Society of Actuaries in Ireland

Complex Numbers

1. Background

A complex number is a number that has a real part and an imaginary part. The form is:

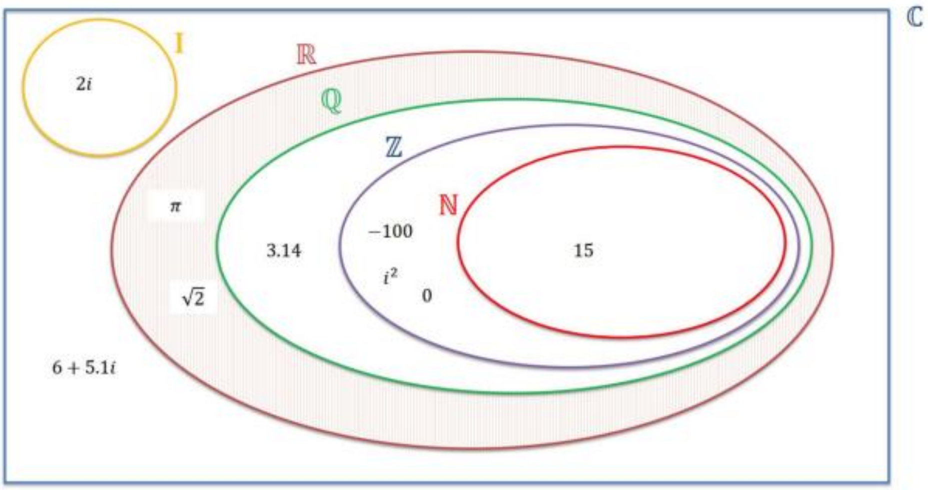
$$Z = a + bi$$

- Where a = real part and is written a = Re(z)
- b = imaginary part and is written b = Im(z)
- $i = \sqrt{-1}$, $i^2 = -1$

With imaginary numbers, we can write the square root of any negative number in terms of i

E.g. $\sqrt{-9} = \sqrt{9} * \sqrt{-1} = 3i$

2. Hierarchy of Sets of Numbers



Where N is all Natural Numbers, Z is all Integers, Q is all Rational Numbers, R is all Real Numbers, I is all Imaginary Numbers and C is all Complex Numbers.

3. Conjugates and Division

The conjugate of z is represented as $\bar{z}$ = a – bi. The sign of the imaginary part is changed. Key properties are:

- $z\bar{z} = |z|^2 = a^2 + b^2$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 * z_2} = \bar{z}_1 + \bar{z}_2$

To divide a complex number, multiply numerator and denominator by the **conjugate** of the denominator.

E.g. $z = \frac{3+2i}{5+i}$

1. Get the conjugate of the denominator = 5 – i

2. Multiply numerator and denominator by conjugate.

$$\frac{3+2i}{5+i} * \frac{5-i}{5-i} = \frac{5*(3+2i)-i(3+2i)}{5(5+i)-i(5+i)}$$

$$= \frac{15+10i-3i-2i^2}{25+5i-5i-i^2} = \frac{15+10i-3i-2(-1)}{25+5i-5i-(-1)}$$

$$= \frac{17+7i}{26} = \frac{17}{26} + \frac{7i}{26}$$

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4. Modulus

The modulus of a complex number is the distance between the number and the origin (0,0). The notation used to represent the modulus of the complex number z is |z|

If z = a+ bi then the modulus is |z| = $\sqrt{a^2 + b^2}$

5. Conjugate Roots Theorem

The conjugate roots theorem states that is a polynomial with **real** coefficients has a complex root z = a + bi, then its conjugate $\bar{z}$ = a – bi is also a root.

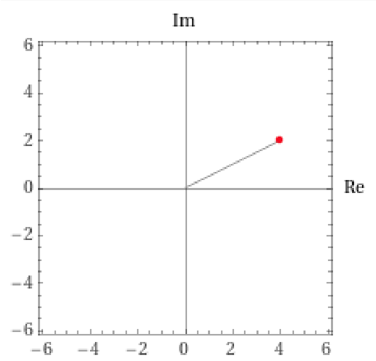
If we are given a cubic where we can work out both complex roots and asked to find the other root:

- If z = x + yi is a root of the equation $az^3 + bz^2 + cz + d = 0$, then z = x – yi is also a root.
- If z = x + yi and z = x – yi are roots, then (z – x - yi) and (z – x +yi) are factors.
- (z – x – yi)*(z – x + yi) is also a factor
- = $z^2 - xz + zyi - xz + x^2 - xyi - zyi + xyi - y^2i^2$
- = $z^2 - 2xz + x^2 + y^2$ is also a factor.
- Now using this term and dividing into $az^3 + bz^2 + cz + d$ will give the final factor and root.

6. Representing Complex Numbers

6.1 Geometric Form

We can represent complex numbers on an argand diagram where the x-axis is the Real axis and the y-axis is the imaginary axis. The numbers are represented as a point on this diagram.
E.g. 4 + 2i



6.2 Polar Form

Z = r(cosθ + i sin θ), where:

- r is the modulus, r = |z|
- θ is called the argument. The formula depends on which quadrant the complex number is in:
  - 1st quadrant: $\theta = Reference\ angle = Tan^{-1}\left(\frac{y}{x}\right)$
  - 2nd quadrant: θ = 180^0 – Reference Angle
  - 3rd quadrant: θ = (180^0 + Reference Angle) <u>or</u> - (180^0 - Reference Angle)
  - 4th quadrant: θ = (360^0 – Reference Angle) <u>or</u> -Reference Angle
- Argument is written as arg(z)

E.g. Write z = 1 + $\sqrt{3}$i in polar form

- r = $\sqrt{1^2 + (\sqrt{3})^2}$ = $\sqrt{1+3}$ = 2

- θ = $\text{Tan}^{-1}(\frac{\sqrt{3}}{1})$ = 60^0.
- z = $2(\cos 60^0 + \sin 60^0)$

7. De Moivre's Theorem

Powers Formula: $z^n = (r\cos\theta + ri\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$

Roots Formula: $z^{\frac{1}{n}} = (r\cos\theta + ri\sin\theta)^{\frac{1}{n}} = r^{\frac{1}{n}}(\cos(\frac{\theta+2k\pi}{n}) + i\sin(\frac{\theta+2k\pi}{n}))$

The key applications are for:

- Calculating powers of a complex number
- Finding the n-th roots of a complex number

E.g. Find the roots of $z^3 = 8 + 8\sqrt{3}i$.

Solution:

1. Find the modulus r:

   r = $\sqrt{a^2 + b^2}$

   r = $\sqrt{8^2 + (8\sqrt{3})^2}$ = $\sqrt{64 + 192}$ = $\sqrt{256}$ = 16.
2. Find the argument θ.

   θ = $\tan^{-1}\left(\frac{b}{a}\right)$

   θ = $\tan^{-1}\left(\frac{8\sqrt{3}}{8}\right)$ = $\tan^{-1}(\sqrt{3})$ = 60^0
3. Polar form

   $z^3 = 16(\cos 60^0 + i\sin 60^0)$
4. Find the cube roots using the roots formula

   $$(16\cos 60^0 + 16i\sin 60^0)^{\frac{1}{3}} = \sqrt[3]{16}(\cos(\frac{60^0 + (k * 360^0)}{3}) + i\sin(\frac{60^0 + (k * 360^0)}{3})) =$$

   For k = 0:

   $Z_0 = \sqrt[3]{16}(\cos(\frac{60^0+(0*360^0)}{3}) + i\sin(\frac{60^0+(0*360^0)}{3}))$

   $= \sqrt[3]{16}(\cos(\frac{60^0}{3}) + i\sin(\frac{60^0}{3}))$

   $= \sqrt[3]{16}(\cos(20^0) + i\sin(20^0))$

   $Z_1 = \sqrt[3]{16}(\cos(\frac{60^0+(1*360^0)}{3}) + i\sin(\frac{60^0+(1*360^0)}{3}))$

   $= \sqrt[3]{16}(\cos(\frac{420^0}{3}) + i\sin(\frac{420^0}{3}))$

   $= \sqrt[3]{16}(\cos(140^0) + i\sin(140^0))$

   $Z_2 = \sqrt[3]{16}(\cos(\frac{60^0+(2*360^0)}{3}) + i\sin(\frac{60^0+(2*360^0)}{3}))$

   $= \sqrt[3]{16}(\cos(\frac{780^0}{3}) + i\sin(\frac{780^0}{3}))$

   $= \sqrt[3]{16}(\cos(260^0) + i\sin(260^0))$
5. Final roots are:

   $Z_0 = \sqrt[3]{16}(\cos(20^0) + i\sin(20^0))$

   $Z_1 = \sqrt[3]{16}(\cos(140^0) + i\sin(140^0))$

   $Z_2 = \sqrt[3]{16}(\cos(260^0) + i\sin(260^0))$

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Proof by Induction

Step 1: Show that the proposition is true for $\boldsymbol{n} = \mathbf{1}$
 (insert your workings to show outcome…..)
Hence, proposition is true for n=1

Step 2: Assume that the proposition is true for $\boldsymbol{n}$=$\boldsymbol{k}$

Step 3: Prove that the proposition is true for $\boldsymbol{n} = \boldsymbol{k} + \mathbf{1}$, given that it is true for $\boldsymbol{n} = \boldsymbol{k}$.
 (insert your workings to show outcome for k+1 & how it relates to outcome for k…..)

∴ The proposition is true for $n = k + 1$, given that it is true for $n = k$.
Step 4: State that proposition is true for $\boldsymbol{n}$ = 1 and if the proposition is true for n=k, it will be true for n = $\boldsymbol{k} + \mathbf{1}$, therefore by induction it is true for all $\boldsymbol{n} \in$ N.