabla\rho_X(u^{t-1}) + \tfrac{1}{2}(\epsilon^t)'H(\upsilon)\epsilon^t \leq \rho^t_{\max} + M(u^t)$$

for all $\upsilon \in U$. By the Mean Value Theorem (Nocedal and Wright, 2006), the latter equation yields

$$\rho_X(u^{t-1} + \epsilon^t) \leq \rho^t_{\max} + M(u^t).$$

Noting that $u^t = u^{t-1} + \epsilon^t$ yields the assertion. □

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