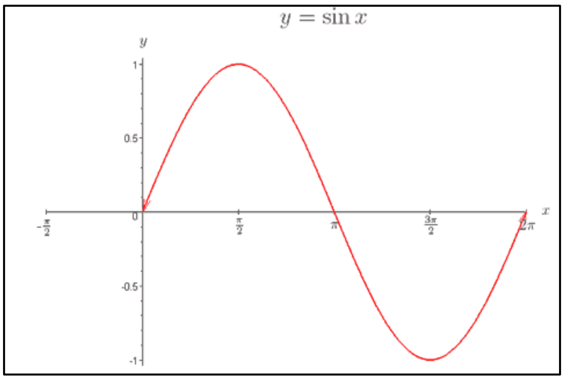
Trigonometry 2

1. Graphing Trigonometric Functions

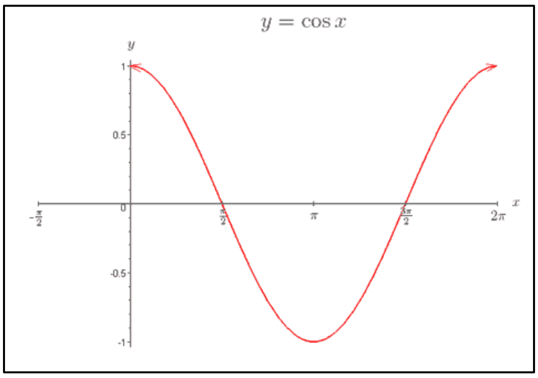
1.1 Basic Sin, Cos, Tan Graphs

You need to be able to draw the below graphs and also know about the range and period.

Sin Graph



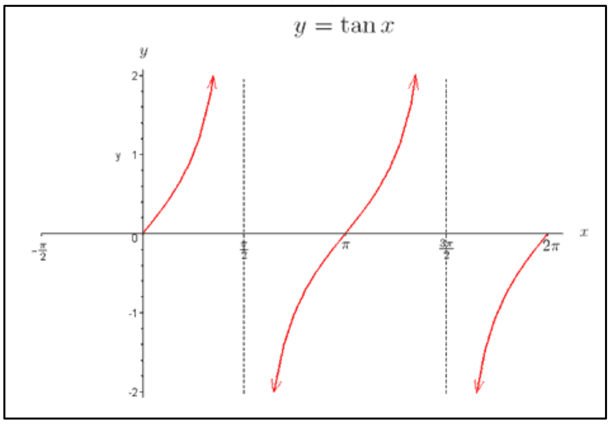
Cos Graph



Range = [-1,1]

Period (how frequently the graph repeats) = 2π radians or 360^0.

Tan Graph



Dotted lines are called asymptotes (lines the graph will never touch).

Range = [-∞,∞]

Period = π radians or 180^0.

1.2 More Advanced Graphs (Sin and Cos only)

1.2.1 Multiplying the function by a constant

- This will **change** the **range** of the function but not the period.
- The range of n*sin(x) or n*cos(x) will be [-n,n].

E.g. the range of y = 2sin(x) is [-2,2].

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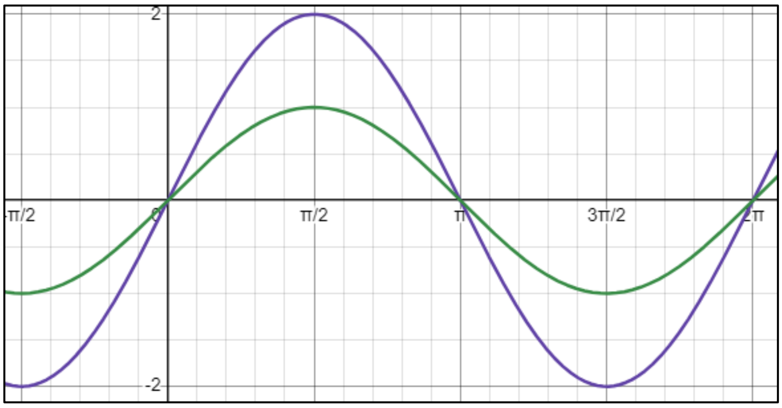
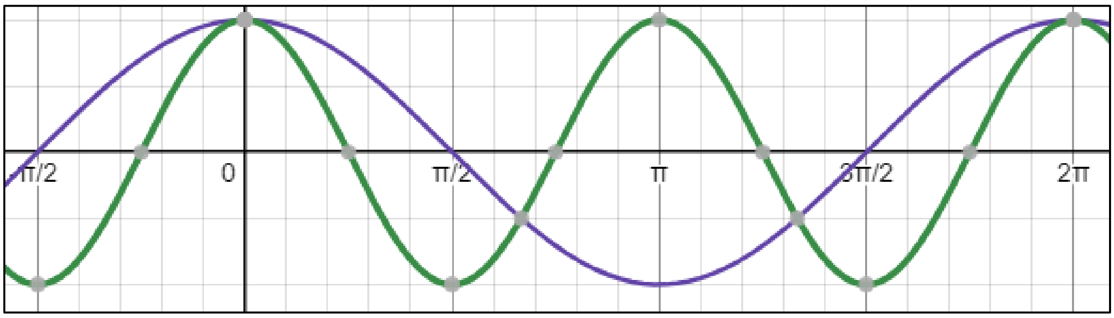


The purple line is y = 2sin(x) and the green line is y = sin(x). The period remains the same for both.

1.2.2 Multiplying the x value by a constant

- This will **change** the **period** of the function but not the range.
- To find the new period, divide by the number in front of the x.
- The period of sin nx or cos nx will be $\frac{2\pi}{n}$ radians or $\frac{360^0}{n}$

E.g. if y = cos 2x, we divide by 2 (number in front of x is 2). The period of cos 2x would be $\frac{2\pi}{2}$ radians = π radians or 180^0

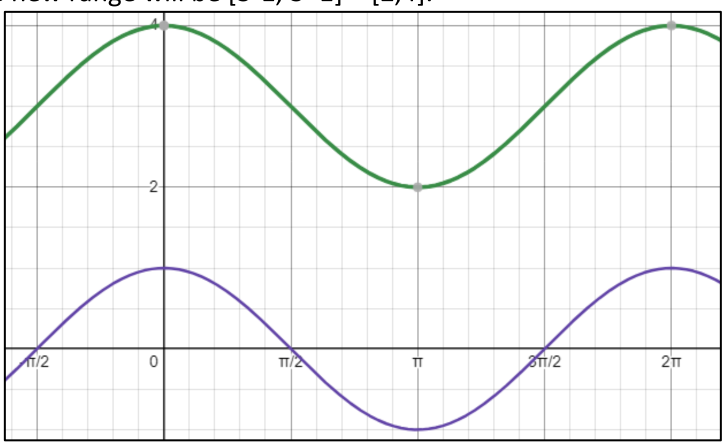


The purple line is y = cos x and the green line is y = cos 2x. As you can see the green line repeats twice as often.

1.2.3 Addition of a constant

- This will shift the graph up or down depending on the number.
- The **range** will **change** but not the period.
- The range of n + Sin(x) or n + cos(x) will be [n-1, n+1].

E.g. for 3+cos(x), the new range will be [3-1, 3+1] = [2,4].



The purple line is y=cos(x) and the green line is 3+cos(x).

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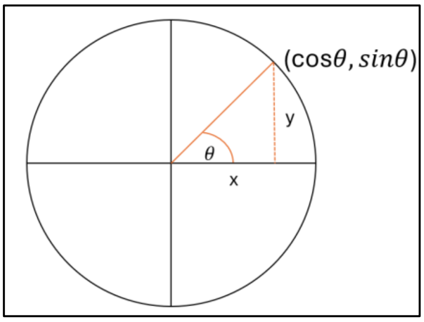
Trigonometry 2

2. Solving Trigonometric Equations

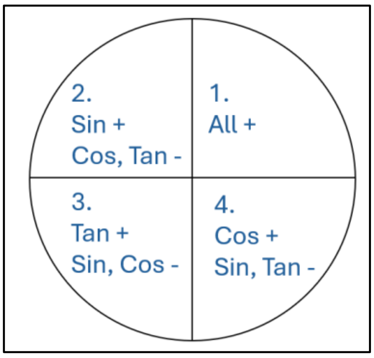
2.1 Unit Circle

The unit circle is just a circle of radius 1 and it can be used to find the value of sin, cos or tan of any angle.

Any point on the unit circle can be represented using (Cosθ, $Sin\theta$). We always measure the angle on a unit circle from the positive x-axis and anticlockwise.



The unit circle is split into 4 quadrants. The signs (positive or negative) or sin, cos and tan in each quadrant can be seen in the below graph.



2.2 General Solutions to Trigonometric Equations

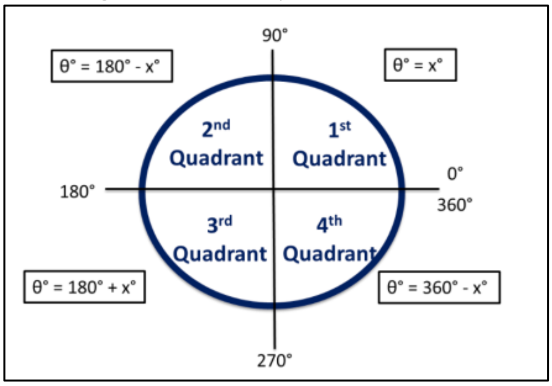
With trigonometric equations, there can be multiple answers. This is because the function repeats itself continuously.

The method to solving these questions is:

(i) Find your reference angle from the initial equation. The reference angle is an angle less than 90^0 that is made with the x-axis in that quadrant. It is used for solving trigonometric equations. When solving for the reference angle, we ignore the signs.

(ii) Find which quadrants the angle is in depending on the sign.

(iii) Use the reference angle and quadrant to find the solution. The below graph shows how to use the reference angle (x) in each quadrant to solve for the solution angle (θ).



E.g. Find the general solution for Sinθ = $-\frac{\sqrt{3}}{2}$.

(i) Find our reference angle.

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=> Sinθ = $\frac{\sqrt{3}}{2}$

=> θ= Sin-1($\frac{\sqrt{3}}{2}$)

=> θ = 60^0

(ii) Find which quadrant the angle lies in.

- ⇨ From our initial equation, Sin θ is negative.
- ⇨ Using the unit circle, Sin is negative in quadrants 3 and 4.

(iii)

Our reference angle is 60^0 and the angle lies in quadrant 3 and 4.

- ⇨ For quadrant 3, the angle is $180^0 + 60^0$ (reference angle) = 240^0
- ⇨ For quadrant 4, the angle is $360^0 – 60^0 = 300^0$

If we are asking for the general solution we need to allow for further 360^0 additions.

So for the above example, our solution is:

$\theta = 240^0 + n*360^0$ or $\theta = 300^0 + n*360^0$

2.3 More Complex Trigonometric Equations

The more difficulty trigonometric equations to solve will be in the form of Sin nθ or Cos nθ. The initial steps remain the same as the previous.

E.g. Solve Cos $2\theta = -\frac{1}{2}$ for $0 \leq \theta \leq 360^0$. (It is important to note if they specific the range of the angle in the question).

(i) Find the reference angle.

=> Cos $2\theta = \frac{1}{2}$.

=> $2\theta = Cos^{-1}(\frac{1}{2}) = 60^0$

We keep it in the same form as it was given in the question for now i.e., we don't divide by 2 yet.

(ii) Find the quadrants for the angle.

- ⇨ From our initial equation, cos was negative.
- ⇨ From the unit circle, cos is negative in quadrants 2 and 3.

(iii) Our reference angle is 60^0 and the angle lies in quadrants 2 and 3

- ⇨ For quadrant 2, the angle is 180-60 = 120^0
- ⇨ For quadrant 3, the angle is 180+60 = 240^0

So we have $2\theta = 120^0 + n*360$ or $2\theta = 240^0 + n*360$

Now we divide by the number in front of the angle.

- ⇨ $\theta = \frac{120^0}{2} + n*\frac{360^0}{2}$ or $\theta = \frac{240^0}{2} + n*\frac{360^0}{2}$
- ⇨ $\theta = 60^0 + n*180^0$ or $\theta = 120^0 + n*180^0$. This is our general solution.

However in this question, we are given the range for θ.

For these types of questions, we need to give the full list of values for θ depending on the range.

N=0: $\theta = 60^0 + 0*180^0$, $\theta = 120^0 + 0*180^0$

$\theta = 60^0$, $\theta = 120^0$

N=1: $\theta = 60^0 + 1*180^0$, $\theta = 120^0 + 1*180^0$

$\theta = 240^0$, $\theta = 300^0$.

N=2: $\theta = 60^0 + 2*180^0$, $\theta = 120^0 + 2*180^0$

$\theta = 420^0$, $\theta = 480^0$. These angles are outside the range we were given for θ.

Our final answer is:

$\theta = 60^0, 120^0, 240^0, 300^0$.

3 Arcs

You need to know and be able to apply the below formulae (page 9 in log tables).

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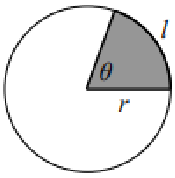

Arc / Sector

$l = r\theta$ $\quad A = \frac{1}{2}r^2\theta$ $\quad$ *when θ is in radians*

$l = 2\pi r\left(\frac{\theta}{360°}\right)$ $\quad A = \pi r^2\left(\frac{\theta}{360°}\right)$ $\quad$ *when θ is in degrees*

4 Trigonometric Identities

You must be able to derive the below trigonometric formulae:

1. $\cos^2 A + \sin^2 A = 1$
2. sine formula: $\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$
3. cosine formula: $a^2 = b^2 + c^2 - 2\,bc\cos A$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
6. $\cos 2A = \cos^2 A - \sin^2 A$
7. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
8. ~~$\sin(A-B) = \sin A\cos B - \cos A\sin B$~~
9. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(No. 8 is not required)

The derivation of these can be found at: http://www.emaths.ie/trigonometry-overview.html

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