Society of Actuaries in Ireland

Algebra 2 – Solutions

Warm Up Questions:

Question 1.

Let $f(x) = -x^2 + 12x - 27,\ x \in \mathbb{R}$.

(a) **(i)** Complete Table 1 below.

Table 1							
x	3	4	5	6	7	8	9
$f(x)$	0	5	8	9	8	5	0

Question 2. Solve for x: $(x+7)/3 \ + 2/x \ = \ 4$

Multiply across:

$$\Rightarrow \frac{(x+7)(x)(3)}{3} + \frac{2(x)(3)}{x} = 4(x)(3)$$

$$\Rightarrow x(x+7) + 2(3) = 12x$$

$$\Rightarrow x^2 + 7x + 6 = 12x$$

$$\Rightarrow x^2 + 7x - 12x + 6 = 0$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow (x - 3)(x - 2)$$

$$\Rightarrow x = 3, x = 2$$

Question 3. Express $\sqrt{48} - \sqrt{12} + \sqrt{27}$ in the form $a\sqrt{b}$

$=> \sqrt{(16\ x\ 3)} - \sqrt{(4\ x\ 3)} + \sqrt{(9\ x\ 3)}$

$=> \sqrt{(16)}\sqrt{(3)} - \sqrt{(4)}\sqrt{(3)} + \sqrt{(9)}\sqrt{3}$

$=> 4\sqrt{(3)} - 2\sqrt{(3)} + 3\sqrt{3}$

$=> 5\sqrt{(3)}$

Question 4. Simplify $(b + 1)^3 - (b - 1)^3$

$(b + 1)^3 = (b + 1)(b + 1)^2$

$= (b + 1)(b^2 + 2b + 1)$

$= b(b^2 + 2b + 1) + 1(b^2 + 2b + 1)$

$= b^3 + 2b^2 + b + b^2 + 2b + 1$

$= b^3 + 3b^2 + 3b + 1$ .....(1)

$(b - 1)^3 = (b - 1)(b - 1)^2$

$= (b - 1)(b^2 - 2b + 1)$

$= b^3 - 3b^2 + 3b - 1$ .....(2)

$(b + 1)^3 - (b - 1)^3 = (1) - (2)$

$= b^3 + 3b^2 + 3b + 1 - (b^3 - 3b^2 + 3b - 1)$

$= 6b^2 + 2$

Side note: It is useful to remember $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Which you can use to check:

$(b + 1)^3 - (b - 1)^3 = b^3 + 3b^2 + 3b + 1 - (b^3 - 3b^2 + 3b - 1)$

$= 6b^2 + 2$

OR *use difference of 2 cubes* method

-b Formula

Question 5.

$10x^2 + 6x - 52 = 0$

$a = 10 \qquad b = 6 \qquad c = -52$

$$\Rightarrow \frac{-6+/-\sqrt{(6^2-4(10)(-52))}}{2(10)}$$

$$\Rightarrow \frac{-6+/-\sqrt{(36-(-2080))}}{20}$$

$$\Rightarrow \frac{-6+/-\sqrt{(2116)}}{20}$$

$$\Rightarrow \frac{-6+/-46}{20}$$

$$\Rightarrow \frac{-6+46}{20} = \frac{40}{20} = 2 \qquad \Rightarrow \frac{-6-46}{20} = \frac{-52}{20} = \frac{-13}{5}$$

$$X = 2 \text{ or } X = (-13)/5$$

Question 6. 2011 Paper 1 Q1

(c) Solve the equation $x^2 - 2\sqrt{3}x - 9 = 0$, giving your answers in the form $a\sqrt{3}$, where $a \in \mathbb{Q}$.

$$x = \frac{2\sqrt{3} \pm \sqrt{\left(-2\sqrt{3}\right)^2 - 4(1)(-9)}}{2(1)} = \frac{2\sqrt{3} \pm \sqrt{48}}{2}$$

$$= \frac{2\sqrt{3} \pm 4\sqrt{3}}{2}$$

$$= \sqrt{3} \pm 2\sqrt{3}$$

$$x = -\sqrt{3} \text{ or } x = 3\sqrt{3}$$

Question 7. 2015 Paper 1 Q2 (25 marks)

Solve the equation $x^3 - 3x^2 - 9x + 11 = 0$.
Write any irrational solution in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$.

$$f(x) = x^3 - 3x^2 - 9x + 11$$
$$f(1) = 1^3 - 3(1)^2 - 9 + 11 = 0$$
$\Rightarrow x = 1$ is a solution.
$(x - 1)$ is a factor

$$\begin{array}{r|l} & x^2 - 2x \quad -11 \\ \hline x - 1 & x^3 - 3x^2 - 9x + 11 \\ & x^3 - x^2 \\ \hline & \quad -2x^2 - 9x + 11 \\ & \quad -2x^2 + 2x \\ \hline & \qquad\quad -11x + 11 \\ & \qquad\quad -11x + 11 \\ \hline \end{array}$$

Hence, other factor is $x^2 - 2x - 11$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm 4\sqrt{3}}{2} = 1 \pm 2\sqrt{3}$$

Solutions: $\{1,\ 1 + 2\sqrt{3},\ 1 - 2\sqrt{3}\}$

Inequalities

Question 8. 2021 Paper 1 Q2(a)

$|-3 + p| = 5$ so $(-3 + p) = \pm 5$

$p - 3 = 5 \Rightarrow p = 8$

or $p - 3 = -5 \Rightarrow p = -2$

OR *square both sides as* $(-3+p)^2 = 5^2$ and solve quadratic

Question 9. Solve the following inequality and graph the solution, $x \in R$: $|3x+4| \leq |x+2|$

$(|3x + 4|)^2 \leq (|x+2|)^2$

$\Rightarrow 9x^2 + 24x + 16 \leq x^2 + 4x + 4$

$\Rightarrow 8x^2 + 20x + 12 \leq 0$

$\Rightarrow 2x^2 + 5x + 3 \leq 0$

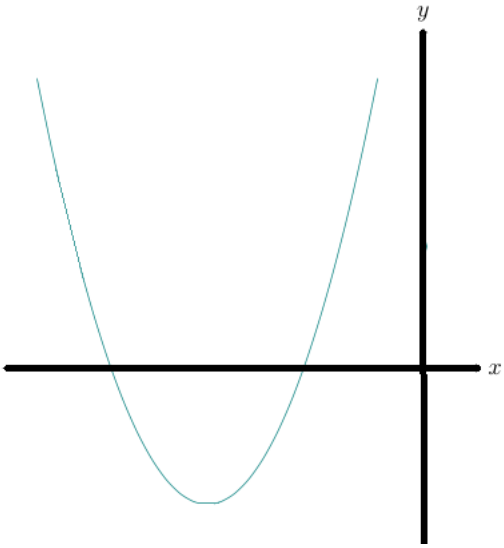
Solve $(2x + 3)(x + 1) = 0$

$\Rightarrow (2x+3) = 0$ *OR* $(x+1) = 0$

$\Rightarrow X = -3/2$ *OR* $x = -1$

Consider $2x^2 + 5x + 3$. This is a quadratic with a positive "a" (the x^2 coefficient).

So it is a quadratic curve with a U shape.



So $2x^2 + 5x + 3$ is less than 0 when:

$$\Rightarrow (-3)/(2) \leq x \leq -1$$

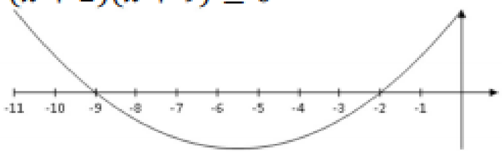
Question 10. 2018 Paper 1 Q1 (b)

(b) Solve the inequality $\frac{2x-3}{x+2} \geq 3$, where $x \in \mathbb{R}$ and $x \neq -2$.

We multiply across by (x + 2)² as it is always non-negative

$$\frac{2x-3}{x+2} \geq 3 \qquad \times\ (x+2)^2$$

$$(2x-3)(x+2) \geq 3(x+2)^2$$
$$2x^2 + x - 6 \geq 3x^2 + 12x + 12$$
$$x^2 + 11x + 18 \leq 0$$
$$(x+2)(x+9) \leq 0$$



$$-9 \leq x < -2$$

Simultaneous Equations

Question 11.

Solve the simultaneous equations:

$$a^2 - ab + b^2 = 3$$
$$a + 2b + 1 = 0$$

$$a = -2b - 1$$

$$(-2b-1)^2 + (2b+1)b + b^2 = 3$$
$$7b^2 + 5b - 2 = 0$$
$$(7b-2)(b+1) = 0$$
$$b = \tfrac{2}{7} \quad \text{or } b = -1$$
$$a = \tfrac{-11}{7} \quad \text{or } a = 1$$

Solution: $\{b = \tfrac{2}{7}$ and $a = \tfrac{-11}{7}\}$ or $\{b = -1$ and $a = 1\}$.

Question 12. **2022 Paper 1 b(ii)**

$28a + 9b + 3c = 879$

Minus $\underline{4a + 3b + 3c = 807}$

24a + 6b = 72

$76a + 15b + 3c = 663$

Minus $\underline{28a + 9b + 3c = 879}$

48a + 6b = -216

24a + 6b = 72

Minus *<u>48a + 6b = -216</u>*

*-24a = 288 so **a = -12***

*24(-12) + 6b = 72 so **b= 60***

*4(-12) + 3(60) + 3c = 807 **so c= 225***

Logs

Question 13. Solve $\log_x 8 = 3$

x^3 $= 8$

$x = 2$

Question 14. Solve 32^$(x - 1) = 28$ for x and give your answer to 2 decimal places

Solution is to take the natural log of both sides

$\ln(32^{x-1}) = \ln(28)$

(x-1)ln32 = ln28 ….. Using $\log(a^b) = b * \log(a)$

x-1 = ln28/ln32

x = ln28/ln32 + 1

x = 1.96

Question 15. 2016 P1 Q4 (10 marks):

Given $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 0$, write each of the following in terms of p and q:

(i) $\log_a \frac{8}{3}$

$$p = \log_a 2\ , \qquad q = \log_a 3$$

$$\log_a \frac{8}{3} = \log_a 8 - \log_a 3$$
$$= \log_a (2)^3 - \log_a 3$$
$$= 3\log_a 2 - \log_a 3$$
$$= 3p - q$$

(ii) $\log_a \frac{9a^2}{16}$.

$$\log_a \frac{9a^2}{16} = \log_a (3a)^2 - \log_a (2)^4$$
$$= 2\log_a 3 + 2\log_a a - 4\log_a 2$$
$$= 2q + 2(1) - 4p$$
$$= 2q + 2 - 4p$$

Question 16. 2014 P1 Q2

Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c (cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2}\log_c x = \frac{1}{2}p$$

using the power law for logarithms.
Also,

$$\log_c (cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c (cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

General Questions

Question 17. 2023 P1 Q1

(a) Find the two values of $m \in \mathbb{R}$ for which $|5+3m| = 11$.

(a) **Method 1**:

$5 + 3m = 11$ $\quad$ $5 + 3m = -11$

$3m = 6$ $\quad$ $3m = -16$

$m = 2$ $\quad$ $m = -\frac{16}{3}$

OR

Method 2:

$(5 + 3m)^2 = 11^2$

$25 + 30m + 9m^2 = 121$

$9m^2 + 30m - 96 = 0$

$3m^2 + 10m - 32 = 0$

$(3m + 16)(m - 2) = 0$

$m = -\frac{16}{3}, \quad m = 2$

(b) For the real numbers h, j, and k:

$$\frac{1}{h} = \frac{k}{j + k}$$

Express k in terms of h and j.

(b)

$j + k = hk$

$k - hk = -j$

$k(1 - h) = -j$

$k = -\frac{j}{1-h}$ or $k = \frac{j}{h-1}$

(c) $x^2 - px + 1$ is a factor of $x^3 - 2x - 3r$, where $p, r \in \mathbb{R}$ and $p < 0$. Find the value of p and the value of r.

Assume other factor is -k so $(x^2 - px + 1)(x+k) = x^3 - 2x - 3r$

$x^3 - px^2 + x + kx^2 - pkx + k = x^3 - 2x - 3r$

$x^3 + (k-p)x^2 + (1-pk)x + k = x^3 - 2x - 3r$ *so* $(k-p) = 0$ *as no* x^2 *term in cubic equation*

$k = p$

$(1-pk) = -2$ *as* $k=p$ *then* $1 - p^2 = -2$ ..... $3 = p^2$ *so* $p = \pm\sqrt{3}$ *but* $p < 0$ *so* $p = -\sqrt{3}$

$k = -3r$ *so* $-\sqrt{3} = -3r$ *so* $\frac{\sqrt{3}}{3} = r$ *or* $r = \frac{1}{\sqrt{3}}$

Question 18. 2023 P1 Q6 (a)(i)

(i) Find the two values of x for which $f(x) = g(x)$.

$x+4 = x^2-2$

$x^2-2-4-x = 0$

$x^2 - x - 6 = (x+2)(x-3)$

$x=3$ or $x = -2$

Question 19. Q1 2020 Paper 1

(a) (i) if (-3) is a root then $f(-3) = 0$

$f(-3) = (-3)^2 + (-3)5 + p = 9 - 15 + p = 0$

$p - 6 = 0$ *so* $p = 6$

(ii) If has 2 roots differing by 3, call them α and α+3 and then factorise the formula

$x^2 + 5x + p = (x - \alpha)(x - \alpha - 3) = x(x - \alpha - 3) - \alpha(x - \alpha - 3)$

$= x^2 + x(-\alpha - \alpha - 3) + \alpha^2 + 3\alpha$

So $\alpha^2 + 3\alpha = p$ *and* $-2\alpha - 3 = 5$

$\alpha = -4$ then $(-4)^2 + 3(-4) = p$ so $p = 4$

(iii) If it does not cross x-axis, then means it has no real roots $\Rightarrow (b^2-4ac) < 0$

$5^2 - 4(1)p < 0 \Rightarrow 25 - 4p < 0$

$25 < 4p$ *or* $p > 25/4$

p must be an integer so could be 7 or 8

(b)

$-1 \le 2x+5 \le 1$ $-6 \le 2x \le -4$ $-3 \le x \le -2$	$2x+5 \le 1$ $2x \le -4$ $x \le -2$ $-1 \le 2x+5$ $-6 \le 2x$ $-3 \le x$ $-3 \le x \le -2$

OR

$(2x+5)^2 \le 1$

$4x^2+20x+25 \le 1$

$4x^2+20x+24 \le 0$

$x^2+5x+6 \le 0$

$(x+2)(x+3) \le 0$

$x=-2, x=-3$

$-3 \le x \le -2$

Question 20 Q3 2021 Paper 1

(a) Area of $xz = 2\sqrt{2}\ cm^2$, *Area of* $yz = 8\sqrt{6}\ cm^2$ *and area of* $xy = 4\sqrt{3}\ cm^2$

$\Rightarrow xz.\ yz.\ xy = x^2y^2z^2 = (2\sqrt{2})^2.(8\sqrt{6})^2.(4\sqrt{3})^2 = 384$

$= 384 = x^2y^2z^2 = (xyz)^2$

$\Rightarrow xyz = \sqrt{384} = 8\sqrt{6}\ cm^3$

OR

$xy = 4\sqrt{3}\ cm^2$ so $y = \frac{4\sqrt{3}}{x}$

$xz = 2\sqrt{2}\ cm^2$ so $z = \frac{2\sqrt{2}}{x}$

$yz = 8\sqrt{6}\ cm^2$ so $\frac{2\sqrt{2}}{x}.\frac{4\sqrt{3}}{x} = 8\sqrt{6}$ $\Rightarrow$ $\frac{8\sqrt{6}}{x^2} = 8\sqrt{6}$ $\Rightarrow$ $x^2 = 1$ $x = 1$ *as must be* > 0

$y = \frac{4\sqrt{3}}{1}$ *and* $z = \frac{2\sqrt{2}}{1}$

Volume $= 1.\frac{4\sqrt{3}}{1}.\frac{2\sqrt{2}}{1} = 8\sqrt{6}\ cm^3$

(b) (i) Given that $f(x) = 3x^2 + 8x - 35$, where $x \in \mathbb{R}$, find the two roots of $f(x) = 0$.

$(3x-7)(x+5) = 0$ *so* $x = \frac{7}{3}$ *and* $x = -5$

(ii) Hence or otherwise, solve the equation $3^{2m+1} = 35 - 8(3^m)$, where $m \in \mathbb{R}$. Give your answer in the form $m = \log_3 p - q$, where $p, q \in \mathbb{N}$.

Set in quadratic format aligns with (b) (i) so $3.3^{2m} + 8(3^m) - 35 = 0$

Factors are $3^{2m} = \frac{7}{3}$ *or* $3^{2m} = -5$ *but* 3^{2m} *must be* > 0 *so* $3^{2m} = \frac{7}{3}$

$Log_3 3^{2m} = Log_3 \frac{7}{3}$ so $m = Log_3 \frac{7}{3}$ (from log tables $Log_3 \frac{7}{3} = Log_3 7 - Log_3 3$)

$m = Log_3 7 - Log_3 3 = Log_3 7 - 1$

Question 21.

(i) *Let x = Stage number.*
There are 4 times as many blue tiles than the Stage number so Blue tiles = 4x

There are 4 white tiles in every stage. This is a constant and remains 4 no matter what stage number we use.

The total number of green tiles is the square of the stage number so Green tiles = x^2

The total number of tiles (T) must be the green tiles + blue tiles + white tiles

Total = $x^2 + 4x + 4$

(ii) $x^2 + 4x + 4 = 324$

Factorise $(x+2)(x+2)=324$

$(x+2)^2=324$

$x+2=18$

$x=16$

There are x^2 *green tiles therefore* $16^2 = 256$ *green tiles.*

(iii) *Mary's kitchen is square. Therefore the length of each side =* $\sqrt{6.76} = 2.6m$ *= 260 cm.*
Each tile has sides of 20 cm each and 13 x 20 = 260. Therefore there are 13 tiles on each side or in each row.
In the first row there are two white tiles and the rest (13-2=11) are blue. Therefore this must be stage 11.

Green = $x^2=121$

Blue =4x =44

White = 4

Check: The total number of tiles = 121+44+4=169.

The area of each tile = 0.20 x 0.20 = 0.04 m^2. *The total number of tiles needed =* $6.76 \div 0.04 = 169$.

Question 22. 2016 Q8 part b

1)

200 m Race:

$$y = a(b - x)^c$$
$$y = 4{\cdot}99087(42{\cdot}5 - 23{\cdot}8)^{1{\cdot}81}$$
$$y = 1000$$

Javelin:

$$y = a(x - b)^c$$
$$y = 15{\cdot}9803(58{\cdot}2 - 3{\cdot}8)^{1{\cdot}04}$$
$$y = 1020$$

2)

$$y = a(x - b)^c$$
$$1295 = 15{\cdot}9803(x - 3{\cdot}8)^{1{\cdot}04}$$
$$81{\cdot}0373 = (x - 3{\cdot}8)^{1{\cdot}04} = z^{1{\cdot}04}$$
$$\log z = \frac{\log 81{\cdot}0373}{1{\cdot}04}$$
$$z = 68{\cdot}4343 = (x - 3{\cdot}8)$$
$$x = 72{\cdot}2343 = 72{\cdot}23 \text{ m}$$

3)

$$y = a(b - x)^c$$
$$1087 = 0{\cdot}11193(254 - 121{\cdot}84)^c$$
$$\frac{1087}{0{\cdot}11193} = (132{\cdot}16)^c$$
$$\log 9711{\cdot}426 = c \log 132{\cdot}16$$
$$c = \frac{\log 9711{\cdot}426}{\log 132{\cdot}16} = 1{\cdot}88$$

Link to SAI website for maths tutorials

https://web.actuaries.ie/students/becoming-actuary/maths-tutorials-higher-level-leaving-certificate-20242025

Link to SAI Instagram:

https://www.instagram.com/saimathsturorials?r=nametag