Geometry 1 Tutorial - Solutions

Q1

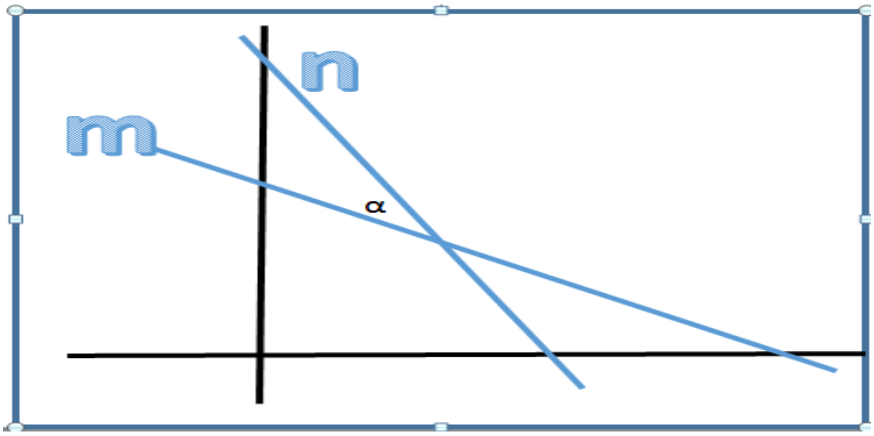
Hint: Rearrange equation to equation of a line i.e. y = mx + c

Line	Equation	y = mx + c	m
h	x = 3 – y	y = -x + 3	-1
i	2x – 4y = 3	$y = \frac{1}{2}x - \frac{3}{4}$	$\frac{1}{2}$
k	$y = -\frac{1}{4}(2x - 7)$	$y = -\frac{1}{2}x + \frac{7}{4}$	$-\frac{1}{2}$
l	4x – 2y – 5 = 0	$y = 2x - \frac{5}{2}$	2
m	$X + \sqrt{3}y - 10 = 0$	$y = -\left(\frac{1}{\sqrt{3}}\right)x + \frac{10}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$
n	$\sqrt{3}x + y - 10 = 0$	$y = -\sqrt{3}x + 10$	$-\sqrt{3}$

(a) Complete table below by matching each description given to one or more of the lines.

Description	Line(s)
A line with a slope of 2	l
A line which intersects the y-axis at $(0, -2\frac{1}{2})$	l
A line which makes equal intercepts on the axes	h …. (0, 3) & (3, 0)
A line which makes an angle of 150° with the positive sense of the x-axis	m …. $\tan 150° = -\frac{1}{\sqrt{3}}$
Two lines which are perpendicular to each other	k, l …. m1 * m2 = -1

(b) Hint: Check P19 of your log tables for a formula.



Society of Actuaries in Ireland

Slopes of lines were calucalted above as m = $-\frac{1}{\sqrt{3}}$ and n = $-\sqrt{3}$

Tan α = $\frac{m1 - m2}{1 + m1m2}$ or $\frac{m2 - m1}{1 + m1m2}$ .... Page 19 of tables

Tan α = $\frac{-\frac{1}{\sqrt{3}} - (-\sqrt{3})}{1 + (-\frac{1}{\sqrt{3}})(-\sqrt{3})}$ or $\frac{-\sqrt{3} - (-\frac{1}{\sqrt{3}})}{1 + (-\frac{1}{\sqrt{3}})(-\sqrt{3})}$ ... discard -neg

Tan α = $\frac{1.154701}{2}$ or $\frac{2}{2\sqrt{3}}$

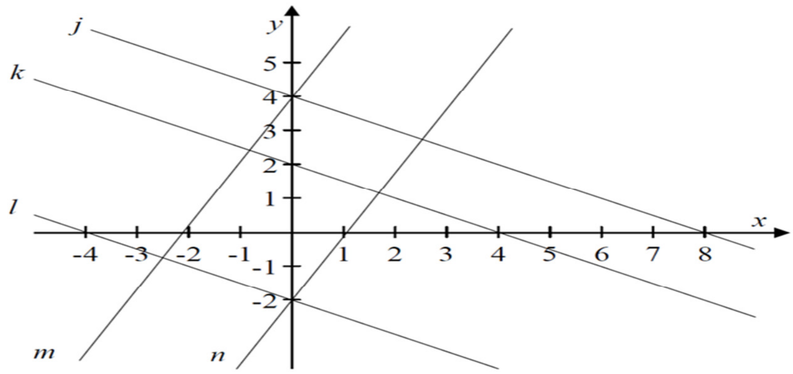
Tan α = 0.57735 or $\frac{1}{\sqrt{3}}$

Answer: α = 30°

Q 2(a)

Line	Equation	y = mx + c	Cuts y-axis	Cuts x-axis
l	x + 2y = -4	y = $-\frac{1}{2}x$ - 2	(0, -2)	(-4, 0)
m	2x – y = -4	y = $2x$ + 4	(0, 4)	(-2, 0)
j	x + 2y = 8	y = $-\frac{1}{2}x + 4$	(0, 4)	(8, 0)
n	2x – y = 2	y = 2x - 2	(0, -2)	(1, 0)

(b) Scale is 6mm per unit – add the numbers to the diagram



(c)

Intercepts for k are (0, 2) and (4,0) ... from observation

Slope of k is $\frac{(y2 - y1)}{(x2 - x1)} = \frac{(0 - 2)}{(4 - 0)} = -\frac{1}{2}$

Equation of k: (y – y1) = m(x – x1)

(y – 2) = $-\frac{1}{2}$(x – 0)

2y – 4 = -x

x + 2y – 4 = 0

Society of Actuaries in Ireland

Q 3(a)

Hints: *- Find the slope of AC*

- Turn the fraction upside-down and multiply by -1

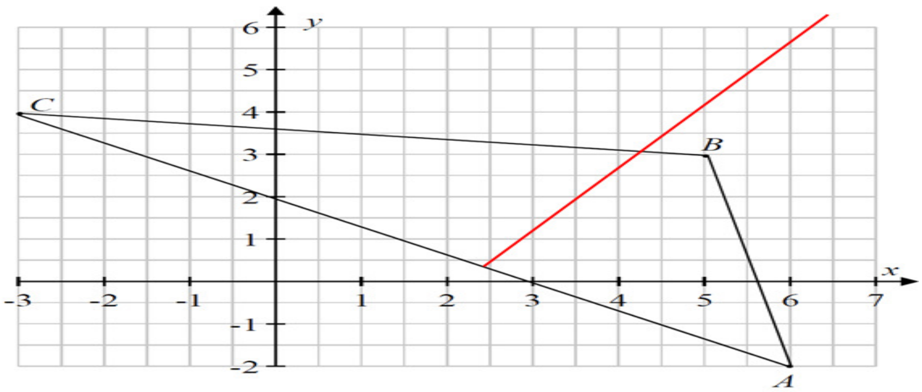
- This gives you the slope of a perpendicular line to AC

Slope of $|AC| = \frac{(y2 - y1)}{(x2 - x1)} = \frac{(-2 - 4)}{(6 - (-3))} = \frac{-6}{9} = -\frac{2}{3}$

Slope of Perpendicular line through B is $\frac{3}{2}$

Use $(y - y1) = m\,(x - x1)$ to find equation of line where $(x1, y1)$ is $(5, 3)$ and m is $\frac{3}{2}$

$$(y - 3) = \frac{3}{2}(x - 5)$$
$$2y - 6 = 3x - 15$$
$$3x - 2y - 9 = 0$$



(b) Hints: *- Find the equation of the line through C which is perpendicular to AB*

- Use your answer from (a) and simultaneous equations to get the answer

Slope of $|BC| = \frac{(y2 - y1)}{(x2 - x1)} = \frac{(4 - 3)}{(-3 - 5)} = \frac{1}{-8} = -\frac{1}{8}$

Slope of Perpendicular line through A is 8

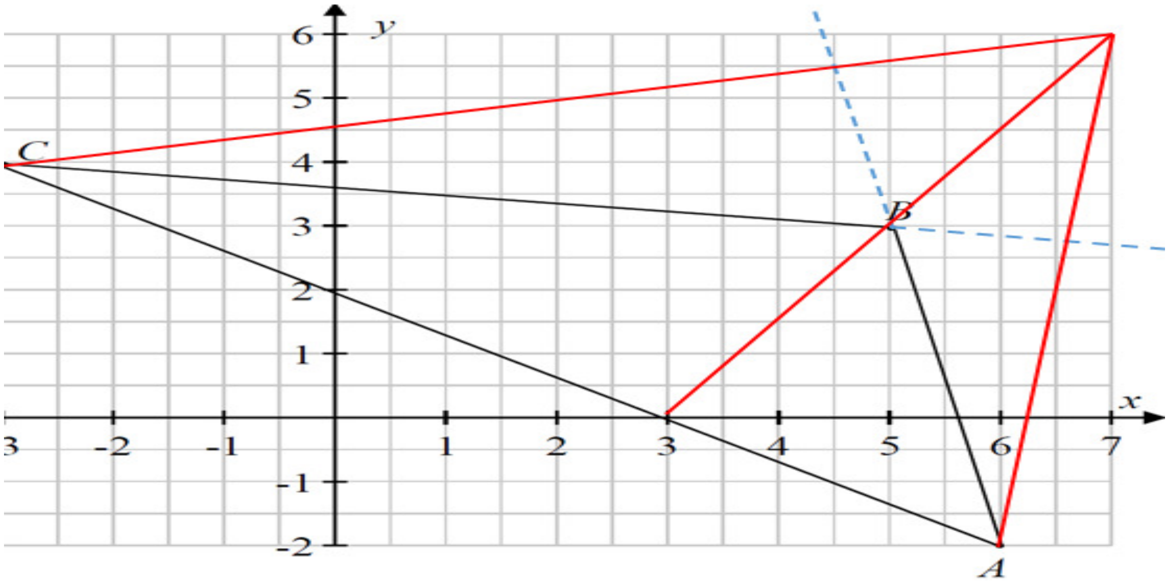
Use $(y - y1) = m\,(x - x1)$ to find equation of line where $(x1, y1)$ is $(6, -2)$ and m is 8

$$(y - (-2)) = 8\,(x - 6)$$
$$y + 2 = 8x - 48$$
$$8x - y - 50 = 0$$

Solve the simultaneous equations

$8x - y - 50 = 0$ and $3x - 2y - 9 = 0$

Point of intersection is $(7, 6)$ = orthocentre



Q4 (a)

Slope of |AB| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(-6-2)}{(6-2)} = \frac{-8}{4} = -2$

Use (y – y1) = m (x – x1) to find equation of line |AB| where (x1, y1) is (2, 2) and m is -2

$(y - 2) = -2\,(x - 2)$
$y - 2 = -2x + 4$
$2x + y - 6 = 0$

(b)

Rewrite |AB| in the format y = mx + c

$2x + y - 6 = 0$
$y = -2x + 6$

c (the y-axis intercept) = 6 OR y = 6 when x = 0

So D is the point (0, 6)

(c)

Use the formula $\frac{|ax1+by1+c|}{\sqrt{a^2+b^2}}$

(x1, y1) is C(-2, -3) and the line (ax +by + c =0) is 2x + y – 6= 0

$$\frac{|2(-2)+1(-3)-6|}{\sqrt{2^2+1^2}} = \frac{13}{\sqrt{5}}$$

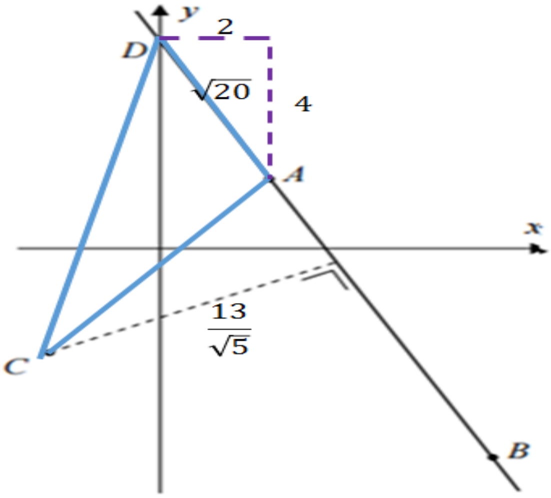
(d)

Area of Triangle = Half the base by the perpendicular height

Base = |AD| = $\sqrt{(x2-x1)^2+(y2-y1)^2} = \sqrt{20}$

Perpendicular Height = $\frac{13}{\sqrt{5}}$

Area = $\frac{1}{2} * \sqrt{20} * \frac{13}{\sqrt{5}}$ = 13 square units



Q5 (a)

Hint: Area = ½ base x perpendicular height

Use the formula for area on page 18 of the tables OR
Use the triangle area formula "half the base by the perpendicular height"

$$\frac{1}{2}\ |OR|.10 = \frac{125}{3}$$

$$|OR| = \frac{25}{3}$$

$$R\ (-\frac{25}{3}, 0)$$

(b)

METHOD 1: Get the equation for the line |RS| and show that E is on the line

Slope of |RS| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(10-0)}{(0-(-\frac{25}{3}))} = \frac{10}{(\frac{25}{3})} = \frac{30}{25} = \frac{6}{5}$

Equation of |RS| is (y – y1) = m (x – x1) .....use $\frac{6}{5}$ for m and (0, 10) for (x1, y1)

$(y-10) = \frac{6}{5}(x-0)$ ➡ $5y - 50 = 6x$ ➡ Equation of |RS| is $6x - 5y + 50 = 0$

Put E (-5, 4) into the equation of the line
6(-5) -5(4) + 50 = - 30 – 20 + 50 = 0 .... So E is on the line |RS|

Society of Actuaries in Ireland

METHOD 2: Show that the slope of |RE| = slope of |ES|

Slope of |ES| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(10-4)}{(0-(-5))} = \frac{6}{5}$

Slope of |RE| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(4-0)}{(-5-\left(-\frac{25}{3}\right))} = \frac{4}{\frac{10}{3}} = \frac{12}{10} = \frac{6}{5}$

So E is on the line |RS|

(c)

y = mx + c

The point E(-5, 4) is on this line, so substituting for x and y

4 = -5m + c .... c = 4 + 5m .... A

This line cuts the y-axis at (0, c) and the x-axis at $(-\frac{c}{m}, 0)$

The area of the triangle is $\frac{125}{3}$

This equals $\frac{1}{2}\left|x1y2 - x2y1\right| = \frac{1}{2}\left|0 - c(-\frac{c}{m})\right| = \frac{1}{2}\left|\frac{c^2}{m}\right|$

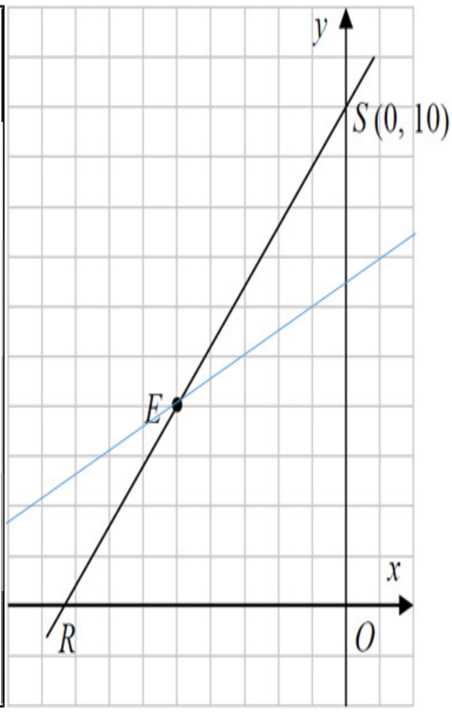
Substituting from A above $\frac{125}{3} = \frac{1}{2}\left|\frac{(4+5m)^2}{m}\right|$

$250m = 75m^2 + 120m + 48$

$75m^2 - 130m + 48 = 0$

$(5m - 6)(15m - 8) = 0$ .... $(5m - 6)$ relates to the line |RS|, so

$m = \frac{8}{15}$ and $c = 4 + 5\left(\frac{8}{15}\right) = \frac{20}{3}$



Q6 (a) *Hint: Find both slopes.*

Find the slope of L1: 3x – 4y -12 = 0

3x – 4y – 12 = 0

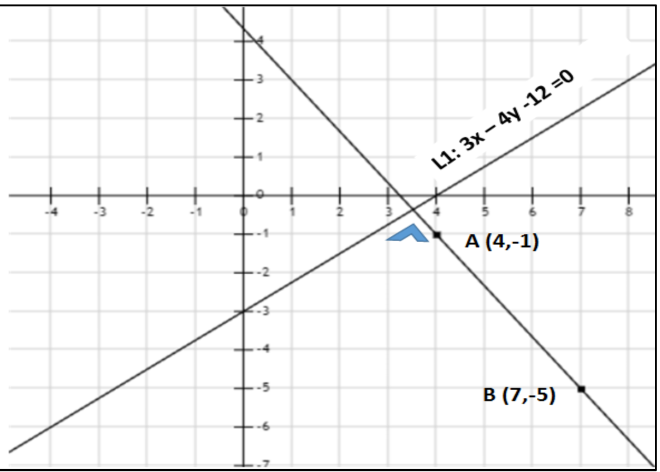
4y = 3x – 12

y = $\frac{3}{4}x - 3$ .... $slope\ of\ L1 = \frac{3}{4}$

|AB| $perpendicular\ to\ L1\ so\ slope\ of\ |AB| is\ -\frac{4}{3}$

Slope of |AB| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(t-(-1))}{(7-4)} = \frac{(t+1)}{(3)}$

$\frac{(t+1)}{(3)} = -\frac{4}{3}$ .... t + 1 = -4 ...... t = -5



Q6 (b) *Hint: Use the perpendicular distance formula*

Use the formula for the perpendicular distance from point (x1, y1) to line ax +by + c =0

$$\frac{|ax1 + by1 + c|}{\sqrt{a^2 + b^2}}$$

(x1, y1) is (10, k) and the line (ax +by + c =0) is 3x – 4y – 12 = 0

$$\frac{|3(10) - 4(k) - 12|}{\sqrt{3^2 + 4^2}}$$

$$\frac{|18 - 4k|}{5}$$

(c) (i)

Use the formula for the perpendicular distance to get the distance in terms of k from P to l2

(x1, y1) is (10, k) and the line (ax +by + c =0) is 5x + 12y – 20 = 0

$$\frac{|5(10) + 12(k) - 20|}{\sqrt{5^2 + 12^2}}$$

$$\frac{|30 + 12k|}{13}$$

P is equidistant from l1 and l2

So $\frac{(18 - 4k)}{5} = \frac{(30+12k)}{13}$ OR $\frac{(18 - 4k)}{5} = -\frac{(30+12k)}{13}$

$k = \frac{3}{4}$ OR $k = -48$

(ii)

k > 0 so $k = \frac{3}{4}$

Use one of the previous results and insert the value for k

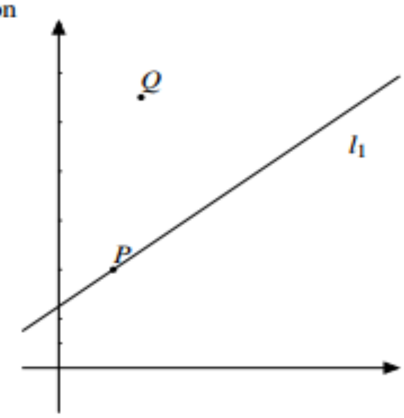
Perpendicular distance

$= \frac{(18 - 4k)}{5} = \frac{(18 - 4(\frac{3}{4}))}{5} = \frac{(18 - 3)}{5} = 3$

Society of Actuaries in Ireland

Q7

(a) Show that, for all $k \in \mathbb{R}$, the point $P(4k-2, 3k+1)$ lies on the line $l_1 : 3x - 4y + 10 = 0$.



If $(x, y) = (4k-2, 3k+1)$ then

$$\begin{aligned} 3x - 4y + 10 &= 3(4k-2) - 4(3k+1) + 10 \\ &= 12k - 6 - 12k - 4 + 10 \\ &= 0 \end{aligned}$$

So the equation of l_1 is satisfied. Therefore $(4k-2, 3k+1)$ lies on l_1.

(b) The line l_2 passes through P and is perpendicular to l_1. Find the equation of l_2 in terms of k.

We have

$$\begin{aligned} 3x - 4y + 10 &= 0 \\ &\Leftrightarrow \\ -4y &= -3x - 10 \\ &\Leftrightarrow \\ y &= \frac{3}{4}x + \frac{5}{2} \end{aligned}$$

Therefore the slope of l_1 is $\frac{3}{4}$. Therefore the slope of l_2 is $\frac{1}{\frac{3}{4}} = -\frac{4}{3}$. So l_2 has slope $-\frac{4}{3}$ and passes through $(4k-2, 3k+1)$. So it has equation

$$y - (3k+1) = -\frac{4}{3}(x - (4k-2))$$

or

$$3y - 3(3k+1) = -4(x - (4k-2)).$$

Rearranging this gives

$$4x + 3y - 25k + 5 = 0.$$

(c) Find the value of k for which l_2 passes through the point $Q(3, 11)$.

The equation of l_2 is

$$4x + 3y - 25k + 5 = 0.$$

Now $(3, 11)$ lies on l_2 if and only if $4(3) + 3(11) - 25k + 5 = 0 \Leftrightarrow 25k = 50 \Leftrightarrow k = 2$. So the $k = 2$ is the required value.

(d) Hence, or otherwise, find the co-ordinates of the foot of the perpendicular from Q to l_1.

When $k = 2$ the equation of l_2 is

$$4x + 3y - 45 = 0.$$

So to find the required point, we solve

$$\begin{aligned} 3x - 4y + 10 &= 0 \\ 4x + 3y - 45 &= 0 \end{aligned}$$

simultaneously.
This is equivalent to

$$\begin{aligned} 12x - 16y + 40 &= 0 \\ 12x + 9y - 135 &= 0 \end{aligned}$$

Subtracting yields

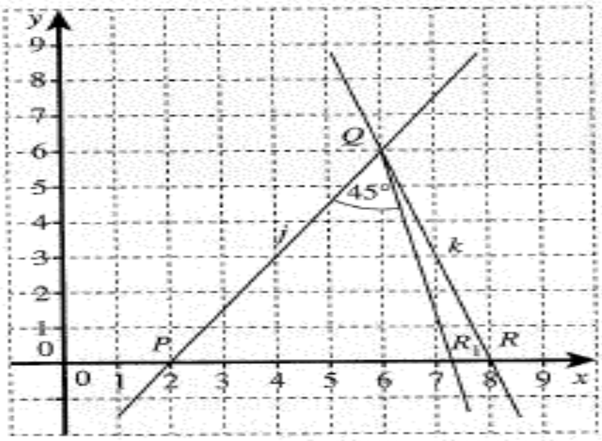
$$-25y + 175 = 0.$$

Therefore $25y = 175$ and $y = \frac{175}{25} = 7$.
Now $3x - 4(7) + 10 = 0 \Leftrightarrow 3x = 4(7) - 10 = 18 \Leftrightarrow x = 6$. So the foot of the perpendicular from Q to l_1 has co-ordinates $(6, 7)$.

Q8

Solutions

Start by drawing a graph to visualise the problem.



(a) Slope of line $j = \frac{-a}{b} = \frac{-(3)}{(-2)} = \frac{3}{2}$, slope of line $k = \frac{-a}{b} = \frac{-(3)}{(1)} = -3$

$$\tan\theta = \left|\frac{\left(\frac{3}{2}\right) - (-3)}{1 + \left(\frac{3}{2}\right)(-3)}\right| = \left|\frac{\frac{3}{2} + 3}{1 - \frac{9}{2}}\right| = \left|\frac{\frac{9}{2}}{\frac{-7}{2}}\right| = \left|-\frac{9}{7}\right| = \frac{9}{7} \Rightarrow \theta = \tan^{-1}\frac{9}{7} \approx 52{\cdot}1°$$

This gives the acute angle. The obtuse angle is $180 - 52{\cdot}1 = 127{\cdot}9°$.

(b) $|\angle PQR| = 52{\cdot}1°$

(c) Slope of PQ = slope of $j = \frac{3}{2}$. Let slope of $QR_1 = m$.

Use the formula to find the angle between two lines. Either slope could be m_1 and m_2 so there are two options:

$$\tan 45° = 1 = \left|\frac{\frac{3}{2} - m}{1 + \frac{3}{2}m}\right| \text{ or } \tan 45° = 1 = \left|\frac{m - \frac{3}{2}}{1 + \frac{3}{2}m}\right|.$$

$$1 = \left|\frac{\frac{3}{2} - m}{1 + \frac{3}{2}m}\right| \Rightarrow 1 + \frac{3}{2}m = \frac{3}{2} - m \Rightarrow \frac{5}{2}m = \frac{1}{2} \Rightarrow m = \frac{1}{5}$$

$$1 = \left|\frac{m - \frac{3}{2}}{1 + \frac{3}{2}m}\right| \Rightarrow 1 + \frac{3}{2}m = m - \frac{3}{2} \Rightarrow \frac{3}{2}m - m = -\frac{3}{2} - 1 \Rightarrow m = -5$$

From the diagram, the slope of QR_1 is negative, so $m = -5$. We know one point on the line: $(x_1, y_1) = Q = (6, 6)$.

Equation of the line: $y - y_1 = m(x - x_1) \Rightarrow y - (6) = (-5)(x - (6))$

$\Rightarrow 5x + y - 36 = 0$

x intercept: $y = 0 \Rightarrow 5x + (0) - 36 = 0 \Rightarrow x = \frac{36}{5} = 7{\cdot}2$

The co-ordinates of R_1 are $(7{\cdot}2, 0)$.

Question 9 - Answer

Q1	Model Solution – 25 Marks
(a)	Slope of BC $m = \frac{3+12}{-4-6} = -\frac{3}{2}$ Equation BC $3x + 2y + 6 = 0.$ Perp. Distance from A to line BC $\frac{3(2)+2(-6)+6}{\sqrt{3^2+2^2}} = \frac{6-12+6}{\sqrt{13}} = \frac{0}{\sqrt{13}} = 0.$ Therefore A, B and C are collinear.

(b)	
	Slope of $a = \frac{1}{2}$ Slope of $b = \tan 60° = \sqrt{3}$ $\tan\theta = \pm\frac{\sqrt{3}-\frac{1}{2}}{1+\frac{\sqrt{3}}{2}} = \pm\frac{2\sqrt{3}-1}{2+\sqrt{3}}$ $= \pm\frac{(2\sqrt{3}-1)(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$ $= \pm(-8+5\sqrt{3})$ $\theta = \tan^{-1}(-8+5\sqrt{3})$ $\theta = 33{\cdot}435°$ **Or** $\theta + \tan^{-1}\frac{1}{2} + 120° = 180°$ $\theta + 26{\cdot}565° + 120° = 180°$ $\theta = 33{\cdot}435°$

Questions 10 - Answer

Q3	Model Solution – 30 Marks
(a)	**Method 1** $(4, 6), (-3, -1), (0, 11)$. -11 $(4, -5), (-3, -12), \ (0, 0)$ AREA $= \frac{1}{2}\lvert 4(-12) - (-3)(-5)\rvert$ $= \frac{1}{2}\lvert -63\rvert$ $= 31 \cdot 5$ **OR** **Method 2** Uses any one of the following formulae: 1. Area = $\frac{1}{2} \boldsymbol{ab\sin c}$ 2. Area $= \frac{1}{2} \times \textbf{base} \times \textbf{perpendicular height}$

Q3	Model Solution – 30 Marks
(b) (i)	Mid-point $= \left(\frac{-1+5}{2}, \frac{k+l}{2}\right)$ $= \left(2, \frac{k+l}{2}\right)$ **OR** -1 to 5 is 6 steps, then $x = -1 + 3 = 2$ k to l is $(l - k)$steps , then $y = k + \frac{l-k}{2} = \frac{k+l}{2}$ Mid-point $= \left(2, \frac{k+l}{2}\right)$

Q3	Model Solution – 30 Marks
(b)(ii)	Slope $AB = \frac{l-k}{5-(-1)} = \frac{l-k}{6}$ Perpendicular slope $= -\frac{6}{l-k}$ Slope of $3x + 2y - 14 = 0$ is $-\frac{3}{2}$ $-\frac{6}{l-k} = -\frac{3}{2}$ so $\boldsymbol{l - k = 4}$ ... Eqn 1 or Slope $AB = \frac{2}{3}$, then $(-1, k)$ and $(5, l) \in$ $y = mx + c$, also gives $\boldsymbol{l - k = 4}$ ... Eqn 1 $\left(2, \frac{k+l}{2}\right) \rightarrow 3(2) + 2\left(\frac{k+l}{2}\right) - 14 = 0$ $\boldsymbol{k + l = 8}$ ... Eqn 2 $\begin{cases} l + k = 8 \\ l - k = 4 \end{cases}$ $2l = 12$ ..... $l = 6$, $k = 2$ **OR** Slope $AB = \frac{l-k}{6}$ and perpendicular $= -\frac{6}{l-k}$ Eqn of perp bisector: $y - \frac{k+l}{2} = -\frac{6}{l-k}(x - 2)$ $2y - k - l + \frac{12}{l-k}x - \frac{24}{l-k} = 0$ or $\frac{12}{l-k}x + 2y - k - l - \frac{24}{l-k} = 0$ Equating coefficients: x: $\frac{12}{l-k} = 3$ so $4 = l - k$ ... **Eqn 1** Const.: $-k - l - \frac{24}{l-k} = -14$ $-k - l - \frac{24}{4} = -14$ So $k + l = 8$ ... **Eqn 2** Solve for $l = 6, k = 2$

Question 11 - Answer

Q2	Model Solution – 30 Marks
(a)	$C = \left(\frac{1(8)+4(-1)}{4+1}, \frac{1(-4)+4(3)}{4+1}\right)$ $C = \left(\frac{4}{5}, \frac{8}{5}\right)$ **OR** x: 9 steps back [8 to -1] So $x_C = 8 - \frac{4}{5}(9) = \frac{4}{5}$ y: 7 steps up [-4 to 3] So $y_C = -4 + \frac{4}{5}(7) = \frac{8}{5}$ $C = \left(\frac{4}{5}, \frac{8}{5}\right)$
(b)	From y-intercept to (q, r): Run $= q$, so rise $= qm$, so y-value $= r - qm$ Answer: $(0, r - qm)$ **OR** $y = mx + c$ So $r = mq + c$ So $c = r - mq$ Answer: $(0, -mq + r)$ **OR** $y - y_1 = m(x - x_1)$ $y - r = m(x - q)$ $y - r = mx - mq$ $y = mx - mq + r$ $x = 0$, so $y = -mq + r$ Answer: $(0, -mq + r)$

Q2	Model Solution – 30 Marks
(c)	$\tan 30° = \frac{-2-m_2}{1+(-2)m_2}$ So $\frac{1}{\sqrt{3}} = \frac{-2-m_2}{1-2m_2}$ $1 - 2m_2 = -2\sqrt{3} - \sqrt{3}m_2$ $m_2 = \frac{1+2\sqrt{3}}{2-\sqrt{3}}$ $m_2 = \frac{1+2\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$ $m_2 = 8 + 5\sqrt{3}$ **OR** $\frac{1}{\sqrt{3}} = \frac{-(-2-m_2)}{1+(-2)m_2}$ $1 - 2m_2 = 2\sqrt{3} + \sqrt{3}m_2$ $m_2 = \frac{1-2\sqrt{3}}{2+\sqrt{3}}$ $m_2 = \frac{1-2\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$ $m_2 = 8 - 5\sqrt{3}$