Society of Actuaries in Ireland

Calculus- Hints and Tips

1. Differentiation

1.1 Definition

- Calculus is the study of change.
- The derivative of a function measures how the function changes as its input changes

1.2 Notation

- If y = f(x), the derivative is denoted as f'(x) or $\frac{dy}{dx}$
- It means how much f(x) changes as x changes.

1.3 Rules of Differentiation

1.3.1 Power Rule

- If f(x) = x^n then f'(x) = $nx^{(n-1)}$. (Bring down the power and reduce the power by 1).
- The derivative of a linear function is the **slope** of that line.
- The derivative of a quadratic function is the slope of the tangent line to the graph of that function at that point.
- E.g. If f(x) = $3x^3$ then f'(x) = $3*3x^{(3-1)}$ = $9x^2$

1.3.2 Sum/Difference Rule

- If f(x) = g(x) + h(x), then f'(x) = g'(x) + h'(x)
- Similarly if f(x) = g(x) – h(x) then f'(x) = g'(x) - h'(x)
- E.g. If g(x) = $3x^4 + 5x^2$ and h(x) = $4x^2 + 6x + 5$, and f(x) = g(x) + h(x) then
  - g'(x) = $4*3x^{(4-1)} + 2*5x^{(2-1)}$ -> g'(x) = $12x^3 + 10x$
  - h'(x) = $2*4x^{(2-1)} + 1*6x^{(1-1)}$ + 0 (Constants go to 0) -> h'(x) = 8x + 6
  - f'(x) = $12x^3$ + 10x + 8x + 6 -> f'(x) = $12x^3$ + 18x + 6

1.3.3 Product Rule (*Page 25 of log tables*)

- This rule is used when you have two functions being **multiplied**.
- If y = uv then $\frac{dy}{dx} = u * \frac{dv}{dx} + v\frac{du}{dx}$
- E.g. If y = $(3x^4 + 5x^2)*(4x^2 + 6x + 5)$ then u = $3x^4 + 5x^2$ and v = $4x^2 + 6x + 5$
  - $\frac{du}{dx}$ = $4*3x^{(4-1)} + 2*5x^{(2-1)}$ -> $\frac{du}{dx}$ = $12x^3 + 10x$
  - $\frac{dv}{dx}$ = $2*4x^{(2-1)} + 1*6x^{(1-1)}$ + 0 (Constants go to 0) -> $\frac{dv}{dx}$ = 8x + 6
  - $\frac{dy}{dx}$ = $(12x^3 + 10x)*(4x^2 + 6x + 5) + (3x^4 + 5x^2)*(8x + 6)$ (Subbing into the formula)
  - Multiplying out the brackets and combining terms gives:
  - $\frac{dy}{dx}$ = $72x^5 + 90x^4 + 140x^3 + 90x^2 + 50x$

1.3.3 Quotient Rule (*Page 25 of log tables*)

- This rule is used when you have a function being **divided** by a function.
- If y = $\frac{u}{v}$, then $\frac{dy}{dx} = \frac{(v*du)-(u*dv)}{v^2}$
- E.g. If y = $\frac{3x^2 + 5x}{2x+6}$ then u = $3x^2 + 5x$ and v = 2x + 6:

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- $\frac{du}{dx}$ = 2*3x$^{(2-1)}$ + 1*5x$^{(1-1)}$ -> $\frac{du}{dx}$ = 6x + 5
- $\frac{dv}{dx}$ = 1*2x$^{(1-1)}$ + 0 (Constants go to 0) -> $\frac{dv}{dx}$ = 2
- $\frac{dy}{dx} = \frac{(6x+5)*(2x+6)-(3x^2+5x)*2}{(2x+6)^2}$ (Subbing into formula)
- Multiplying out the brackets and combining terms gives:
- $\frac{dy}{dx} = \frac{6x^2+36x+30}{(2x+6)^2}$

1.3.4 Chain Rule (*Page 25 of log tables*)

- This rule is used when you have a function **within** a function.
- If f(x) = u(v(x)) then f'(x) = $\frac{du}{dv} * \frac{dv}{dx}$
- E.g. If f(x) = $(5x^2 + 7)^3$, then u = v^3 and v = $5x^2 + 7$,
  - $\frac{du}{dv}$ = 3*v$^{(3-1)}$ = $3v^2$
  - $\frac{dv}{dx}$ = 2*5x$^{(2-1)}$ + 0 = 10x
  - f'(x) = $3v^2$ * 10x (Subbing into formula)
  - f'(x) = $3(5x^2 + 7)^2$ *10x (Subbing back in value for v)
  - f'(x) = $30x(5x^2 + 7)^2$

1.4 Common Derivatives of Functions (*Page 25 of log tables*)

Function	**Derivative**
Ln(x)	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
Cos(x)	-Sin(x)
Sin(x)	Cos(x)
Tan(x)	$Sec^2(x)$
$Sin^{-1}(\frac{x}{a})$	$\frac{1}{\sqrt{a^2-x^2}}$
$cos^{-1}(\frac{x}{a})$	$\frac{-1}{\sqrt{a^2-x^2}}$
$tan^{-1}(\frac{x}{a})$	$\frac{a}{a^2+x^2}$

1.5 Differentiation by First Principles

- This is a formula to learn off as it is not in the log tables.
- Given f(x) the derivative is $lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- E.g. If f(x) = $3x^2 + 5$:
  - f(x+h) = $3(x+h)^2 + 5$. (Sub in x+h for every x).
  - Multiplying out the brackets gives $3x^2 + 6xh + 3h^2 + 5$
  - Simplify the numerator: f(x + h) – f(x):
  - $(3x^2 + 6xh + 3h^2 + 5) - (3x^2 + 5) = 6xh + 3h^2$

- Divide by h: $\frac{6xh+3h^2}{h}$ = 6x + 3h
- Take the limit as h goes to 0: 6x + 3(0) (Sub in 0 for h)
- The final answer is 6x.

- You need to be able to apply this method to linear and quadratic functions.

1.6 Minimum, Maximum and Inflection Points

- The value of x for which $f'(x) = 0$, identifies either a maximum or a minimum point on a curve (i.e. the slope of the tangent to the curve at that point is 0)
- In order to determine if it's a maximum or a minimum, we look at the second derivative.
- If $f''(x) > 0$ -> Minimum turning point.
- If $f''(x) < 0$ -> Maximum turning point.
- If $f''(x) = 0$ -> Point of inflection. Point of inflection is the point where the slope is increasing/decreasing the most.
- E.g. If $f(x) = x^3 - 4x^2 - 3x + 5$, differentiate the equation, set equal to 0 and solve the quadratic equation for x:
    - $f'(x) = 3x^2 - 8x - 3$
    - $3x^2 - 8x - 3 = 0$. Solving for X gives x = 3 and $x = \frac{-1}{3}$
    - Sub back in the values to the original equation to find the y coordinate:
        - For x =3, $f(3) = 3^3 - 4(3)^2 - 3(3) + 5 = -13$.
            - => (3, -13) is a turning point
        - For $x = \frac{-1}{3}$, $f(\frac{-1}{3}) = (\frac{-1}{3})^3 - 4(\frac{-1}{3})^2 - 3(\frac{-1}{3}) + 5 = (\frac{149}{27})$
            - => $(\frac{-1}{3}, \frac{149}{27})$ is a turning point
    - To find which point if minimum and maximum, calculate the second derivative and sub in the value for x and determine if the result is < 0 (max) or > 0 (min).
    - $f''(x) = 6x - 8$ (differentiating $3x^2 - 8x - 3$).
        - For x = 3, $f''(3) = 6(3) - 8 = 18 - 8 = 10$. This is greater than 0 so (3, -13) is the minimum turning point.
        - For $x = \frac{-1}{3}$, $f''(\frac{-1}{3}) = 6(\frac{-1}{3}) - 8 = -2 - 8 = -10$. This is less than 0 so $(\frac{-1}{3}, \frac{149}{27})$ is the maximum turning point.
    - To find the point of inflection, set the second derivative equation equal to 0 and solve.
    - $f''(x) = 6x - 8 = 0$ -> $6x = 8$
    - $x = \frac{4}{3}$. Sub back into the original equation to find the y coordinate.
        - $f(\frac{4}{3}) = (\frac{4}{3})^3 - 4(\frac{4}{3})^2 - 3(\frac{4}{3}) + 5 = (\frac{-101}{27})$
        - $(\frac{4}{3}, \frac{-101}{27})$ is the point of inflection.

