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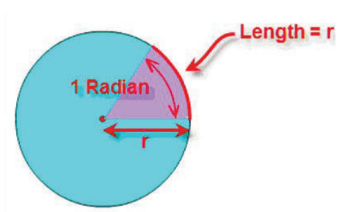
__Please note:__ All attempts have been made to ensure the accuracy and reliability of the information provided in this document.

Trigonometry 1 – Hints & Tips

- Read the question carefully.
- Know the vocabulary.
- **Draw a diagram every time.**
- Label all diagrams.
- Check the mode on your calculator – radians or degrees.
- Check for the format of your answer; e.g. in terms of π.
- Include the units.
- Know how to round your answer.
- Know when to round your answer.
- Do not round early.

A radian is the measure of the angle at the centre of a circle subtended by an arc equal in length to the radius.



Sine, Cosine and Tangent are well known but are given on page 16 of your Formulae book – along with Pythagoras' Theorem.

Cosecant, Secant and Cotangent are the reciprocals of these and are given on page 13.

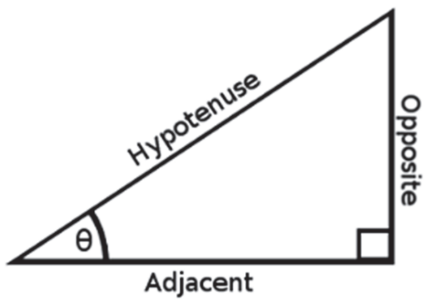
Handy way to remember the ratios:

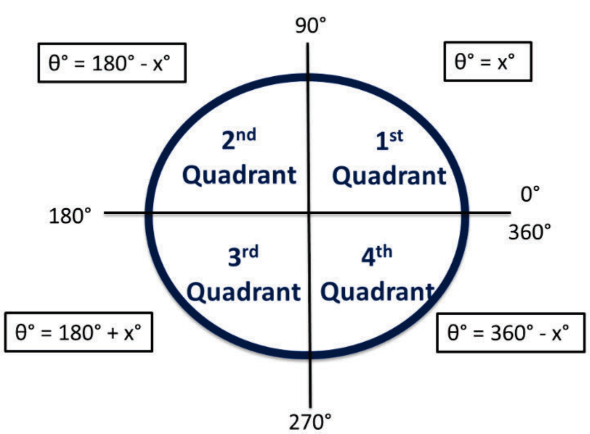
$$\textbf{Sin}\, x = \frac{\textbf{\textit{O}}pposite}{\textbf{\textit{H}}ypotenuse} \quad \textbf{Cos}\, x = \frac{\textbf{\textit{A}}djacent}{\textbf{\textit{H}}ypotenuse} \quad \textbf{\textit{T}}an\, x = \frac{\textbf{\textit{O}}pposite}{\textbf{\textit{A}}djacent}$$

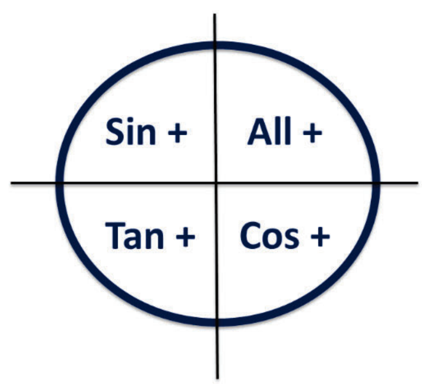
SOH **CAH** **TOA**

Silly **O**ld **H**arry **C**aught **A** **H**erring **T**rawling **O**ff **A**merica

Sides of a right angle triangle:







Refer to page 13 in your log tables.

Trigonometry Exam Tips:

- Draw a diagram every time.
- Label all diagrams.
- Check the mode on your calculator – radians or degrees.
- Know how and when to round your answer.

Important Formulae

1. Formulae for sin, cos and tan
2. Pythagoras' Theorem
3. Cosine Rule
4. Sine Rule
5. Area of Triangle
6. Length of Arc
7. Area of section of circle
8. Formula for radians to degrees

Important Formulae

1. Formulae for sin, cos and tan

$\sin\theta = \frac{opp}{hyp}$; $\cos\theta = \frac{adj}{hyp}$; $\tan\theta = \frac{opp}{adj}$

2. Pythagoras Theorem $hyp^2 = opp^2 + adj^2$

3. Cosine Rule $a^2 = b^2 + c^2 - 2bcCosA$

4. Sine Rule $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ *or* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

5. Area of Triangle $= \frac{1}{2}ab\sin C$

Important Formulae

6. Length of Arc $l = r\theta$

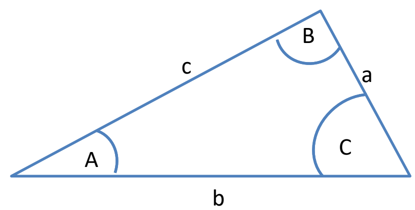
7. Area of section of circle Area $= \frac{1}{2}r^2\theta$

8. Formula for radians to degrees $\pi\ radians = 180°$

Log tables – Pages 13, 14, 15 and 16

Labelling in sine and cosine rules and area formula

- a, b, c are the lengths of the sides
- A, B, C are the angles
- A opposite a
- B opposite b
- C opposite c



Using $\sin^{-1}$ and $\cos^{-1}$ to solve equations

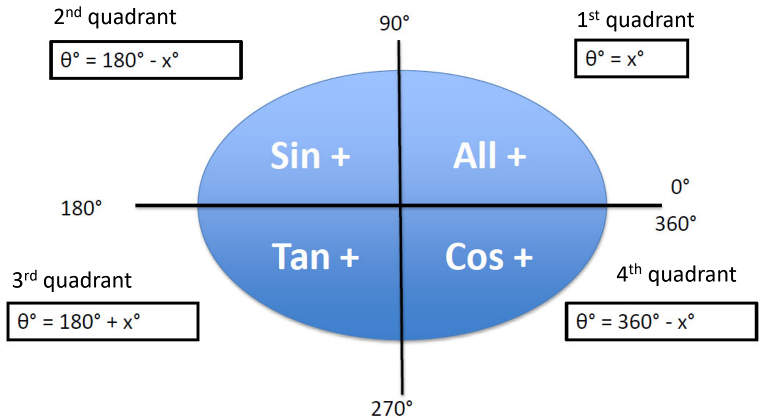
In general, an equation like cos(x) = 0.5 has two solutions.
In this case the solutions are x=60° and x= 300°.

<u>Procedure for getting both solutions:</u>

1. Get reference angle using $\cos^{-1}$ $\cos^{-1} 0.5 = 60°$
2. Check which quadrant the other solution could be in.
   cos is positive in 1st and 4th quadrant
3. Apply the correct formula for that quadrant.
   in 4th quadrant the formula is θ = 360° - reference angle
   so the solutions are x = 60° and x = 300°

See next slide for details

Using $\sin^{-1}$ and $\cos^{-1}$ to solve equations



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Coordinate Geometry: The Line – Hints & Tips

General Hints and Tips

1 **Always draw diagrams.** This is useful in every question, but it is particularly helpful with questions relating to the circle or more difficult questions.

2 Make sure you **know which formulae are in the tables,** and where in the tables they are. Formulae in the tables:
- **Slope of a line** $\frac{(y2-y1)}{(x2-x1)}$
- **Distance between 2 points** $\sqrt{(x2-x1)^2+(y2-y1)^2}$
- **Midpoint formula** $\left(\frac{x1+x2}{2},\frac{y1+y2}{2}\right)$
- **Equation of a line (2 different formats)** $(y-y1)=m(x-x1)$
- **Area of a triangle with one point at the origin** $(½|x1y2-x2y1|)$
- **Point dividing a line segment in the ratio a:b**
-
- **To find the angle between 2 lines:** $Tan\theta=\pm(m1-m2)/(1+m1.m2)$
- **Perpendicular distance from a point to a line** $\frac{|ax1+by1+c|}{\sqrt{a^2+b^2}}$

3 **Learn other formulae** off by heart.

The Line

1 To get an **equation of a line** you always need 2 things:
- **A point**
- **A slope**

Once you have these, use the formula $y - y_1 = m(x - x_1)$

2 To check if a point is on a line, substitute it into the equation.
If the answer = 0, then the point is on the line, otherwise it is not.

3 To plot a line, you need two points on the line.
An easy way to find points on a line is:
Let x = 0, solve for y. This will give you a point (0,y)
Let y = 0, solve for x. This will give you a point (x,0)
Use these two points to plot the line.

4 If a line **intersects the x-axis**, then **y = 0** at that point.
If a line **intersects the y-axis**, then **x = 0** at that point.

5 Use simultaneous equations to find the point of intersection between 2 lines.

6 If lines are **parallel**, their **slopes are equal**.
If lines are **perpendicular**, then multiplying their slopes together equals -1 (**$m_1.m_2 = -1$**)
An example - you want the slope of a line and are told it is perpendicular to another line with slope 2/3
Turn it upside down and change the sign of it. So in this case, the slope of the line you want is -3/2

8 To use the area of a triangle formula ($½|x_1y_2 - x_2y_1|$) one of the points needs to be (0,0).
If you are looking for the area of a triangle, where no points are at the origin (0,0), use translations to bring one of the points to (0,0) and then use the formula as normal.
Alternatively, you can use the area = ½ base x perpendicular height formula.

9 If 3 or more points lie on the same line, they are said to be collinear.
To check if 3 points (e.g. a, b, c) are collinear, see what the slopes of |ab| and |bc| are.
If they are the same, then the points are collinear, otherwise they are not.
An alternative way of doing this is to calculate the area of the triangle using the 3 points.
If the area = 0, then the points are collinear, otherwise they are not.

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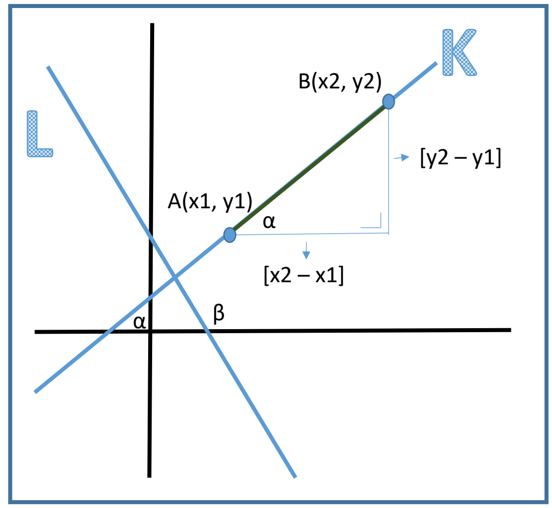
The Basics – Pages 18 & 19 of "Formulae and Tables"

Definitions of slope (m)
$m = \frac{dy}{dx}$
$m = \frac{(y2 - y1)}{(x2 - x1)}$
$m = \tan \alpha = \frac{opposite}{adjacent}$
m = "the rise over the run"
m, when line written y = mx + c L1: 4x – 2y + 7 = 0 2y = 4x + 7 y = 2x + 7/2 m = 2

When you have a point (x1, y1) and the slope m find the equation of the line using:
(y – y1) = m (x – x1)

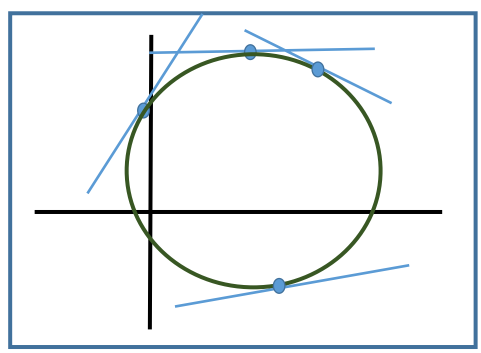


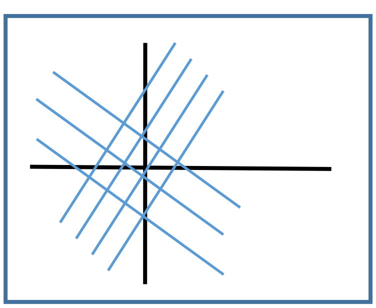
Line K has a positive slope (α < 90°)
Line L has a negative slope (β > 90°)

The slope continued

A line has a constant slope, more complex functions, like a circle or a quadratic, don't

But the slope at any point on a curve has the same slope as the tangent to the curve at that point





Lines with the same slope are parallel (and vice versa!)

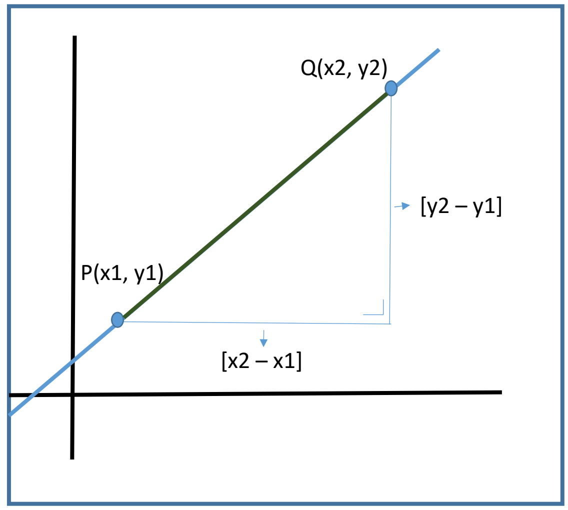
Lines with the slopes **m** and **-1/m** are perpendicular. For example if a line has the slope -2/3 then all lines perpendicular to this line will have a slope 3/2

The Basics – Pages 18 & 19 of "Formulae and Tables"

Distance:

$|PQ| = \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$

(Pythagoras's Theorem with the line segment as the hypotenuse)



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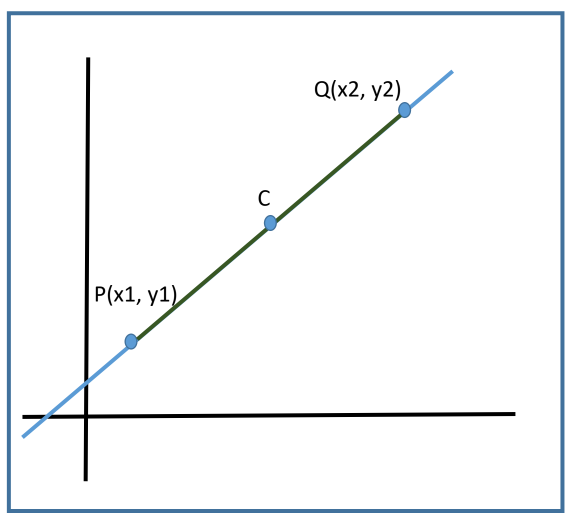
The Basics – Pages 18 & 19 of "Formulae and Tables"

Point dividing [PQ] in the ratio a:b

$= \left(\frac{bx1+ax2}{a+b}, \frac{by1+ay2}{a+b}\right)$

Midpoint:

$C = \left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$



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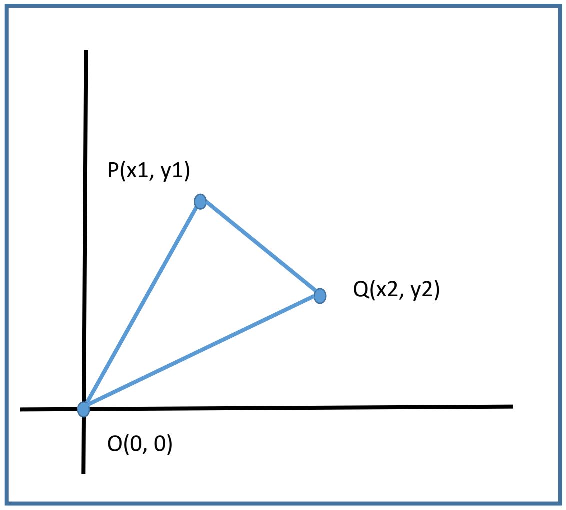
The Basics – Pages 18 & 19 of "Formulae and Tables"

Area of Triangle: OPQ

Area = $\frac{1}{2}\,|x1y2 - x2y1|$



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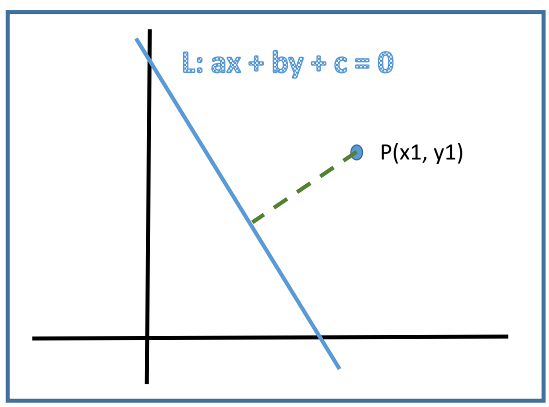
The Basics – Pages 18 & 19 of "Formulae and Tables"

Perpendicular distance from point $(x1,\ y1)$ to line ax +by + c = 0

$$\frac{|ax1 + by1 + c|}{\sqrt{a^2 + b^2}}$$



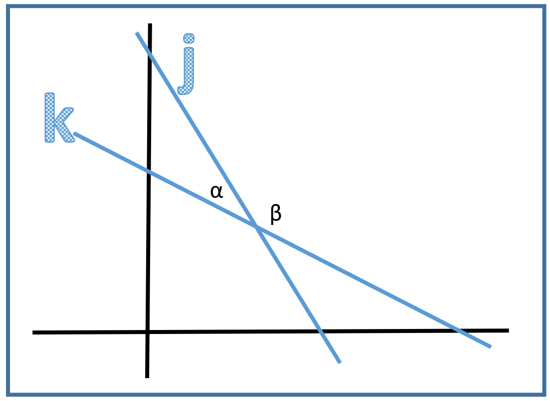
The Basics – Pages 18 & 19 of “Formulae and Tables”

The angle formed between two lines of slope m1 and m2 can be found using:

tan α = $\pm \frac{m1 - m2}{1 + m1m2}$

The positive answer gives the tan of the acute angle α

The negative answer gives the tan of the obtuse angle β



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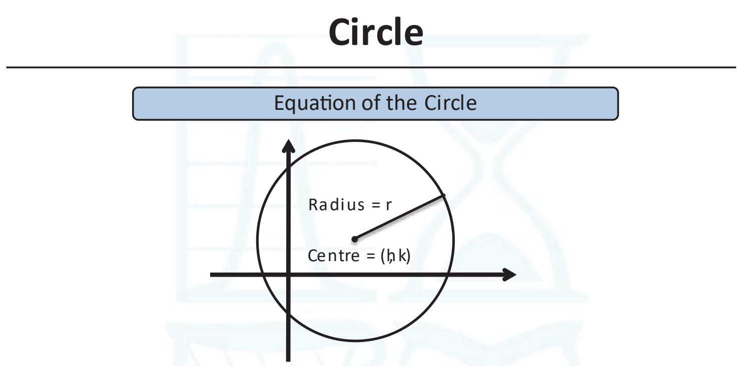
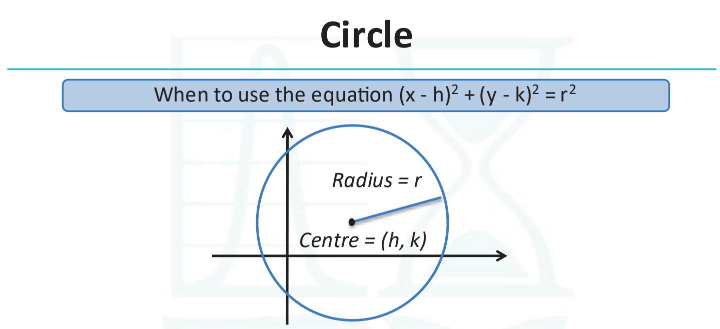
Coordinate Geometry: The Circle – Hints & Tips

General:



- The equation of a circle has an x^2 term and a y^2 term.
- Equation of a circle with centre (0, 0) and radius r: $x^2 + y^2 = r^2$.
- Equation of a circle with centre (h, k) and radius r: $(x – h)^2 + (y – k)^2 = r^2$ (Page 19)
- General equation of a circle: $x^2 + y^2 + 2gx + 2fy + c = 0$ (Page 19).
  - Centre = (-g, -f).
  - Radius = $\sqrt{g^2 + f^2 - c}$

When to use $(x - h)^2 + (y - k)^2 = r^2$:

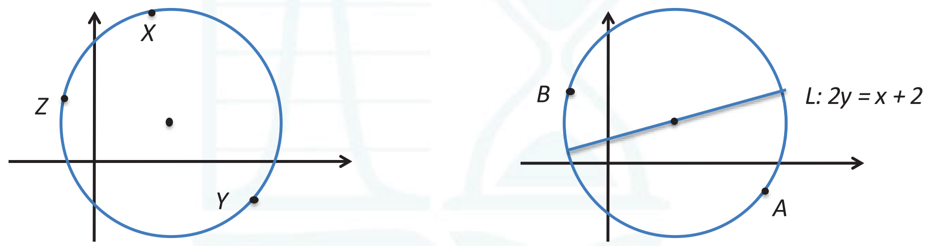


- If we know centre (h, k) and the radius use this equation.
- If we know the centre and a point on the circle:
  - Calculateradiusas distance between centre and the point on the circle.
- If we know the centre of the circle, it's easier to use the equation $(x - h)^2 + (y - k)^2 = r^2$

When to use $x^2 + y^2 + 2gx + 2fy + c = 0$:

Circle

When to use the equation $x^2 + y^2 + 2gx + 2fy + c = 0$

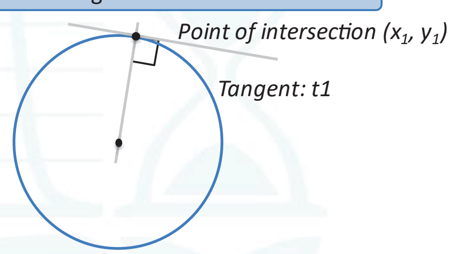


- If we have 3 points on the circle:
  - Substitute each point into the equation of the circle.
  - This will give 3 equations with 3 unknowns (g, f and c). Solve simultaneous equations.
- If we have 2 points on the circle and equation of a line containing centre of circle:
  - Substitute each point into the equation of the circle.
  - Substitute (-g, -f) into the equation of the line.
  - This will given 3 equations with 3 unknowns (g, f and c). Solve simultaneous equations.
- There may be other scenariosgiven but if you can either get <u>3 points on the circle</u>or <u>2 points on the circle and the equation of a line with the centre</u>, then you can use the formula $x^2 + y^2 + 2gx + 2fy + c = 0$.

Tangents:

Circle

Tangents



- A tangent is a line which touches the circle at exactly one point.
- The slope of the tangent and the slope of the radius at the point of intersection are perpendicular.
- If we have the equation of a tangent and the point of intersection then we can find an equation of a line containing the centre of the circle.

- If a circle has the xaxis as a tangent then: $g^2 = c$
- If a circle has the yaxis as a tangent then: $f^2 = c$
- If a circle has the xaxis and yaxis as tangents then:
  - $g^2 = f^2 = c$
  - $g = +/- f$

Circles Touching:

Circle

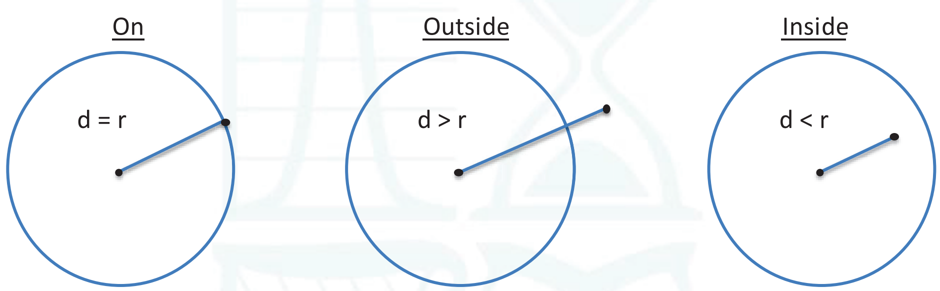
Circles Touching

- If two circles touch externally, the distance between the centres is the sum of their radii i.e. r1 + r2 = d
- If two circles touch internally, the distance between the centres is the difference of their radii i.e. r1 - r2 = d

Points on, outside or inside a circle:

Circle

Points on, outside or inside circle

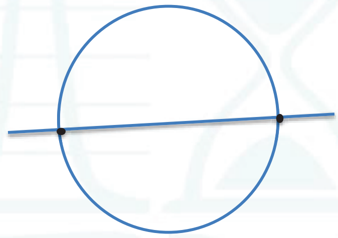


- If a point is **on** the circle: distance from the centre to the point **is equal** to the radius
- If a point is **inside** the circle: distance from the centre to the point **is less** to the radius
- If a point is **outside** the circle: distance from the centre to the point **is greater** to the radius

Point(s) of Intersection between Line and Circle:

Circle

Finding point(s) of intersection between line and circle

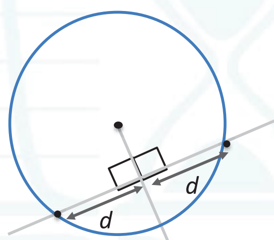


- If we know the equation of the circle and know the equation of the line that intersects the circle:
  - Get x in terms of y from the equation of the line, or y in terms of x. E.g. If the equation of the line is $2y = x + 3$. Then $x = 2y – 3$.
  - Substitute this new term for x into the equation of the line. E.g. If the equation of the circle is $(x – 3)^2 + (y – 6)^2 = 36$ the substitute $(2y – 3)$ for each x term in the equation of the circle.
  - Solve the quadratic equation to find the y coordinates at the points of intersection.
  - Sub back in the values of y from above into the equation of the line to find the x coordinates.
  - If the line is a tangent then there will only be one point of intersection

Chords:

Circle

Perpendicular bisector of a chord

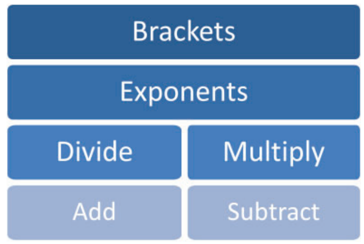


- A chord is a line joining two points on the circumference of a circle.
- A diameter is a type of chord.
- The perpendicular bisector of a chord is a line containing the centre of the circle.
- A bisector divides the chord into two equal parts.

Common Chord/ Tangent

- If two circles have the same chord/tangent in common then to find the equation of the tangent:
  - Use the equation $C_1 – C_2 = 0$ where C_1 and C_2 are the equations of the circles in the form $x^2 + y^2 + 2gx + 2fy + c = 0$
  - You can only use this if the x^2 and y^2 terms have the same coefficient for both circles. (i.e. when you subtract $C_1 – C_2$, the x^2 and y^2 terms cancel out.

Simplifying an expression. What order should you follow?



To manipulate an equation, you can:

- Add or subtract the same number to/from both sides
- Multiply or divide both sides by the same number
- Square both sides, cube both sides, etc.
- Take the square root of both sides or cube root, etc

Simple Algebraic Rules

- Multiplying Terms
  (a) x (bc) = abc
  (a) x (b+c) = ab + ac
  (a+b) x (c+d) = a(c+d) + b(c+d)

- Anything to the power of 0 is 1.

  $A^0 = 1$, $473^0 = 1$, $\pi^0 = 1$, $1{,}000{,}000^0 = 1$

- Inverse Powers

  $X^{-1} = \frac{1}{X}$, $Y^{-6} = \frac{1}{Y^6}$

- Indices

  $Y^3 + Y^2 = Y \times Y \times Y + Y \times Y$ ..... Can't be simplified

  $(Y^2)^3 = (Y \times Y)^3 = (Y \times Y) \times (Y \times Y) \times (Y \times Y) = Y^6$ ..... 2 x 3 = 6

  $Y^2 \times Y^3 = (Y \times Y) \times (Y \times Y \times Y) = Y^5$ ..... 2 + 3 = 5

  $\frac{Y^3}{Y^2} = \frac{Y \times Y \times Y}{Y \times Y} = Y^1$ ..... 3 – 2 = 1

<u>Cancelling terms in fractions:</u>

$$\frac{A}{A \times B} = \frac{1}{B} \quad ✓ \qquad \frac{X^2+2X}{X} = \frac{X(X+2)}{X} = \frac{X+2}{1} = X + 2 \quad ✓$$

$$\frac{A}{A + B} = \frac{1}{B} \quad ✗ \qquad \frac{X+2}{X} = 2 \quad ✗$$

<u>Indices and Logarithms</u> (From your Log Tables – Very Important!!!)

Indices

$a^p a^q = a^{p+q}$

$a^{\frac{1}{q}} = \sqrt[q]{a}$

$\frac{a^p}{a^q} = a^{p-q}$

$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

$(a^p)^q = a^{pq}$

$(ab)^p = a^p b^p$

$a^0 = 1$

$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

$a^{-p} = \frac{1}{a^p}$

Logarithms

General rule of logs: $a = b^c <=> \log_b a = c$

$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\frac{1}{x}\right) = -\log_a x$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(a^x) = x$

$\log_a x^q = q \log_a x$

$a^{\log_a x} = x$

$\log_a 1 = 0$

$\log_b x = \frac{\log_a x}{\log_a b}$

Common forms of ***<u>algebraic equations you might need to factorise</u>***:

Take out common terms ab + ad = a(b + d)	**Factorise by grouping** ab +ad + cb + cd = a(b + d) + c(b + d) = (a + c)(b + d)
Factorise a trinomial $a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2$	**Difference of two squares** $a^2 - b^2 = (a + b)(a - b)$
Difference of two cubes $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	**Sum of two cubes** $a^3 + b^3 = (a + b)(a^2 - 2ab + b^2)$

Solving Quadratic Equations (i.e. finding the roots of a quadratic equation)

You can either:

A. Factorise & let each factor = 0 ; OR

B. Use the formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$; OR

C. Complete the square (write in vertex form) and set = 0

All three approaches will work. Some quadratics are easier to factorise than others, but you can always use the formula approach – even for simple quadratics.

Regarding the formula:

- If $b^2 - 4ac > 0$, the equation has two real distinct roots
- If $b^2 - 4ac = 0$, the equation has two equal real roots
- If $b^2 - 4ac < 0$, the equation has no real roots

Factor theorem

"If an algebraic expression is divided by one of its factors, then the remainder is zero. The expression (x – k) is a factor of a polynomial f(x) if the remainder when we divide f(x) by (x – k) is zero."

- If f(k) = 0, then (x – k) is a factor of f(x)
- If (x – k) is a factor of f(x), then f(k) = 0

Solving cubic equations ($ax^3 + bx^2 + cx + d = 0$)

- Find the first root, k, by trial and error
- If x = k is a root, then (x – k) is a factor
- Divide f(x) by (x – k), which always gives a quadratic expression
- Find the factors of the quadratic and then find the roots of the quadratic

Inequalities

Always remember that multiplying or dividing both sides of an inequality by a negative number reverses the direction of the inequality symbol

Quadratic inequalities

Replace >, <, ≥, ≤ with = to make an equation

- Solve the equation to find the roots -> these are called the critical values of the inequality & the solution either will be between these critical values or outside these critical values
- Test a number between the critical values (often 0) in the original inequality

Two possibilities arise:

- If the inequality is true, then the solution lies between the critical values
- If the inequality is false, then the solution does not lie between the critical values

(**Note:** that if the inequality uses < or >, the critical values are not included in the solution set, whereas if the inequality uses ≤ or ≥, the critical values are included in the solution set)

Modulus inequalities

If $|x| \leq a$, then $-a \leq x \leq a$ (**Note:** "mod x" is the same as "|x|")

If $|x| \geq a$, then $x \leq -a$ or $x \geq a$

Surds

Properties of surds…

$\sqrt{ab} = \sqrt{a}\sqrt{b}$
$\sqrt{\left(\frac{a}{b}\right)} = \frac{\sqrt{a}}{\sqrt{b}}$
$\sqrt{a}\sqrt{a} = a$

To simplify surds…

Find the largest possible perfect square number greater than 1 that will divide evenly into the number under the square root.

Then use the first property above. E.g. $\sqrt{63} = \sqrt{(9x7)} = \sqrt{9} \text{ x } \sqrt{7} = 3\sqrt{7}$

Lowest Common Denominator

When you want to sum (or subtract) two fractions, you need to find a common denominator. The easiest way is usually to multiply the two denominators (this will give you a common denominator, but it won't necessarily be the lowest one).

E.g.

$$\frac{1}{7} + \frac{3}{5}$$

Lowest Common denominator = 7 x 5 = 35

Now multiply so that you have the same denominator in each fraction:

$\frac{5}{5} x \frac{1}{7} + \frac{3}{5} x \frac{7}{7}$ (effectively multiplying by 1)

$= \frac{5}{35} + \frac{21}{35}$ (you can combine once they have the same denominator)

$= \frac{26}{35}$

The same applies for algebraic fractions, e.g.

$$\frac{x}{y} + \frac{2x}{z}$$

Lowest Common denominator = yz

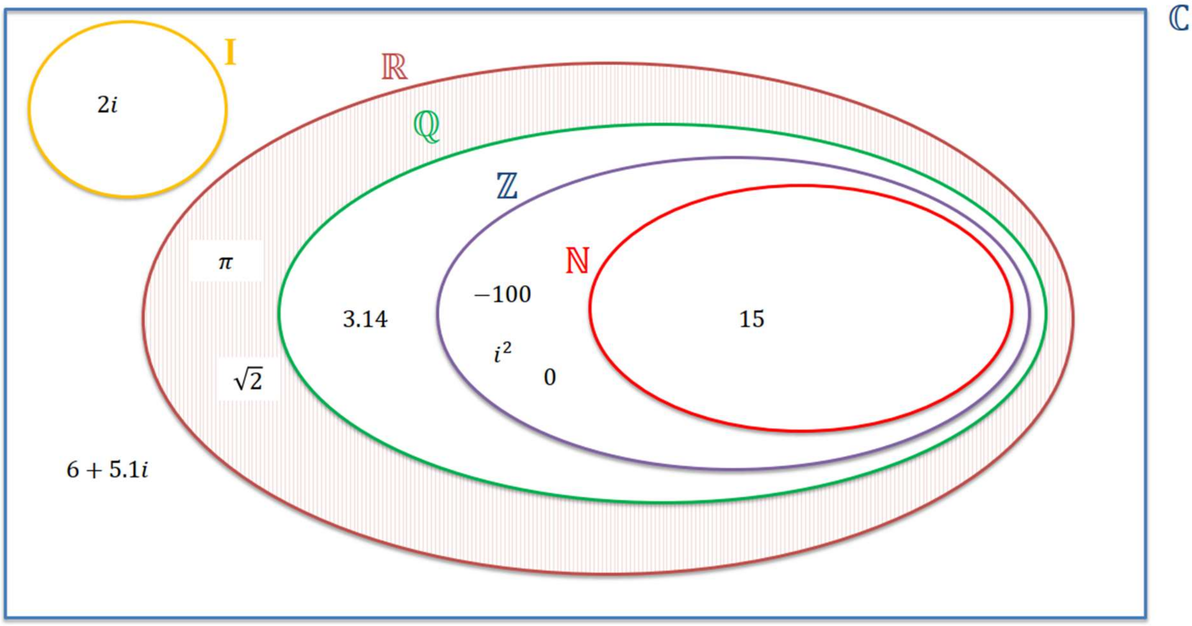
$\frac{z}{z} x \frac{x}{y} + \frac{2x}{z} x \frac{y}{y}$

$= \frac{zx}{zy} + \frac{2xy}{zy}$

$= \frac{zx+2xy}{zy}$

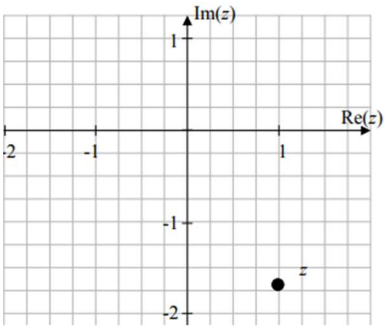
Hints and Tips – Complex Numbers & Proof by Induction

$i = \sqrt{(-1)}$



Where N is all Natural Numbers, Z is all Integers, Q is all Rational Numbers, R is all Real Numbers, I is all Imaginary Numbers and C is all Complex Numbers

Plotting a complex number on Argand diagram z = 1 - 1.8i



Conjugates

Conjugate of a+bi = $\overline{a+bi}$ = a-bi

$$\overline{Z_1 . Z_2} = \overline{Z_1} . \overline{Z_2}$$

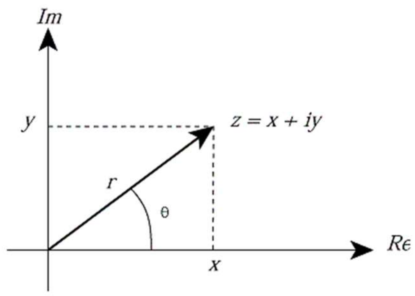
$$\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$$

To divide by a complex number – multiply by 1 using conjugate above & below line i.e. :

$$\frac{a+bi}{e+fi} \Rightarrow \frac{a+bi}{e+fi} \times \frac{e-fi}{e-fi}$$

Conjugate is equivalent to reflection in the real axis.

Modulus is the distance of point from point 0 (i.e. from 0 + 0i) on Argand diagram $|x+iy| = \sqrt{(x^2 + y^2)}$



Cos θ = x/r   Sin θ = y/r

Tan θ = y/x

$|z_1 z_2| = |z_1|.|z_2|$

Polar Form:

Polar Form of z, where z = x+iy is rCosθ+irSinθ where r = |x+iy| so r>=0

Using Cos θ = x/r   Sin θ = y/r

De Moivre's Thoerom

~~$[r\,\text{Cos}\theta + ir\,\text{Sin}\theta]^k = [r^k$~~$\text{Cos}(k\theta) + ir^k\,\text{Sin}(k\theta)] = r^k(\text{Cos}(k\theta)+\text{Sin}(k\theta))$

θ or angle/argument is measured in radians

To find k roots use using De Moivre's Theorem set:

$r^{1/k}(\text{Cos}(\theta/k + 2n\pi/k)+\text{Sin}(\theta/k+2n\pi/k))$ for n =0, 1,…,k-1

Proof by Induction

Step 1: Show that the proposition is true for $n = 1$

(insert your workings to show outcome…..)

Hence, proposition is true for n=1

Step 2: Assume that the proposition is true for n=k

Step 3: *Prove* that the proposition is true for $n = k + 1$, given that it is true for $n = k$.

(insert your workings to show outcome for k+1 & how it relates to outcome for k…..)

∴ The proposition is true for $n = k + 1$, given that it is true for $n = k$.

Step 4: State that proposition is true for $n = 1$ and if the proposition is true for n=k, it will be true for n = k + 1, therefore by induction it is true for all $n \in \mathrm{N}$