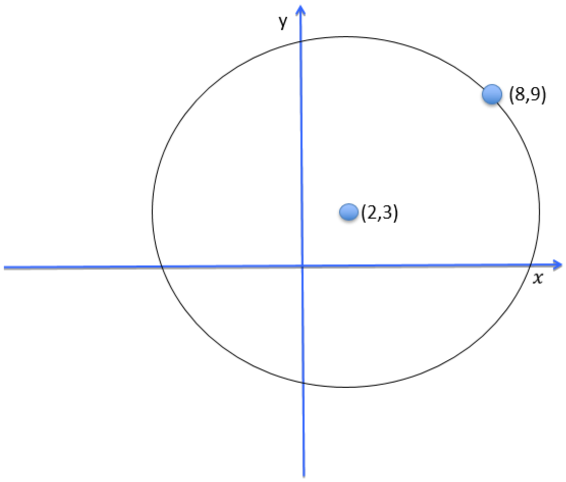
Geometry 2 Tutorial – Solutions

Q1.

a)



b) Distance between (8,9) and (2,3) is radius

$\Rightarrow \sqrt{(\mathbf{2}-\mathbf{8})^2+(\mathbf{3}-\mathbf{9})^2}$

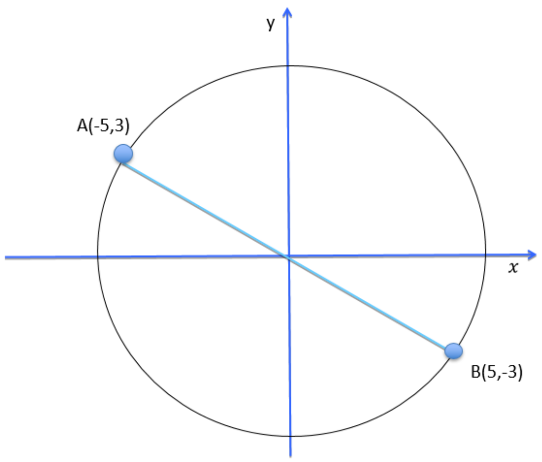
$\Rightarrow \sqrt{(-\mathbf{6})^2+(-\mathbf{6})^2} = \sqrt{\mathbf{36}+\mathbf{36}} = \sqrt{\mathbf{72}}$

c) Equation of circle with centre (h,k) and radius r is: $(x-h)^2+(y-k)^2=r^2$

$\Rightarrow (x-2)^2+(y-3)^2=\sqrt{72}^2=72$

Q2.

a)



b) Midpoint of A(-5,3) and B(5,-3) is the centre of the circle

$\Rightarrow$ Midpoint is (0,0)

c) Distance between (-5,3) and (0,0) is radius

$\Rightarrow \sqrt{(-5-0)^2+(3-0)^2}$

$\Rightarrow \sqrt{5^2+3^2}$

$\Rightarrow \sqrt{34}$

d) $x^2+y^2=r^2$

$\Rightarrow x^2+y^2=(\sqrt{34})^2$

⇒ $x^2 + y^2 = 34$

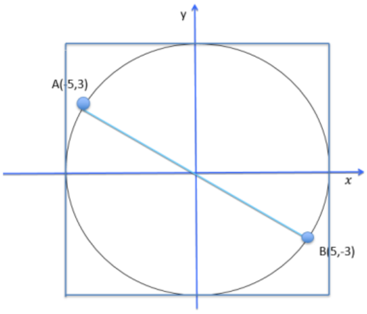
e) *Area* = πr^2

⇒ $\pi\sqrt{34}^2$

⇒ 34π

⇒ 106.81 units squared.

f)



Side of square is equal to the diameter, area of square is diameter ^{2}

⇒ *Diameter = 2r =* $2\sqrt{34}$

⇒ Area of square = $(2\sqrt{34})^2$

⇒ *136 units squared*

Q3.

a) *Solution circle c1*

$x^2 + y^2 + 2gx + 2fy + c = 0$

⇒ $x^2 + y^2 - 6x - 10y + 29 = 0$ *(Formulae p. 19)*

⇒ *2g = -6 => -g = 3*

⇒ *2f = -10 => -f = 5*

⇒ centre (3, 5)

⇒ r= $\sqrt{g^2 + f^2 - c}$, c = 29 => $\sqrt{3^2 + 5^2 - 29}$

⇒ $\sqrt{9 + 25 - 29}$ => $\sqrt{5}$

Solution circle c2

$x^2 + y^2 - 2x - 2y - 43 = 0$

⇒ 2g = -2 => -g = 1

⇒ 2f = -2 => -f = 1

⇒ centre (1,1)

⇒ r = $\sqrt{1^2 + 1^2 + 43}$

⇒ $\sqrt{45} = 3\sqrt{5}$

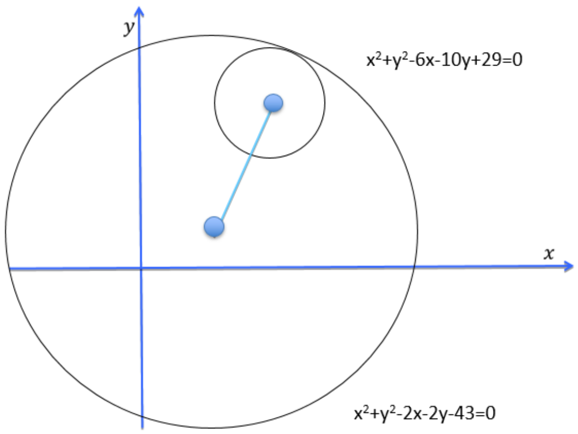
b) Distance between centres:

⇒ $\sqrt{(3-1)^2 + (5-1)^2}$

⇒ $\sqrt{20}$

⇒ $2\sqrt{5}$

⇒ The distance between the centres is the difference of the radii => circles touch (internally).



c) *Substitute (4,7) into c_1 and c_2*

$\Rightarrow$ $4^2 + 7^2 - 6(4) - 10(7) + 29 = 0$

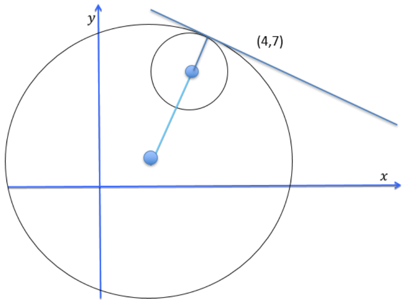
$\Rightarrow$ *(4, 7) is point on c_1*

$\Rightarrow$ $4^2 + 7^2 - 2(4) - 2(7) - 43 = 0$

$\Rightarrow$ *(4, 7) is point on c_2*

$\Rightarrow$ *Circles touch internally at (4,7)*

d)



Slope from (3, 5) to (4, 7) is: $\frac{7-5}{4-3} = 2$

$\Rightarrow$ *Slope of tangent* $= -\frac{1}{2}$ *(as perpendicular)*

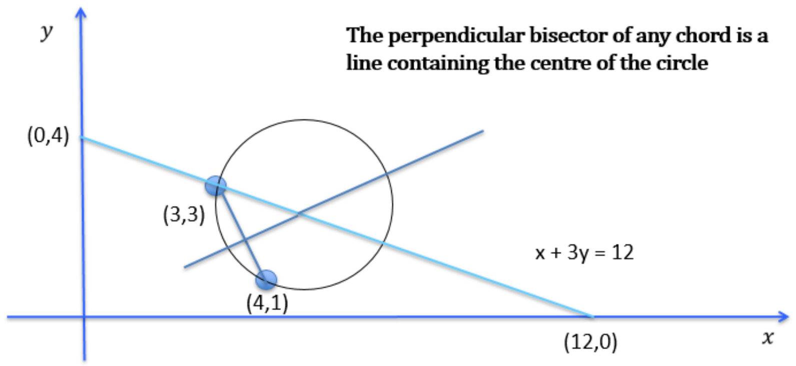
$\Rightarrow$ *Equation of tangent with slope* $-\frac{1}{2}$ *and point(4,7)*

$\Rightarrow$ $y - 7 = -\frac{1}{2}(x - 4)$

$\Rightarrow$ $2y - 14 = -x + 4$

$\Rightarrow$ $x + 2y - 18 = 0$

Q4.



The perpendicular bisector of any chord is a line containing the centre of the circle

- $\Rightarrow$ End points of chord: (3,3) and (4,1)
- $\Rightarrow$ Midpoint of Chord: (3.5, 2)
- $\Rightarrow$ slope of chord: (1-3)/(4-3) = -2
- $\Rightarrow$ slope of perpendicular of chord: $\frac{1}{2}$

Eqn of perpendicular of chord: slope = $\frac{1}{2}$, point (3.5,2)

- $\Rightarrow$ $\mathrm{y} - 2 = \frac{1}{2}(x - 3.5)$
- $\Rightarrow$ $2\mathrm{y} - 4 = x - 3.5$
- $\Rightarrow$ $\mathrm{x} - 2\mathrm{y} = -0.5$

- $\Rightarrow$ $\underline{\mathrm{x} + 3\mathrm{y} = 12}$ point of intersection is centre of circle
- $\Rightarrow$ $-5\mathrm{y} = -12.5$ *subtracting*
- $\Rightarrow$ $\mathrm{y} = 2.5;\ x = 4.5$
- $\Rightarrow$ *Centre is (4.5,2.5)*

Now, need to find the radius

- $\Rightarrow$ Distance between (4.5,2.5) and (3,3)
- $\Rightarrow$ $\mathrm{r} = \sqrt{(3 - 4.5)^2 + (3 - 2.5)^2}$
- $\Rightarrow$ $\mathrm{r} = \sqrt{2.5}$

Equation of the circle:

- $\Rightarrow$ $(x - 4.5)^2 + (y - 2.5)^2 = 2.5$
- $\Rightarrow$ $x^2 + y^2 - 9x - 5y + 24 = 0$

OR

Using equation $x^2 + y^2 + 2gx + 2fy + c = 0$.

Substitute in (3, 3):

$(3)^2 + (3)^2 + 2g(3) + 2f(3) + c = 0$

$9 + 9 + 6g + 6f + c = 0$

$6g + 6f + c = -18$ *(1)*

Substitute in (4, 1):

$(4)^2 + (1)^2 + 2g(4) + 2f(1) + c = 0$

$16 + 1 + 8g + 2f + c = 0$

$8g + 2f + c = -17$ *(2)*

We know that the line $x + 3y = 12$. The centre of the circle is $(-g, -f)$.
Substitute in $(-g, -f)$:
$(-g) + 3(-f) = 12$
$-g - 3f = 12$
$g + 3f = -12$ ***(3)***

We know have 3 equations with 3 unknowns that we can solve simultaneously.
(1) - (2):
$6g + 6f + c = -18$ *(1)*
$-8g - 2f - c = 17$ *(-2)*
$-2g + 4f = -1$ ***(4)***

We can know use equations **(3)** and **(4)** to find g and f.
2**(3)* + *(4)*
$2g + 6f = -24$ *(2*3)*
$-2g + 4f = -1$ *(4)*
$10f = -25$
$f = -5/2$

Substitute back in to **(3)** or **(4)** find g:
$g + 3f = -12$ *(3)*
$g + 3(-5/2) = -12$
$g - (15/2) = -12$
$g = -12 + (15/2)$
$g = -9/2$

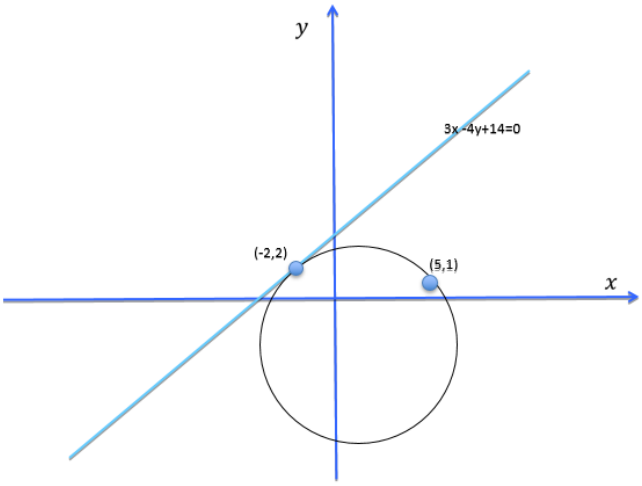
Substitute back into (1) or (2) to find c:
$6g + 6f + c = -18$ *(1)*
$6(-9/2) + 6(-5/2) + c = -18$
$-27 - 15 + c = -18$
$-42 + c = -18$
$c = -18 + 42$
$c = 24$

$g = (-9/2)$, $f = -(5/2)$, $c = 24$
Equation of the circle:
$x^2 + y^2 + 2(-9/2)x + 2(-5/2)y + (24) = 0.$
$x^2 + y^2 - 9x - 5y + 24 = 0.$

Society of Actuaries in Ireland

Q5.

a)



b) Need to find the centre of the circle to find the equation of the circle
We know the radius is perpendicular to the tangent at the point of intersection.
We know the perpendicular bisector of any chord is a line containing the centre of the circle.
The point of intersection of these two lines is the centre of the circle

Tangent: 3x -4y+14=0

$\Rightarrow$ Slope of tangent = ¾

$\Rightarrow$ slope of perpendicular = -(4/3)

Equation of perpendicular: slope -4/3, point (-2,2)

$\Rightarrow$ $y - 2 = -(\frac{4}{3})(x - (-2))$

$\Rightarrow$ $3y - 6 = -4x - 8$

$\Rightarrow$ $4x + 3y = -2$

To find perpendicular bisector of chord:

$\Rightarrow$ Midpoint (-2,2) and (5,1): (1.5, 1.5)

$\Rightarrow$ Slope of (-2,2) and (5,1): (1-2)/(5-(-2))

$\Rightarrow$ Slope: -(1/7)

$\Rightarrow$ Slope of perpendicular = 7

Equation of perpendicular: slope 7, point (1.5,1.5)

$\Rightarrow$ $y - 1.5 = 7(x - 1.5)$

$\Rightarrow$ $7x - y = 9$

Simultaneous equations

$\Rightarrow$ $7x - y = 9$

$\Rightarrow$ $4x + 3y = -2$

$\Rightarrow$ $\underline{21x - 3y = 27}$

$\Rightarrow$ Adding: $25x = 25$;

$\Rightarrow$ $x = 1$ and $y = -2$

$\Rightarrow$ centre $(1, -2)$

$\Rightarrow$ So, centre (1,-2)

Radius is distance between (1,-2) and (5,1)

$\Rightarrow$ $r = \sqrt{(5-1)^2 + (1-(-2))^2}$

$\Rightarrow$ $r = \sqrt{16 + 9}$

- ⇒ r = 5
- ⇒ Equation of circle: $(x-1)^2 + (y-(-2))^2 = 5^2$
- ⇒ $x^2 + y^2 - 2x + 4y - 20 = 0$

Q6 (2022 Paper 2, Q3)

(a)

Centre $= (1, -4)$

Radius $= \sqrt{(-1)^2 + (4)^2 - k} = 5\sqrt{3}$

$17 - k = 75$

$k = -58$

(b)

Centre = (5, -2)

Slope from centre to the point (9, -4):

(-4 – (-2)) / (9 – 5)

= - 2/4

= - ½

Slope of tangent is perpendicular to the slope of the radius at the point of tangency:

Slope of tangent = 2

(c)

Centre = (r, -r)

⇒ $(1-r)^2 + (-8-(-r))^2 = r^2$

⇒ $1 - 2r + r^2 + 64 - 16r + r^2 = r^2$

⇒ $r^2 - 18r + 65 = 0$

⇒ $(r-13)(r-5) = 0$

⇒ r = 13 or r = 5

Answers:

$(x-13)^2 + (y+13)^2 = 169$

$(x-5)^2 + (y+5)^2 = 25$

OR

$x^2 + y^2 + 2gx + 2fy + c = 0$

$1^2 + (-8)^2 + 2g(1) + 2f(-8) + c = 0$

$2g - 16f + c = -65$

f = -g and |g| = r = $\sqrt{g^2 + f^2 - c}$

$g^2 = g^2 + g^2 - c$

⇒ $g^2 = c$

⇒ $2g - 16(-g) + g^2 = -65$

⇒ $g^2 + 18g + 65 = 0$

⇒ $(g+13)(g+5) = 0$

⇒ g = -13 or g = -5

⇒ f = 13 or f = 5

⇒ r = 13 and 5

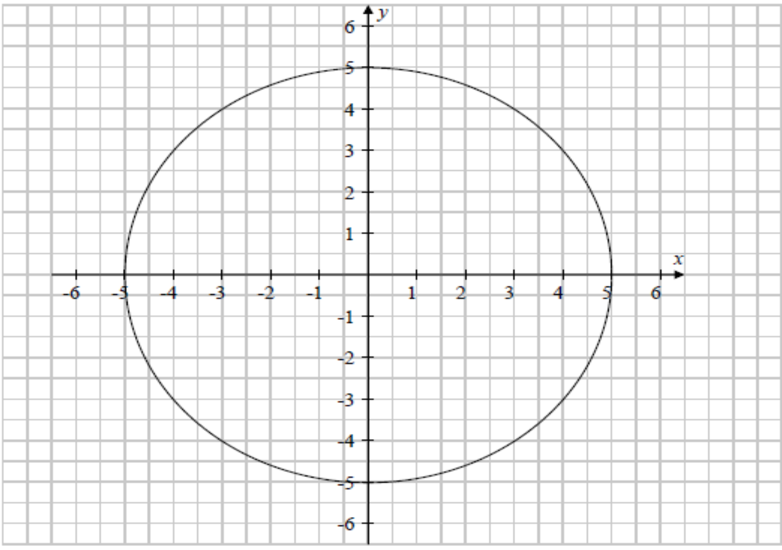
Answers:

$x^2 + y^2 - 26x + 26y + 169 = 0$

$x^2 + y^2 - 10x + 10y + 25 = 0$

Q7.

a)



(b) Verify, using algebra, that $A(-4, 3)$ is on c.

$x^2 + y^2 = 25$

$(-4)^2 + 3^2 = 16 + 9 = 25 = \text{RHS}$

or

Centre of c: $O\,(0, 0)$

$|OA| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(-4-0)^2 + (3-0)^2}$

$= \sqrt{25} = 5 =$ radius of c

(c) Find the equation of the circle with centre $(-4, 3)$ that passes through the point $(3, 4)$.

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3+4)^2 + (4-3)^2} = \sqrt{49+1} = \sqrt{50}$

$(x-h)^2 + (y-k)^2 = r^2 \Rightarrow \quad (x+4)^2 + (y-3)^2 = \left(\sqrt{50}\right)^2 = 50$

Q8.

a)

For the radius:

$$\sqrt{g^2 + f^2 - c} = \sqrt{1+1+7} = 3$$

Radius for c_2 = 3

(a) Complete the following table:

Circle	Centre	Radius	Equation
c_1	$(-3, -2)$	2	$(x+3)^2+(y+2)^2=4$ **OR** $x^2+y^2+6x+4y+9=0$
c_2	$(1, 1)$	3	$x^2+y^2-2x-2y-7=0$

(b) **(i)** Find the co-ordinates of the point of contact of c_1 and c_2.

Divide line segment joining $(-3, -2)$ and $(1, 1)$ in ratio 2 : 3

$$\left(\frac{2(1)+3(-3)}{2+3}, \frac{2(1)+3(-2)}{2+3}\right)=\left(-\frac{7}{5}, -\frac{4}{5}\right)$$

OR

Slope line of centres $=\frac{3}{4}$.

Equation line of centres: $y-1=\frac{3}{4}(x-1) \Rightarrow 3x-4y+1=0$

$c_1-c_2=4x+3y+8=0$

$4x+3y+8=0 \cap 3x-4y+1=0 \Rightarrow x=-\frac{7}{5}, y=-\frac{4}{5}$

(ii) Hence, or otherwise, find the equation of the tangent, t, common to c_1 and c_2.

Slope of line of centres: $\frac{1+2}{1+3}=\frac{3}{4}$

Slope of tangent: $m=-\frac{4}{3}$

Equation of tangent: $y+\frac{4}{5}=-\frac{4}{3}\left(x+\frac{7}{5}\right)$

$\Rightarrow 3y+\frac{12}{5}=-4x-\frac{28}{5}$

$\Rightarrow 4x+3y+8=0$

OR

$c_1-c_2=x^2+y^2+6x+4y+9-(x^2+y^2-2x-2y-7)=0$

$\Rightarrow 6x+4y+9-(-2x-2y-7)=0$

$\Rightarrow 8x+6y+16=0 \Rightarrow 4x+3y+8=0$

OR

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

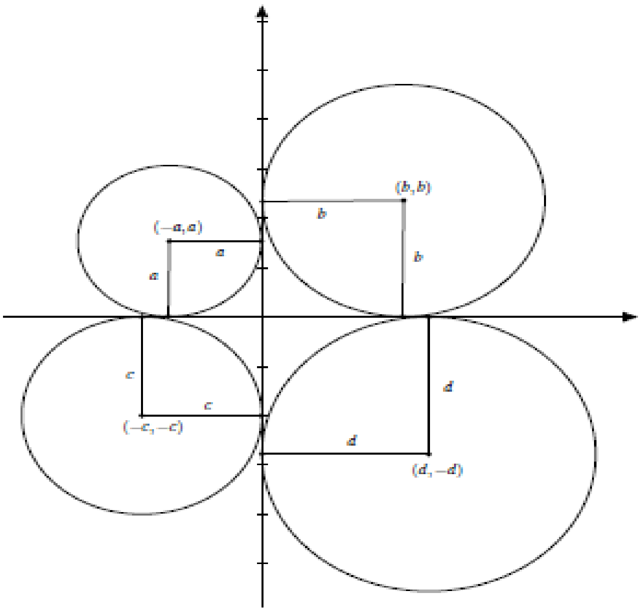
$x(-\frac{7}{5})+y(-\frac{4}{5})+3(x+(-\frac{7}{5}))+2(y+(-\frac{4}{5}))+9=0$

$\Rightarrow 4x+3y+8=0$

Q9.

The centre of a circle lies on the line $x+2y-6=0$. The x-axis and the y-axis are tangents to the circle. There are two circles that satisfy these conditions. Find their equations.

Consider the diagram below:



Note that this diagram is not meant to represent the solution. It is just meant to illustrate the possibilities for circles that are tangent to both axes. In the diagram, a, b, c and d are the radii of the circles. We can see from this diagram that if (x,y) is the centre of a circle that has both the x-axis and the y-axis as tangents, then either

- Case 1: $y=x$
- Case 2: $y=-x$

In either case the radius is either x or $-x$. Since the radius is positive, we can say that the radius is $|x|$ in either case...

Case 1: $y=x$. We are also told that $x+2y-6=0$. Subsituting x for y in the latter equation gives

$$x+2x-6=0 \Leftrightarrow 3x-6=0 \Leftrightarrow x=2.$$

Now $y=x$ so $y=2$. Therefore the centre of the circle has co-ordinates $(2,2)$ and the radius is 2. Therefore in this case the circle has equation

$$(x-2)^2+(y-2)^2=4.$$

Case 2: $y=-x$. As before we use this to substitute $-x$ for y in the equation $x+2y-6=0$. This gives

$$x+2(-x)-6=0 \Leftrightarrow -x-6=0 \Leftrightarrow x=-6.$$

It follows that $y=-(-6)=6$. So in this case the centre has co-ordinates $(-6,6)$ and the radius is 6. So this circle has equation

$$(x+6)^2+(y-6)^2=36.$$

Question 10 **(2023 Paper 2, Question 9 (c))**

(i)

If $|\angle QOP|$=45° and in the second quadrant, then for the line OQ:

m = -1 and

$y=-x$

Find where the Circle: $x^2 + y^2 = 1$ and the line $y=-x$ intersect:-

∴ $x^2 + x^2 = 1$

$2x^2 = 1$

$x = \pm 1/\sqrt{2}$

But x in 2nd quadrant. ∴ $x = -^1/_{\sqrt{2}}$, and $y = ^1/_{\sqrt{2}}$.

(ii)

Consider both Tangent Lines and where they intersect.

Tangent at P

$P = (-1,0)$ Tangent at P: $x = -1$

Tangent at Q

$Q = (-^1/_{\sqrt{2}}, ^1/_{\sqrt{2}})$

Slope of tangent at Q =1 (using the slope of the line OQ and multiplying it by -1

∴ Tangent at Q: $y - ^1/_{\sqrt{2}} = 1(x -- ^1/_{\sqrt{2}})$

=> $y = x + \sqrt{2}$

At $x = -1$: $y = -1+\sqrt{2}$

∴ Centre $=(-1, -1+\sqrt{2})$

And Radius $= -1+\sqrt{2}$