Simplified stage-based modeling of multi-stage stochastic programming problems

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Overview

- Introduction: Multi-stage decision process optimization
- Multi-stage stochastic programing - Main issues
- Modeling multi-stage stochastic programming problems
- Simplified stage-based multi-stage modeling
- Modeling Examples
- Conclusion

Multi-stage Stochastic Decision Process

Discrete-time decision processes considered:

- **Sequence of decisions.** At each decision stage $t = 0, \ldots, T$ do:
  - Observe the realization of random variable ξ_t.
  - Take a decision x_t based on all observed values $\xi_0, \ldots, \xi_t$.
- **At stage T.** Sequence of decisions $x = (x_0, \ldots, x_T)$ with realizations $\xi = (\xi_0, \ldots, \xi_T)$ leads to cost $f(x, \xi)$.
- **Goal.** Find a sequence of decisions $x(\xi)$, which minimizes a functional (commonly the expectation) of the cost $f(x(\xi), \xi)$.

Multi-stage: At least one intermediary stage between root and terminal stage.

Multi-stage Stochastic Programming

$$\begin{array}{ll} \text{minimize } x: & \mathbb{F}\Big(f(x(\xi),\xi)\Big) \\ \text{subject to} & (x(\xi),\xi) \in \mathcal{X} \\ & x \in \mathcal{N} \end{array}$$

- Multi-variate, multi-stage stochastic process ξ.
- Constraint-Set $\mathcal{X}$ defining feasible (x, ξ).
- Set $\mathcal{N}$ of functions $\xi \mapsto x$, such that x_t is based on realizations up to stage t $(\xi_0, \ldots, \xi_t)$ only (non-anticipativity constraints).

Remark. The scenario tree approximation of the underlying stochastic process will inherently fulfill the non-anticipativity constraints.

Multi-stage stochastic programming - Main issues

Issue One - Modeling underlying decision problem - Multi-stage models and scenario model (tree) handling are considered to be too complex to be used in companies for real-world applications. Communication of tree-based models to non-experts is complicated.

Issue Two - Modeling underlying uncertainty - A discrete tree approximation of the underlying stochastic process has to be generated in order to numerically compute a solution. The quality of the scenario model severely affects the quality of the solution (garbage in $\rightarrow$ garbage out).

Both issues are valid since the inception of stochastic programming. However, are they properly solved?

Modeling multi-stage stochastic programming problems (1)

Most stochastic programming modeling environments summarized in:

- Wallace, S. W. and W. T. Ziemba (Eds.) *Applications of stochastic programming.* MPS/SIAM Series on Optimization, Vol. 5. SIAM. 2005.

Some recent developments were reported by:

- Lopes, L. - PhD, Northwestern 2003
- van Delft, Ch. and Vial, J.-P. - Automatica 2004
- Fourer, R. and Lopes, L. - Optimization Online 2006
- Thénié, J. et al. - Computational Management Science 2007
- Valente, C. et al. - Optimization Online 2007

Modeling multi-stage stochastic programming problems (2)

Design philosophy. Complete decoupling of scenario tree modeling and handling from the decision problem modeling process. Three-layered approach: explicit decoupling of modeling and (scenario) tree handling.

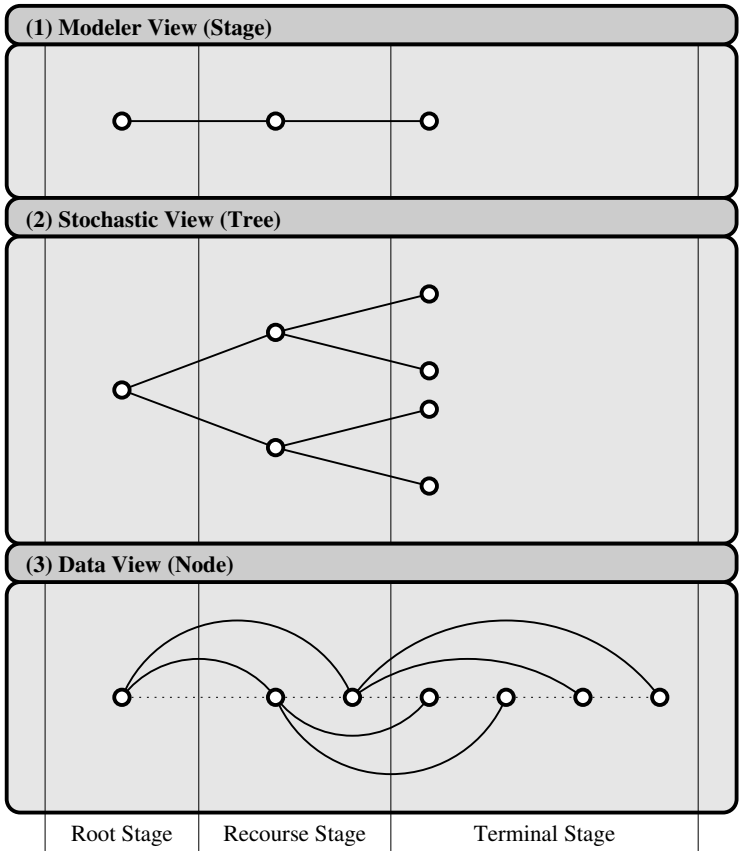
1. Decision problem layer. Decision problem modeler only concerned with actions/decisions at stages.

2. Scenario tree layer. Creating a scenario tree which optimally represents the subjective beliefs of the decision taker at each node.

3. Data layer. Data structures, how to (memory-)optimally store large trees, and access ancestor tree nodes fast, ...

Design goal. Focus on usability, and model readability.

Three-layered approach (1)



Three layered approach (2)

Scenario Tree Handling. Need for a coherent interface to handle scenario trees. Still no common standard for representing discretized stochastic processes available. Lack of commercial interest?!

Node-based vector/matrix data format of a scenario tree:

$V(n, d)$	d-dimensional value of node n
$A(n)$	ancestor node of node n
$P(n)$	probability to reach node n from its ancestor
$Z(n)$	probability of scenario terminating at node n
$T(n)$	stage of node n

Underlying concept. Node-sets $\mathcal{N}_t$, include all tree nodes of stage t. Deterministic root-node and root-stage indexed with 0.

A simple multi-stage model

Stylized multi-stage stochastic programming example from (Heitsch el al., 2006):

- Optimal purchase over time under cost uncertainty,
- uncertain prices are given by σ_t.
- Decisions x_t: amounts to be purchased at each time period t.
- Minimize expected costs such that prescribed amount a is achieved at T.

$$\begin{array}{ll} \text{minimize} & \mathbb{E}\left(\sum_{t=1}^{T} \xi_t x_t\right) \\ \text{subject to} & s_t - s_{t-1} = x_t \forall t = 2, \ldots, T \\ & s_1 = 0, s_T = a, x_t \geq 0, s_t \geq 0, \end{array}$$

where s_t is a state variable containing the amount at time t.

A simple multi-stage model MusMod - formulation

```
deterministic a: T;
stochastic V, x, s: 0..T;
stochastic nonAnticitpativity: 1..T;
stochastic constraintRootStage: 0;
stochastic constraintTerminalStage: T;

param a, V;
var x >= 0, s >= 0;

maximize objFunc: E(V * x, 0..T);
subject to nonAnticitpativity: s - s(-1) = x;
subject to constraintRootStage: s = 0;
subject to constraintTerminalStage: s = a;
```

MusMod - AMPL extension - stochastic additions

Additional keywords (for parameters, variables, and constraints)

- `deterministic` *variable-name*: *stage-set*;
- `stochastic` *variable-name*: *stage-set*;
- Stochastic parameters are defined on the underlying tree node structure.
- Deterministic parameters are definied on the stage structure, i.e. same value for all nodes in the respective stage.

Remark. Stage-sets may be single stages, ranges, or lists.

MusMod - AMPL extension - direct modeling changes

Additional functions:

- stochastic variables (recourse)
  `variable-name(recourse-depth, parameters).`
  Recourse-depth equals number of stages, commonly -1.
- expectation `E(`*stochastic-variable-name, stage-set*`).`

Possible extensions:

- quantiles `Q(`*stochastic-variable-name, stage,* α`).`
- probabilistic constraints
  `P(`*stochastic-variable-name, stage,* $\leq$`,` α`).`

MusMod - Stage-set parsing, node-set creation

Node-sets parsing & creation.

- Add one stage-set for the whole horizon ($0..T$).
- Parse all *stage-sets* $\mathcal{T}$ defined
  - direcly with keywords `stochastic` and `deterministic`, and
  - within the objective special function `E()`.
- For each *stage-set* - given one specific scenario tree - create appropriate *node-sets* containing all nodes of the respective stages.

MusMod - Conversion example - Stage and node-sets

Example: Simple three-stage ($t = 0, 1, 2$) binary tree, (uni-variate) starting value: 10. Up 1 with $p = 0.6$ and down 1 with $p = 0.4$, i.e.

n	0	1	2	3	4	5	6
V[n]	10	11	9	12	10	10	8
A[n]		0	0	1	1	2	2
T[n]		1	1	2	2	2	2
P[n]	1	0.6	0.4	0.6	0.4	0.6	0.4
Z[n]	1	0.6	0.4	0.36	0.24	0.24	0.16

Node- and stage-sets: Using the above inventory example:

Model	Node-Set	Stages	Nodes
(0..T)	0	0 1 2	0 1 2 3 4 5 6
0	1	0	0
T	2	2	3 4 5 6
1..T	3	1 2	1 2 3 4 5 6

MusMod - Conversion example - Variables and Parameters

1. Replace stochastic parameters and variables by a node-set definition, and
2. replace deterministic parameters and variables by stage-set definitions.

```
deterministic a: T;
stochastic V, x, s: 0..T;
param a, V;
var x >= 0, s >= 0;

param a[stageSet2], V[nodeSet0];
var x[nodeSet0] >= 0, s[nodeSet0] >= 0;
```

MusMod - Conversion example - Constraints

For each stochastic constraint add as many deterministic equivalent constraints as nodes in the respective node set, i.e.

```
stochastic constraintRootStage: 0;
subject to constraintRootStage: s = 0;

subject to constraintRootStage: s[0] = 0;
```

MusMod - Conversion example - Recourse constraints

Deterministic parameters in stochastic constraints make use of the stage mapping information T[n]:

```
stochastic constraintTerminalStage: T;
subject to constraintTerminalStage: s = a;

subject to constraintRootStage: s[3] = a[T[3]];
subject to constraintRootStage: s[4] = a[T[4]];
subject to constraintRootStage: s[5] = a[T[5]];
subject to constraintRootStage: s[6] = a[T[6]];
```

MusMod - Conversion example - Recourse constraints

Recourse constraints make use of the anchestor information A[n]. Higher depths are integrated recursively.

```
stochastic nonAnticitpativity: 1..T;
subject to nonAnticitpativity: s - s(-1) = x;

subject to nonAnticitpativity: s[1] - s[A[1]] = x[1];
subject to nonAnticitpativity: s[2] - s[A[2]] = x[2];
...
subject to nonAnticitpativity: s[6] - s[A[6]] = x[6];
```

Further advantage: No explicit tree formulation in the model anymore.

MusMod - Conversion example - Objective function

Objective function replacements based replacing `E()` by sums using the stage probabilities Z[n]:

```
maximize objFunc: E(V * x, 0..T);
```

```
maximize objFunc:
    ( sum{n in nodeSet0}: Z[n] * ( V[n] * x[n] ) );
```

Multi-stage stochastic Asset Liability Management

$$
\begin{array}{lll}
\text{maximize}_x & \sum_{n\in\mathcal{N}_{(T)}} Z(n)w_n + \kappa(\gamma - \sum_{n\in\mathcal{N}_{(T)}} \frac{Z(n)z_n}{1-\alpha}) & \\
\text{subject to} & \sum_{a\in\mathcal{A}} x_{n,a} \leq \beta = w_n & \forall n \in \mathcal{N}_{(0)} \\
& \forall a \in \mathcal{A} : x_{n,a} \leq V(n,a)x_{A(n),a} + b_{n,a} - s_{n,a} & \forall n \in \mathcal{N}_{(1..T-1)} \\
& \sum_{a\in\mathcal{A}} b_{n,a} \leq \sum_{a\in\mathcal{A}} s_{n,a} & \forall n \in \mathcal{N}_{(1..T-1)} \\
& w_n = \sum_{a\in\mathcal{A}} x_{n,a} + f_{S(n)} & \forall n \in \mathcal{N}_{(1..T-1)} \\
& w_n = (\sum_{a\in\mathcal{A}} V(n,a)x_{A(n),a}) + f_{S(n)} & \forall n \in \mathcal{N}_{(T)} \\
& z_n \geq \gamma - w_n & \forall n \in \mathcal{N}_{(T)}
\end{array}
$$

$x_{n,a}$	amount of money held in each asset a
$b_{n,a}$, $s_{n,a}$	amount bought and sold
w_n	current wealth
f_t	(deterministic) liabilities
β	initial budget
κ	risk aversion parameter
α	AVaR quantile level
z_n, γ	auxiliary variables (AVaR)

ALM example - Stage-based notation

$$
\begin{array}{lll}
\text{maximize}_x & \mathbb{E}(w_T) + \kappa(\gamma - \mathbb{E}(\frac{z_T}{1-\alpha})) & \\
\text{subject to} & \sum_{a\in\mathcal{A}} x_a \leq \beta = w & (t = 0) \\
 & \forall a \in \mathcal{A} : x_a \leq V_a x_a^{(-1)} + b_a - s_a & (t = 1, \ldots, T-1) \\
 & \sum_{a\in\mathcal{A}} b_a \leq \sum_{a\in\mathcal{A}} s_a & (t = 1, \ldots, T-1) \\
 & w = \sum_{a\in\mathcal{A}} x_a + f & (t = 1, \ldots, T-1) \\
 & w = \sum_{a\in\mathcal{A}} V_a x_a^{(-1)} + f & (t = T) \\
 & z \geq \gamma - w & (t = T)
\end{array}
$$

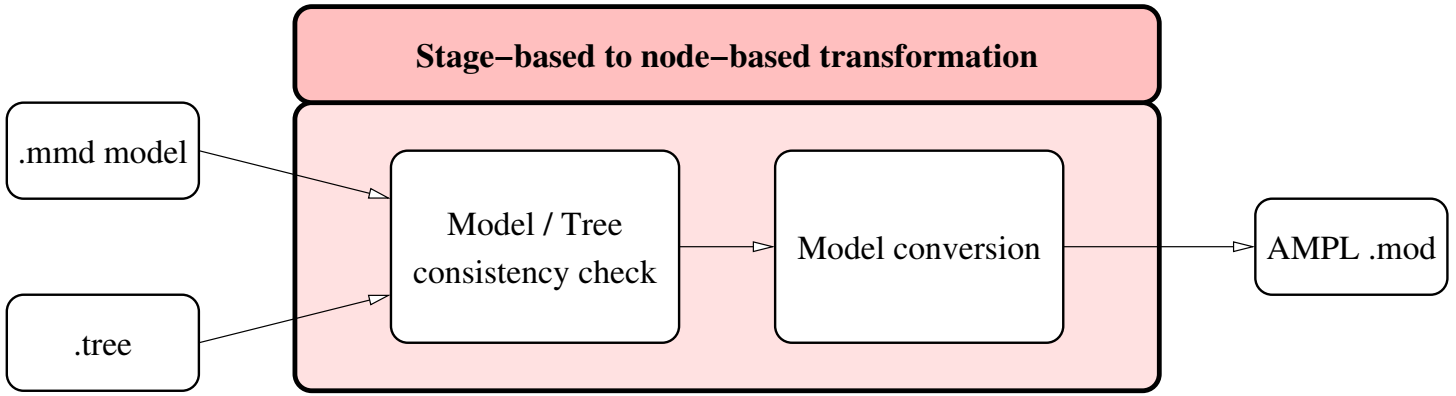
ALM example - Simplified notation (AMPL extension, 1)

```
param alpha; param beta; param kappa;
param assets; set ASSET := 1 ..  assets;
param V{ASSET};


var x{ASSET} >= 0, b{ASSET} >= 0, s{ASSET} >= 0, w >= 0;
var f; var g; var z >= 0;


maximize objFunc:  E(wealth, T) + kappa * ( g - ( E(z / ( 1 - alpha ), T) ) );


subject to cInitBudget:  ( sum{a in ASSET} x[a] ) <= beta;

subject to cInitWealth:  ( sum{a in ASSET} x[a] ) == w;

subject to cTradeStages{a in ASSET}:  x[a] <= ( V[a] * x(-1, a) ) + b[a] - s[a];

subject to cBuySell:  ( sum{a in ASSET} b[a] ) <= ( sum{a in ASSET} s[a] );

subject to cNodeWealth:  w <= ( sum{a in ASSET} x[a] ) + f;

subject to cFinalStageWealth:  w <= ( sum{a in ASSET} V[a] * x(-1, a) ) + f;

subject to cAVaR: z >= g - w;
```

ALM example - Simplified notation (AMPL extension, 2)

```
deterministic f:  1..T;


stochastic cInitBudget, cInitWealth:  0;
stochastic x, w:  0..T;
stochastic b, s:  1..T-1;
stochastic V, cTradeStages, cBuySell, cNodeWealth:  1..T;
stochastic z, cAVaR, cFinalStageWealth:  T;
```

MusMod - Workflow



Model / Tree consistency check - Examples:

- Number of stages smaller than highest recourse-depth.
- Equal/Odd number of stages required by model.

Conclusion, Contact & More Information

Conclusion

- Multi-stage modeling completely decoupled from the scenario tree handling (stage-based modeling view).
- AMPL extension implemented as Web application.
- Framework for teaching and selling multi-stage models.
- Guideline for designing stochastic programming (XML) formats.

Contact & More Information

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