Calculus 2 Tutorial Answers

Exercises:

1) (i) $2x^2 + c$ (ii) $\frac{5y^4}{4} + c$ (iii) $\frac{7x^6}{6} + 3x^2 + 2x + c$ (iv) $\ln(x) + c$

2) $\int_2^4 6x^2\,dx = 2x^3|_2^4 = 2(4^3) - 2(2^3)$ = 112 units squared

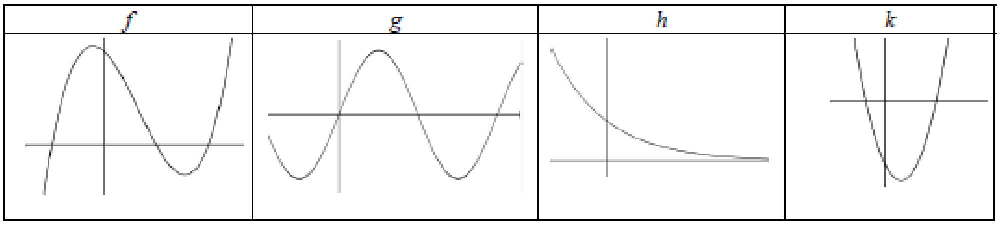
3) $\int_1^5 4x^3 + 2\,dx = (x^4 + 2x)|_1^5 = (5^4 + 2(5)) - (1^4 + 2(1))$= 632 units squared

4) $\frac{1}{3-0}\int_0^3 5 - 2x\,dx = \frac{1}{3}(5x - x^2)|_0^3 = \frac{1}{3}[(5(3) - 3^2) - (5(0) - 0^2)] = \frac{1}{3}(6) = 2$

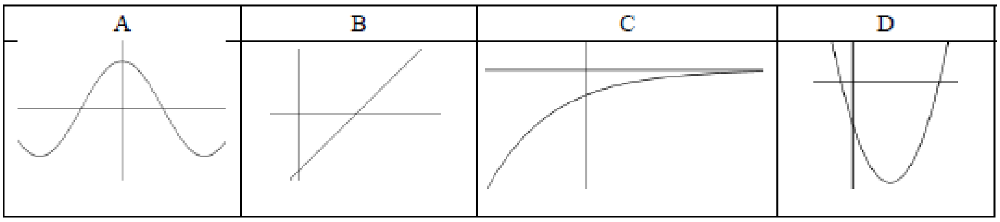
Past Exam Questions

Question 1 [2013 Sample Paper 1, Q5] (25 marks)

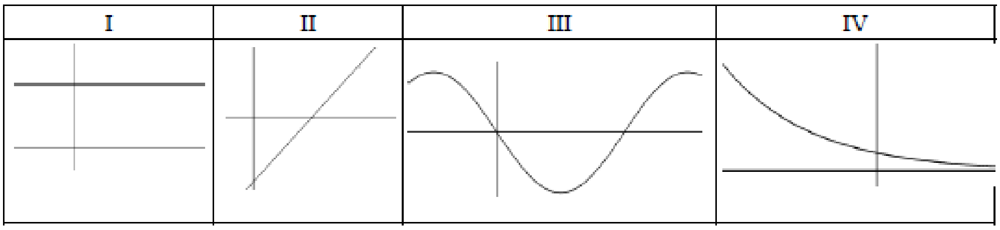
Each diagram below shows part of the graph of a function. Each is one of these: quadratic, cubic, trigonometric or exponential (not necessarily in that order).



Each diagram below shows part of the graph of the first derivative of one of the above functions (not necessarily in the same order).



Each diagram below shows part of the graph of the second derivative of one of the original functions (not necessarily in the same order).



(a) Complete the table below by matching the function to its first derivative and its second derivative.

Type of function	Function	First derivative	Second derivative
Quadratic	*k*	B	I
Cubic	*f*	D	II
Trigonometric	*g*	A	III
Exponential	*h*	C	IV

(b) For one row in the table, explain your choice of first derivative and second derivative.

A quadratic function differentiates to a line which differentiates to a constant.

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Question 2 [2014 Paper 1, Q5] (25 marks)

(b) The slope of the tangent to a curve $y = f(x)$ at each point (x, y) is $2x - 2$. The curve cuts the x-axis at $(-2, 0)$.

(i) Find the equation of $f(x)$.

$$\int dy = \int (2x-2)dx$$
$$\Rightarrow y = x^2 - 2x + c$$
At $x=-2, y=0 \Rightarrow 0 = 4 + 4 + c \Rightarrow c = -8$

Hence, $y = x^2 - 2x - 8$

(ii) Find the average value of f over the interval $0 \le x \le 3, x \in \mathbb{R}$.

Average value: $\frac{1}{b-a}\int_a^b f(x)dx$

$$\frac{1}{3-0}\int_0^3 (x^2 - 2x - 8)dx = \frac{1}{3}\left[\frac{x^3}{3} - x^2 - 8x\right]_0^3$$
$$= \frac{1}{3}\left[\frac{27}{3} - 9 - 24\right] = -8$$

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Question 3 [2015 Paper 1, Q3] (25 marks)

Let $f(x) = -x^2 + 12x - 27$, $x \in \mathbb{R}$.

(a) (i) Complete Table 1 below.

Table 1							
x	3	4	5	6	7	8	9
$f(x)$	0	5	8	9	8	5	0

(ii) Use Table 1 and the trapezoidal rule to find the approximate area of the region bounded by the graph of f and the x-axis.

$$A = \frac{h}{2}\left[y_1 + y_n + 2(y_2 + y_3 + \cdots + y_{n-1})\right]$$
$$= \frac{1}{2}\left[0 + 0 + 2(5 + 8 + 9 + 8 + 5)\right]$$
$$= 35 \text{ square units}$$

(b) (i) Find $\int_3^9 f(x)dx$.

$$\int_3^9 \left(-x^2 + 12x - 27\right) dx$$
$$= \left[\frac{-x^3}{3} + \frac{12x^2}{2} - 27x\right]_3^9$$
$$= (-243 + 486 - 243) - (-9 + 54 - 81)$$
$$= 36$$

(ii) Use your answers above to find the percentage error in your approximation of the area, correct to one decimal place.

$$\frac{1}{36} \times 100 = 2 \cdot 8\%$$

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Question 4 [2016 Paper 1, Q7] (40 marks)

Q7	Model Solution – 40 Marks	Marking Notes
(a) (i)	$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dr} = 4\pi r^2$ $\frac{dv}{dt} = 250 \text{ cm}^3/\text{s}$ $\frac{dr}{dt} = \frac{dr}{dv}.\frac{dv}{dt} = \frac{1}{4\pi r^2}.250$ $\frac{dr}{dt} = \frac{250}{4\pi 400} = \frac{5}{32\pi} \text{ cm/s}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • work towards $\frac{dv}{dr}$ or $\frac{dv}{dt}$ or $\frac{dr}{dt}$ *High Partial Credit* • correct expression for $\frac{dr}{dt}$
(ii)	$a = 4\pi r^2 \Rightarrow \frac{da}{dr} = 8\pi r$ $\frac{da}{dt} = \frac{da}{dr}.\frac{dr}{dt} = 8\pi r.\frac{5}{32\pi}$ $= \frac{5(20)}{4}$ $= 25 \text{ cm}^2/\text{s}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • work towards $\frac{da}{dr}$ or $\frac{da}{dt}$ *High Partial Credit* • correct expression for $\frac{da}{dt}$
(b) (i)	$-x^2 + 10x = 0$ $x(-x + 10) = 0$ $x = 0 \quad \text{or} \quad x = 10$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • quadratic equation formed • gets $x = 0$ only *High Partial Credit* • quadratic factorised Note: $f'(x) = 0 \Rightarrow 2x - 10 = 0 \Rightarrow x = 5$ *merits 0 marks*
(ii)	$\frac{1}{10 - 0}\int_0^{10} (-x^2 + 10x)dx$ $= \frac{1}{10}\left[\frac{-x^3}{3} + 5x^2\right]_0^{10}$ $= \frac{1}{10}\left[\left(\frac{-1000}{3} + 500\right) - 0\right]$ $= \frac{-100}{3} + 50 = \frac{50}{3} \text{m}$	Scale 10C (0, 3, 7, 10) *Low Partial Credit* • integration set up *High Partial Credit* • correct integration with some substitution

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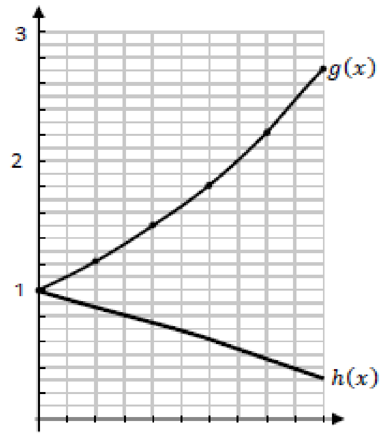
Question 5 [2017 Paper 1, Q6] (25 marks)

Q6	Model Solution – 25 Marks	Marking Notes
(a)	See below	Scale 15C (0, 5, 10, 15) *Low Partial Credit:* • one point correct *High Partial Credit* • Graph not in required domain



$$g(x) = e^x \quad h(x) = e^{-x} = \frac{1}{e^x}$$

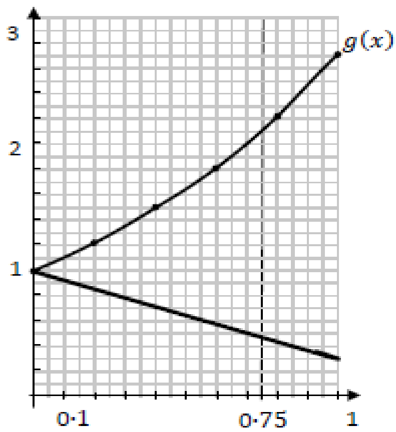
$g(x) = e^x$:

x	0	0·2	0·4	0·6	0·8	1·0
y	1	1·22	1·49	1·82	2·23	2·72

$h(x) = \frac{1}{e^x}$:

x	0	0·2	0·4	0·6	0·8	1·0
y	1	0·82	0·67	0·55	0·45	0·37

(b)		
	See below	Scale 10C (0, 5, 8, 10) *Low Partial Credit:* • Formulates integration for area under one curve with limits *High Partial Credit* • integrates twice for correct area under both curves **Note:** Trapezoidal rule must have at least 5 divisions <u>AND</u> fully correct work gets Low Partial Credit



$$A = \int_0^{0\cdot75} e^x dx - \int_0^{0\cdot75} e^{-x} dx$$

$$= \int_0^{0\cdot75} (e^x - e^{-x}) dx$$

$$= e^x + e^{-x}$$

$$e^{0\cdot75} + e^{-0\cdot75} - [e^0 + e^0]$$

$$= 0\cdot5894$$

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Question 6 [2022 Paper 1, Q2] (15 marks)

(a)	$\frac{2x^3}{3}+\frac{5x^2}{2}+6x+c$
(b) (i)	$\int_0^2 (ax^2+bx+c)\,dx = 538$ $\left.\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right\rvert_{x=0}^{x=2} = 538$ $\frac{a(2^3)}{3}+\frac{b(2^2)}{2}+c(2) = 538$ $4a+3b+3c = 807$

Question 7 [2022 Paper 1, Q7] (45 marks)

(a) (b)	**(a)** $h(4) = 2(4^3) - 28{\cdot}5(4^2) + 105(4) + 70$ $= 162$ BPM **(b)** $h'(x) = 3(2x^2) - 2(28\cdot 5x) + 105$ $= 6x^2 - 57x + 105$
(c)	$h'(2) = 6(2^2) - 57(2) + 105 = 15$ Explanation: It is the rate at which Hannah's heart rate is increasing after / at 2 minutes.

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(d)	Least value of $h(x) = h(0) = 70$ [from graph] $h'(x) = 6x^2 - 57x + 105 = 0$ at local max $2x^2 - 19x + 35 = 0$ $(2x - 5)(x - 7) = 0$ $x = 2{\cdot}5$ or 7 Max $= h(2{\cdot}5)$ [from graph] $= 2(2 \cdot 5^3) - 28 \cdot 5(2 \cdot 5^2) + 105(2 \cdot 5) + 70$ $= 185{\cdot}625$
(e)	$h'(x) = 6x^2 - 57x + 105$ Decreasing most quickly at $h''(x) = 0$ So $12x - 57 = 0$ So $x = 4{\cdot}75$ minutes $= 4$ mins 45 secs **OR** Decreasing most quickly at midpoint of local max/min, that is, $x = \frac{2{\cdot}5+7}{2} = 4{\cdot}75$ minutes $= 4$ mins 45 secs
(f)	$b'(x) = h'(x)$ $k'(x) = 0{\cdot}9\, h'(x)$

