Society of Actuaries in Ireland

SAI Tutorial on Complex Numbers and Proof by Induction - Answers

Question A – Exercise Questions

(i)

1. $2z_1 + z_2$ = 2(2+3i)+(1-2i) =5 +4i
2. $z_1 * z_2$ = (2+3i) (1-2i) = 8 - i
3. $z_1 - iz_2$ = (2+3i)-i(1-2i) =2i
4. | $2+z_1$|=|2+(2+3i)|=$\sqrt{(4^2+3^2)}=5$
5. $\frac{z_1}{z_2} = \frac{2+3i}{1-2i}$

$= \frac{2+3i}{1-2i} * \frac{1+2i}{1+2i}$ (Multiply above and below by conjugate of denominator)

$= \frac{2+4i+3i+6i^2}{1-2i+2i-4i^2} = \frac{2+7i+6(-1)}{1-4(-1)}$

$= \frac{-4+7i}{5} = \frac{-4}{5} + \frac{7i}{5}$

6. $\overline{2z_1 + z_2}$ = 5 – 4i (conjugate of question 1)

(ii)

1. 2 + 2i

Find the modulus and argument

r = $\sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8}$

Complex Number is in 1st quadrant:

⇨ $\theta = Reference\ Angle$

$\theta = Tan^{-1}\left(\frac{2}{2}\right) = Tan^{-1}(1) = 45^0$.

Polar Form = $\sqrt{8}(Cos45^0 + iSin45^0)$

2. $-2\sqrt{2} + 2\sqrt{2}i$

Find the modulus and argument

r = $\sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = 4$

Complex Number is in 2nd quadrant:

⇨ $\theta = 180^0$ – Reference Angle

⇨ Reference Angle = $Tan^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{2}}\right) = Tan^{-1}(1) = 45^0$

⇨ $\theta = 180^0 - 45^0 = 135^0$

Polar Form = $4(Cos135^0 + iSin135^0)$

Question B (Question 3 Paper 1 2022)

(a)

(i)

$z - iz = 6+2i - i(6+2i)$

$= 6+2i - 6i - 2i^2$

$= 6 - 4i - 2(-1) = 8 - 4i$

(ii)

$|z|^2 = 6^2 + 2^2 = 40$

$|iz|^2 = 2^2 + 6^2 = 40$

$|z|^{2}+|iz|^{2}$ = 40 + 40 = 80
$|z-iz|^2$ = |8-4i|^2 = 8^2+ 4^2 = 80

(iii)

Right angled triangle so Pythagorean therefore diameter = $|z|^{2}+|iz|^{2}$ = 80 from pervious
Radius = √80 ÷2= √20
Area = πr^2 = 20π square units
OR
Calculate centre as midpoint of iz & z
= 2+4i
then Radius is distance to z.

Radius = $\sqrt{(6-2)^2+(2-4)^2}$ = √20
Area = πr^2 = 20π square units

(b)

De Moivre's Thoerom => [rCosθ+irSinθ]^n = r^n(Cos(nθ)+Sin(nθ))
Write complex number in polar form first i.e. (√3 – i) = rCosθ+irSinθ

r = $\sqrt{(\sqrt{3})^2+(-1)^2}$ = √4 = 2

Argument: Complex Number is in 4th quadrant

⇨ θ = - Reference Angle

Reference Angle = $Tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^0$

$\theta = -30^0$

(√3 – i) = 2(Cos(-30^0)+iSin(-30^0))

(√3 – i)^9 = 2^9 (Cos(-30^0 * 9)+iSin(-30^0 * 9))

(√3 – i)^9 = 512 (0 +i) = 512i

a = 0, b = 512

Question C (Question 1 Paper 1 2021)

(a)

$$\frac{4-2i}{2+4i} = \frac{4-2i}{2+4i}\cdot\frac{2-4i}{2-4i} = \frac{8-4i-16i+8i^2}{4+8i-8i-16i^2} = \frac{8-20i+8(-1)}{4-16(-1)} = \frac{-20i}{20}$$

k= −1

(b)

Use De Moivres Theorem:

Polar Form of -5+12i = rCosθ+irSinθ

Modulus: r = √(5^2+12^2)= √169 = 13

Argument: Complex number is in 2nd quadrant:

⇨ θ = 180^0 – Reference Angle

Reference Angle = $Tan^{-1}\left(\frac{12}{5}\right)$ = 67.38^0

θ = 180^0 – 67.38^0 = 112.62

$-5 + 12i = 13(\cos 112.62^0 + i \sin 112.62^0)$

$(-5 + 12i)^{1/2}$ = $[13(\cos 112.62^0 + i \sin 112.62^0)]^{1/2}$

= $\sqrt{13}$ $(\cos(\frac{112.62}{2} + n\,\frac{360}{2}) + i \sin(\frac{112.62}{2} + n\,\frac{360}{2}))$ for n=0 or 1

<u>n = 0:</u>

= $\sqrt{13}$ $(\cos(\frac{112.62}{2} + 0 * \frac{360}{2}) + i \sin(\frac{112.62}{2} + 0 * \frac{360}{2}))$

= $\sqrt{13}$ $(\cos(56.31^0) + i \sin(56.31^0))$

= $\sqrt{13}$ (0.554+ 0.832i)

= 2 + 3i

<u>n = 1:</u>

= $\sqrt{13}$ $(\cos(\frac{112.62}{2} + 1 * \frac{360}{2}) + i \sin(\frac{112.62}{2} + 1 * \frac{360}{2}))$

= $\sqrt{13}$ $(\cos(56.31^0 + 180^0) + i \sin(56.31^0 + 180^0))$

= $\sqrt{13}$ (-0.554 - 0.832i)

= -2 - 3i

(c)

$$z^3 = r(\cos\theta + i\sin\theta)$$

$$z = \left(r(\cos\theta + i\sin\theta)\right)^{\frac{1}{3}}$$

$$= 2\left(\cos\frac{\pi+2n\pi}{3} + i\sin\frac{\pi+2n\pi}{3}\right)$$

$$n = 0: \quad z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}i$$

$$n = 1: \quad z = 2\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right) = -2$$

$$n = 2: \quad z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 1 - \sqrt{3}i$$

<u>Question D (Question 2 Paper 1 2020)</u>

(a)

Multiply iz_1 by i to get $i^2z_1 = -z_1$ and then substitute in formulae

$$iz_1 = -4 + 3i$$

$$i(iz_1) = i(-4 + 3i)$$

$$-z_1 = -4i + 3i^2$$

$$z_1 = 3 + 4i$$

$$3z_1 - z_2 = 3(3 + 4i) - z_2 = 11 + 17i$$

$$z_2 = 9 + 12i - 11 - 17i$$

$$z_2 = -2 - 5i$$

(b) (i)

$$\frac{5-i}{3+2i} = \frac{5-i}{3+2i}\cdot\frac{3-2i}{3-2i} = \frac{15-3i-10i+2i^2}{9+6i-6i-4i^2} = \frac{15-13i+2(-1)}{9-4(-1)} = \frac{13-13i}{13}$$

<u>Question E (Question 5 Paper 1 2019)</u>

(a) Substitute into the polynomial

$$(3+2i)^2 + p(3+2i) + q = 0$$
$$5 + 12i + 3p + 2pi + q = 0$$
$$2p = -12 \Rightarrow p = -6$$
$$5 + 3p + q = 0 \Rightarrow q = 13$$

(b)(i)

$$|v| = \sqrt{4+12} = 4$$
$$\theta = 300°$$
$$v = 4(\cos 300° + i \sin 300°)$$

Or

$$|v| = \sqrt{4+12} = 4$$
$$\theta = \frac{5\pi}{3}$$
$$v = 4\left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right)$$

(b)(ii)

$$w = \pm v^{\frac{1}{2}}$$
$$w = \pm 2(\cos 300 + i \sin 300)^{\frac{1}{2}}$$
$$w = \pm 2(\cos 150 + i \sin 150)$$
$$w = \pm(-\sqrt{3} + i)$$
$$w = -\sqrt{3} + i \text{ or } \sqrt{3} - i$$

Or

$$w = [4(\cos(300 + 360n) + i\sin(300 + 360n)]^{\left(\frac{1}{2}\right)}$$

$$w = 4^{\frac{1}{2}}[\cos(150 + 180n) + i\sin(150 + 180n)]$$

$\underline{n = 0}$

$$w = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$\underline{n = 1}$

$$w = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

<u>Question F (Question 1 Paper 1 2013)</u>

(a)

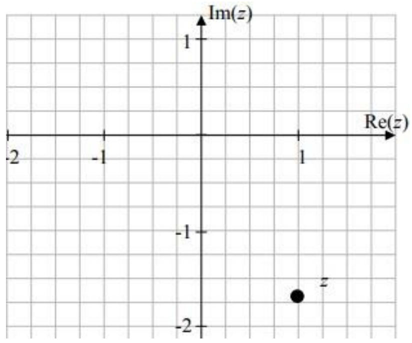
$$z = \frac{4}{1+\sqrt{3}i} = \frac{4}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{4-4\sqrt{3}i}{1+3} = 1 - \sqrt{3}i$$

(b)

$$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \theta = \frac{5\pi}{3}$$

$$r = |1-\sqrt{3}i| = \sqrt{1+3} = \sqrt{4} = 2$$

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$



(c)

$$\begin{aligned} z^{10} &= \left[2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\right]^{10} \\ &= 2^{10}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2^{10}\left(\cos\frac{50\pi}{3} + i\sin\frac{50\pi}{3}\right) \\ &= 2^{10}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2^{10}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2^{9}\left(1-\sqrt{3}i\right) \end{aligned}$$

<u>Question G (Question 1 Paper 1 2016)</u>

(a) -4 – 3i

(b)

$$r = \sqrt{1^2+1^2} = \sqrt{2} \qquad \theta = \frac{\pi}{4}$$

$$(1+i)^8 = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^8$$

$$(1+i)^8 = \{16(\cos 2\pi + i\sin 2\pi)\}$$

$$(1+i)^8 = 16(1) = 16$$

(c)

$$\begin{aligned} z &= \frac{(2-i) \pm \sqrt{(-2+i)^2 - 4(3-i)}}{2} \\ &= \frac{(2-i) \pm \sqrt{4-4i-1-12+4i}}{2} \\ &= \frac{2-i \pm \sqrt{-9}}{2} \\ &= \frac{2-i \pm 3i}{2} \\ &= 1-2i \textbf{ or } 1+i \end{aligned}$$

Question H (Question 4 Paper 1 2021)

$P(1)$: $2^{3(1)-1}+3=7$, which is div. by 7

$P(k)$: Assume $2^{3k-1}+3$ is div. by 7

$$2^{3k-1}+3=7M$$

$$2^{3k-1}=7M-3$$

$P(k+1)$: $2^{3k+2}+3$

$$=2^3(2^{3k-1})+3$$

$$=8(7M-3)+3$$

$$=56M-21$$

$P(k+1)$ is divisible by 7

True for $n=1$ and, if true for $n=k$, then true for $n=k+1$. Therefore, true for all $n\geq 1$.

OR

$P(k+1)$: $2^{3k+2}+3$

$$=2^3(2^{3k-1})+3$$

$$=(7+1)(2^{3k-1})+3$$

$$=(7.2^{3k-1})+(2^{3k-1}+3)$$

Both divisible by 7

True for $n=1$ and, if true for $n=k$, then true for $n=k+1$. Therefore, true for all $n\geq 1$.

Question I (Question 7(d) Paper 1 2020)

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$P(1)$: $1=\frac{1(2)(3)}{6}$

$P(k)$: $1+4+9+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$

$P(k+1)$: $1+4+9+\cdots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

$$LHS=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$LHS=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$LHS=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$LHS=\frac{(k+1)[2k^2+7k+6]}{6}$$

$$\frac{(k+1)(k+2)(2k+3)}{6}=RHS$$

Thus the proposition is true for $n=k+1$ provided it is true for $n=k$ but it is true for $n=1$ and therefore true for all positive integers.

Question J 2023 Q4 paper 1

(a)

Method 1

$(1+i)^2 + (3-2i)(1+i) + p = 0$

$1 + 2i + i^2 + 3 + i - 2(i)^2 + p = 0$

$5 + 3i + p = 0$

$p = -5 - 3i$

Method 2

Let the second root $= z_2$

Sum of roots:

$1 + i + z_2 = -3 + 2i$

$z_2 = -4 + i$

Product of roots:

$(1+i)(-4+i) = p$

$p = -5 - 3i$

Method 3

$z = \frac{-(3-2i) \pm \sqrt{(3-2i)^2 - 4p}}{2}$

$2z = -(3-2i) \pm \sqrt{(3-2i)^2 - 4p}$

$2z + 3 - 2i = \pm\sqrt{(3-2i)^2 - 4p}$

$[2z + 3 - 2i]^2 = (3-2i)^2 - 4p$

$z = 1 + i$ satisfies this equation

(b)

Reference Angle:

$\alpha = \tan^{-1}\frac{\sqrt{3}}{1} = 60^\circ \left(\frac{\pi}{3} \text{ rads}\right)$

Argument:

$\theta = 180^o - 60^o = 120^o \left(\frac{2\pi}{3} \text{ rads}\right)$

Modulus:

$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$

$= \sqrt{4}$

$= 2$

General Polar Form:

$2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)$

$w^2 = 2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)$

$w = \left[2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)\right]^{\frac{1}{2}}$

De Moivre:

$w = 2^{\frac{1}{2}}\left[\left(\cos\frac{1}{2}\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3} + 2n\pi\right)\right)\right]$

$= 2^{\frac{1}{2}}\left[\cos\left(\frac{\pi}{3} + n\pi\right) + i\sin\left(\frac{\pi}{3} + n\pi\right)\right]$

$n = 0$:

$w = \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

$= \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

$n = 1$:

$w = \sqrt{2}\left(\cos\left(\frac{\pi}{3} + \pi\right) + i\sin\left(\frac{\pi}{3} + \pi\right)\right)$

$= \sqrt{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$