

# **Enhancing insurer value through reinsurance, dividends and capital optimization: an expected utility approach**

Topic 1: Risk Management of an Insurance Enterprise

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## **Abstract**

The paper investigates the existence of risk management incentives in insurance in the presence of insolvency cost using an expected utility approach. The insurer's objective is to maximize shareholder value under a solvency constraint imposed by a regulatory authority. In this paper we show that the maximization of shareholder value under solvency restrictions is approximately equivalent to the maximization of shareholder value using utility approach with a special isoelastic utility function. Using this isoelastic utility function we maximize the expected present value of utility of future dividends by three uncontrolled variables: dividend rate, leverage ratio and retention level of quota share proportional reinsurance, for the surplus which follows a geometric Brownian motion.

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# Enhancing insurer value through reinsurance, dividends and capital optimization: an expected utility approach

Yuriy Krvavych \*

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## 1 Introduction

In finance the existence of corporate risk management is due to imperfections in financial markets. One of the main imperfections is associated with the cost of corporate risk that firms assume. Costly corporate risk creates a set of frictional costs and thereby decreases corporate value. Financial corporations manage their risk to reduce the expected value of frictional costs and enhance shareholder value, and do so using a wide variety of tools. This study considers an insurance company as a special type of financial corporation leveraged by risky debt, and investigates the existence of risk management incentives in insurance in the presence of frictional costs such as insolvency cost. An insurer risk management problem is investigated here in a dynamic setting under solvency constraints using an expected utility approach, where the main objective is to find optimal reinsurance, dividend payments and capital structure under which the expected utility value of discounted dividend payments is maximized.

Many studies of reinsurance optimization in the classical actuarial literature assume that the insurer objective is to minimize its ruin probability. This assumption has limitations from the point of view of the modern theory of integrated risk management for an insurance company, since it focuses on risk minimization only, without any explicit regard to the company's economic value. Other more recent studies (e.g. see Paulsen (2003 [4], Gerber and Shiu (2004 [1],

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Taksar (2000 [7]) and references therein) that maximize the expected value of discounted future dividends (company's value) by dividends and/or reinsurance do not take into consideration frictional costs such as corporate tax, costs of financial distress and/or insolvency cost.

In Krvavych and Sherris (2006 [2]), authors studied the demand for reinsurance in the presence of corporate tax and costs of financial distress. They considered a constrained problem of maximization of future consumptions subject to the constraint on risk which is measured by the probability of insolvency. It was clearly explained there why the main insurer's objective is to maximize shareholder value, and why without any constraints on underwriting risk the insurer would behave as a risk-neutral agent. As we know insurers do operate in a regulatory environment with risk constraints. This is because the insurer's intention to maximize its shareholder value usually shifts the insurer to the state with a higher leverage level, which is achievable by issuing more insurance debt (or by assuming more underwriting risk). But at the same time assuming more underwriting risk will require the insurer to be more financially reliable. And this financial reliability (strength) is controlled by a regulator via imposing some specific constraints on underwriting risk, such as for example maximally acceptable levels of ruin (insolvency) probability or volatility. Regulators impose solvency requirements in order to protect insureds who accumulate their insurance credit exposure with one or few insurers and thus are sensitive to the insolvency risk.

Continuing this logic further we argue here in this paper that the insurer should consider some preference value as a function of its wealth, which is associated with the fixed level of insolvency risk. This preference value should be higher for a larger monetary amount of wealth, and that the gain of the preference value resulting from the same monetary gain is a decreasing function of initial wealth. In other words the preference value is an increasing function of wealth and its marginal function is a decreasing function of wealth, i.e. the preference value represents a concave utility function of the insurer (i.e. the insurer is risk-averse). It is worth noticing that this analysis is consistent with the fact that the mean-variance criterion can be reconciled with the expected utility approach using a quadratic utility function (see eg Markowitz (1959 [3])).

In this paper we will show that the maximization of shareholder value under solvency restrictions is approximately equivalent to the maximization of shareholder value using utility approach with a specific isoelastic utility function.

Using this isoelastic utility function we maximize the expected present value of utility of future dividends by three uncontrolled variables: dividend rate (coefficient of proportionality to the surplus), leverage ratio and retention level of quota share proportional reinsurance, for the surplus which follows a geometric Brownian motion.

## **2 Insurer preference ordering under solvency constraints: a derivation of the corresponding utility function**

Consider a discrete time ruin (Cramér-Lundberg) model of the surplus for which the following equalities hold

$$S_t = S_{t-1} + c - (L_t - L_{t-1}) = L_{t-1} + c - \Delta L_{t-1}, \quad t = 1, 2, \dots$$

where the annual total claims  $\Delta L_t$ ,  $t = 1, 2, \dots$  are independent and identically compound Poisson distributed, say  $\Delta L_t \sim L$ ; the initial surplus  $S_0$  is equal  $x$  and  $c$  is the annual premium. There exists a minimum annual premium  $c = P^-$  such that for the fixed initial surplus  $x$  the insurer's ruin probability attains its maximally acceptable level  $\varepsilon$  defined by a regulator. We may assume that this level can be approximated by Lunberg upper bound  $e^{-\rho x}$ , where  $\rho$  denotes the adjustment coefficient and is thus the root of the equation  $e^{\rho c} = \mathbb{E}[e^{\rho L}]$ . Hence,

$$P^- = \frac{1}{\rho} \ln(\mathbb{E}[e^{\rho L}]),$$

where  $\rho = \frac{|\ln \varepsilon|}{x}$ . From practice we now that  $0 < \rho \lll 1$  and therefore using Taylor's expansion we get

$$\begin{aligned} \ln(\mathbb{E}[e^{\rho L}]) &= \ln\left(1 + \mathbb{E}[L]\rho + \frac{1}{2}\mathbb{E}[L^2]\rho^2 + o(\rho^2)\right) \\ &= \mathbb{E}[L]\rho + \frac{1}{2}\mathbb{E}[L^2]\rho^2 + o(\rho^2) - \frac{1}{2}\left(\mathbb{E}[L]\rho + \frac{1}{2}\mathbb{E}[L^2]\rho^2 + o(\rho^2)\right)^2 + o(\rho^2) \\ &= \mathbb{E}[L]\rho + \frac{1}{2}\mathbb{E}[L^2]\rho^2 + o(\rho^2) - \frac{1}{2}(\mathbb{E}^2[L]\rho^2 + o(\rho^2)) + o(\rho^2) \\ &= \mathbb{E}[L]\rho + \frac{1}{2}\text{Var}[L]\rho^2 + o(\rho^2). \end{aligned}$$

From here we conclude that the minimal insurer's annual premium rate is

$$P^- = \mathbb{E}[L] + \frac{1}{2}\text{Var}[L]\rho + o(\rho)$$

and hence it can be approximated by

$$P^- \approx \mathbb{E}[L] + \frac{|\ln \varepsilon|}{2x} \text{Var}[L].$$

On the other hand the same minimal insurer's annual rate  $P^-$  can be calculated using expected utility approach for the period from 0 to 1. Let the unknown insurer's utility function is  $U$ , then  $P^-$  must satisfy the following equation

$$\mathbb{E}[U(x + P^- - L)] = U(x). \tag{1}$$

The fact that the insurer is risk-averse implies that the marginal utility  $U'$  is positive decreasing function. Therefore, the latter equation determines  $P^-$  uniquely, but has no explicit solution in general. However, using the parametrization of  $L : L(t) = \mathbb{E}[L] + tY$ , with  $\mathbb{E}[Y] = 0$ , and expansion of  $P^-$  in powers of  $t$ , i.e.  $P^-(t) = \sum_{k=0}^{\infty} p_k t^k$ , we can approximate  $P^-$  by the mean of  $L$  plus variance of  $L$  with coefficient of proportionality determined by Arrow-Pratt absolute risk aversion coefficient  $r(x) = -\frac{U''(x)}{U'(x)}$ .

In order to find the first three terms in series of  $P^-(t)$  we first set  $t = 0$  in (1) and obtain

$$U(x) = U(x + P^-(0) - L(0)) = U(x + p_0 - \mathbb{E}[L]),$$

which is equivalent to  $p_0 = \mathbb{E}[L]$ .

If we differentiate both sides of (1) with respect to  $t$  and then set  $t = 0$ , we get

$$\mathbb{E}[(p_1 - Y) U'(x)] = 0,$$

or  $p_1 U'(x) = 0$ , which gives us  $p_1 = 0$ .

Finally, by differentiating twice both sides of (1) and setting  $t = 0$ , we get

$$\begin{aligned} 0 &= \mathbb{E}[2p_2 U'(x)] + \mathbb{E}[(p_1 - Y)^2 U''(x)] \\ &= 2p_2 U'(x) + \text{Var}[Y] U''(x), \end{aligned}$$

or equivalently  $p_2 = \frac{1}{2} r(x) \text{Var}[Y]$ .

So, for small  $t$  we can write

$$\begin{aligned} P^-(t) &= \mathbb{E}[L(t)] + \frac{1}{2} r(x) \text{Var}[Y] t^2 + o(t^2) \\ &= \mathbb{E}[L(t)] + \frac{1}{2} r(x) \text{Var}[L(t)] + o(t^2) \end{aligned}$$

and thus we get another approximation of  $P^-$

$$P^- \approx \mathbb{E}[L] + \frac{1}{2} r(x) \text{Var}[L].$$

Comparing this with the first approximation of  $P^-$  we get

$$r(x) = \frac{|\ln \varepsilon|}{x},$$

which means that the insurer's utility function must satisfy the following ODE

$$U''(x) + \frac{|\ln \varepsilon|}{x} U'(x) = 0.$$

This ODE gives us the solution  $U(x) = \frac{x^{1-m}}{1-m}$  with  $m = |\ln \varepsilon| > 0$ . Note that  $m \neq 1$  for all reasonable values of  $\varepsilon < 10\%$ .

So, we conclude that the corresponding insurer's utility function has the constant relative risk aversion (CRRA) coefficient  $0 < m \neq 1$  and thus this utility function is isoelastic.

### 3 Maximization of shareholder value using utility approach

#### 3.1 Dynamic model of the insurer's surplus

We consider the diffusion model of insurer's surplus formulated by Powers in (1995 [5]). In practice insurance firms receive insurance premiums and must establish liability accounts for the loss (outstanding claims) reserve and the unearned premium reserves (unexpired risk reserve). The unexpired risk reserve can be used to fund a surrender value to the policyholder in the case of withdrawal from the insurance policy. The loss reserve is the reserve to pay anticipated losses, including incurred but not reported (IBNR) losses and outstanding amounts (open case estimates) of reported losses. These losses create the company's main liabilities. An insurance company's assets consist of its invested funds arising from premiums and current value of surplus, that generate an investment income. The underwriting profit can be estimated by deducting the current value of incurred losses (reported losses plus IBNR) and earned expenses from earned premiums. The insurance company's expenses may consist of acquisition costs that are approximately proportional to premiums and administrative costs (claims management expenses) that are approximately proportional to current incurred losses. On the one hand the insurance liabilities (reserves) are increased by instantaneous policy writing and at the same time are decreased by the instantaneous loss payment outflow. On the other hand, assets are increased by instantaneous earned premiums and the investment income inflow but are decreased by instantaneous loss payments.

We will use the following notation:

- $S_t$  denotes the surplus of the insurance company at time  $t$ ;
- $L_t$  denotes the expected current loss reserves at time  $t$ ;
- $dP_t$  denotes the instantaneous earned premium inflow at time  $t$ .

Let  $\pi_L$  be the underwriting profit and expense loading charged by the insurer, expressed as a proportion of expected losses, and

$$dP_t = (1 + \pi_L)L_t dt,$$

that is  $1 + \pi_L$  is the gross insurance premium rate. Then the instantaneous underwriting profit at time  $t$  follows the diffusion process

$$\begin{aligned} d\Pi_t &= dP_t - L_t dt - \varepsilon_L L_t dt - \varepsilon_P dP_t + s_L dW_t^L \\ &= (\pi_L(1 - \varepsilon_P) - (\varepsilon_L + \varepsilon_P)) L_t dt + s_L dW_t^L, \end{aligned}$$

where  $\varepsilon_L > 0$  and  $\varepsilon_P > 0$  denote, respectively, the expense ratio for administrative expenses that are proportional to expected losses and the expense ratio for acquisition expenses that are proportional to premiums; the volatility of this diffusion is assumed to be  $s_L = \sigma_L L_t$ .

By definition the insurer's assets are equal to the insurance liabilities (loss reserves) plus surplus, and thus investment income is generated by investing these assets in the capital market. It is assumed that the total investment return follows a geometric Brownian motion with drift  $r_I$  and volatility  $\sigma_I$ , i.e. the instantaneous investment income inflow at time  $t$  is

$$dI_t = r_I(L_t + S_t)dt + \sigma_I(L_t + S_t)dW_t^I.$$

Denote by  $\lambda_t = \frac{L_t}{S_t}$  the current leverage ratio at time  $t$ . For the sake of simplicity and ability to illustrate results we assume that  $W^I$  and  $W^L$  are independent, the insurer can control its leverage ratio over the period to keep it constant at level  $\lambda$ . Now, taking into account the fact that the instantaneous insurer's surplus at time  $t$ ,  $dS_t$ , is the sum of instantaneous values of investment income and underwriting profit (i.e.  $d(I_t + \Pi_t)$ ) we can write the diffusion process of the insurer's surplus as

$$\begin{aligned} dS_t &= d(I_t + \Pi_t) = (\pi_L(1 - \varepsilon_P) - (\varepsilon_L + \varepsilon_P)) L_t dt \\ &\quad + r_I(1 + \lambda)S_t dt + \sigma_L L_t dW_t^L + \sigma_I(1 + \lambda)S_t dW_t^I \\ &= \mu S_t dt + \sigma S_t dW_t, \end{aligned} \tag{2}$$

where  $\mu = \lambda(\pi_L(1 - \varepsilon_P) - (\varepsilon_L + \varepsilon_P) + r_I) + r_I$ ,  $\sigma = \sqrt{\lambda^2 \sigma_L^2 + (\lambda + 1)^2 \sigma_I^2}$ , and  $W$  is a standard Brownian motion associated with the surplus.

In the case of taking quota share proportional reinsurance with the cedent's retention level  $\alpha \in (0, 1)$  the drift and diffusion of  $S$  in (3) are respectively equal  $\mu = \alpha\lambda[\pi_L(1 - \varepsilon_P) - (\varepsilon_L + \varepsilon_P) + r_I - p_{re}(1 - \alpha)] + r_I$  and  $\sigma = \sqrt{\alpha^2 \lambda^2 \sigma_L^2 + (\alpha\lambda + 1)^2 \sigma_I^2}$ , where  $p_{re}$  is the reinsurance premium rate.

Now, taking into account the dividend payments, dividend rate  $d_t$  of which is proportional to the surplus with constant coefficient of proportionality (i.e.  $d_t = \delta S_t$ ), we obtain the *reflected Itô diffusion process* of the company's surplus

$$dS_t = \mu S_t dt + \sigma S_t dW_t - dD_t = (\mu - \delta)S_t dt + \sigma S_t dW_t, \tag{3}$$

where  $D_t$  denote the cumulative amount of dividend paid-out up to time  $t$  and is equal to  $D_t = \int_0^t d_t dt = \delta \int_0^t S_t dt$ .

### 3.2 Maximization of shareholder value using utility approach

Let  $x^* < S_0 = x$  denote the minimal capitalization level (regulatory surplus or risk based capital) at which the insurer is considered financially solvent by an insurance regulator, and the insurer will be announced insolvent at time  $\tau = \inf \{t : S_t = x^*\}$ . Let there also be an insolvency (bankruptcy) cost  $B$ . From the regulator's point of view, the insolvency cost may be defined to include both policyholders' and shareholder losses associated with the insolvency event, however, it may also include payments made to third parties, as well as legal fees and other costs (see Rajani (2002 [6])). At the time of insolvency  $\tau$  the residual surplus net of insolvency cost is distributed among shareholders as terminal dividends.

Here we define the value function (shareholder value) as the expected present value of future dividend payments up to bankruptcy (or ruin) by

$$V(x) = V(x; a, \lambda, \delta) = \mathbb{E} \left[ \int_0^\tau e^{-\gamma s} U(d_s) ds + e^{-\gamma \tau} U(B) \right]$$

with the boundary condition  $V(x^*) = U(B)$  for  $U(x) = \frac{x^{1-m}}{1-m}$  with  $m = |\ln \varepsilon| > 0$ , the force of interest  $\gamma$  and where the dynamics of the surplus are defined by the diffusion model (3). The important property of  $V$  that determines the second boundary condition is that  $V$  is bounded.

The corresponding Hamilton-Jacobi-Bellman (HJB) equation which the value function must satisfy is

$$\frac{1}{2} \sigma^2 x^2 V''(x) + (\mu - \delta) x V'(x) - \gamma V(x) + U(\delta x) = 0, \quad x \geq x^*$$

subject to the boundary condition  $V(x^*) = U(B)$ . One can notice that the latter HJB equation is a Cauchy-Euler second order ODE, which can be solved using the transformation of the variable  $x = e^y$ . Doing so, we get

$$\frac{1}{2} \sigma^2 V''_{yy} + \left( \mu - \delta - \frac{1}{2} \sigma^2 \right) V'_y - \gamma V = -\frac{(\delta e^y)^{1-m}}{1-m}.$$

The general solution to the latter ODE is

$$V(y; a, \lambda, \delta) = C_- e^{-\theta_- y} + C_+ e^{-\theta_+ y} + \widehat{V}(y; a, \lambda, \delta),$$

where  $\theta_\pm = \frac{-(\mu - \delta - \frac{1}{2} \sigma^2) \pm \sqrt{(\mu - \delta - \frac{1}{2} \sigma^2)^2 + 2\gamma \sigma^2}}{\sigma^2}$ , and  $\widehat{V}(y; a, \lambda, \delta)$  is the partial solution. By introducing the partial solution in the form  $\widehat{V}(y; a, \lambda, \delta) = g(a, \lambda, \delta) e^{(1-m)y}$  and substituting it to the ODE we get

$$\begin{aligned} \frac{1}{2} g(a, \lambda, \delta) \sigma^2 (1-m)^2 e^{(1-m)y} + g(a, \lambda, \delta) \left( \mu - \delta - \frac{1}{2} \sigma^2 \right) (1-m) e^{(1-m)y} \\ - \gamma g(a, \lambda, \delta) e^{(1-m)y} = -\frac{(\delta e^y)^{1-m}}{1-m}, \end{aligned}$$

or equivalently

$$g(a, \lambda, \delta) = -\frac{\delta^{1-m}}{1-m} \left( \frac{1}{2} \sigma^2 (1-m)^2 + \left( \mu - \delta - \frac{1}{2} \sigma^2 \right) (1-m) - \gamma \right)^{-1}.$$

It should be noticed that  $\theta_- < 0$ ,  $\theta_+ > 0$  and  $1-m < 0$  for  $\varepsilon < 10\%$ . Therefore, when  $x \rightarrow \infty$  and thus when  $y \rightarrow \infty$  the term  $C_- e^{-\theta_- y}$  is unbounded, implying that  $C_- = 0$ .

Hence, putting  $\theta = \theta_+$  the solution to the HJB equation becomes

$$V(x; a, \lambda, \delta) = C_+ x^{-\theta} + g(a, \lambda, \delta) x^{1-m}.$$



Finally, imposing the first boundary condition,  $V(x^*) = \frac{B^{1-m}}{1-m}$ , gives us

$$C_+ = \frac{B^{1-m}}{1-m} (x^*)^\theta - g(a, \lambda, \delta) (x^*)^{\theta+1-m},$$

and thus

$$V(x; a, \lambda, \delta) = \left[ \frac{B^{1-m}}{1-m} - g(a, \lambda, \delta) (x^*)^{1-m} \right] \left( \frac{x}{x^*} \right)^{-\theta} + g(a, \lambda, \delta) x^{1-m}.$$

Now to find optimal uncontrolled variables such as retention level of quota share proportional reinsurance, leverage ratio and dividend rate one maximizes the value function  $V$  to obtain

$$(a^*, \lambda^*, \delta^*) = \arg \max_{a \in (0,1); \lambda > 0; \delta \in (0,1)} V(x; a, \lambda, \delta).$$

## 4 Conclusion

In this article, we have shown that the maximization of shareholder value under solvency constraints is equivalent to the maximization of an isoelastic utility function of shareholder value. Analyzing the value function, i.e. the utility function of shareholder value, one can notice that the optimal leverage, reinsurance and dividend rate are, in particular, dependent on the insolvency cost. It is well-known that in corporate financial theory, one possible reason for the existence of an optimal leverage is the bankruptcy cost. Likewise, in the insurance model of maximization of shareholder value considered above, the insolvency cost can be a factor that leads to an optimal leverage and especially to an optimal reinsurance.

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