Society of Actuaries in Ireland

Revision (2) Tutorial Questions – 17/04/24

Question 1 - CALCULUS

(a) The function f is defined as follows, for $x \in \mathbb{R}$:

$$f(x) = \frac{1}{5x^2 + 7}$$

Find $f'(x)$, the derivative of f. Give your answer in its simplest form.

Q5	Model Solution – 30 Marks
(a)	$f(x) = (5x^2 + 7)^{-1}$ $f'(x) = -1(5x^2 + 7)^{-2}(10x)$ $= -10x(5x^2 + 7)^{-2}$ $f'(x) = \frac{-10x}{(5x^2 + 7)^2}$ **OR** $u(x) = 1$ so $u'(x) = 0$ $v(x) = 5x^2 + 7$ so $v'(x) = 10x$ $f'(x) = \frac{(5x^2+7)(0)-1(10x)}{(5x^2+7)^2}$ $= \frac{-10x}{(5x^2+7)^2}$

(b) The function $g(x)$ is defined as follows, for $x \in \mathbb{R}$, $0 < x < \pi$:

$$g(x) = \left(\tan\left(\frac{x}{2}\right)\right)(\ln x)$$

Find the value of $g'\left(\frac{\pi}{2}\right)$. Give your answer in the form $a + \ln b$, where $a, b \in \mathbb{R}$.

(b)	$u = \tan\frac{x}{2}$ $\quad v = \ln x$ $\frac{du}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ $\quad \frac{dv}{dx} = \frac{1}{x}$ $g'(x) = \left(\tan\frac{x}{2}\right)\left(\frac{1}{x}\right) + (\ln x)\left(\frac{1}{2}\sec^2\frac{x}{2}\right)$ At $x = \frac{\pi}{2}$: $g'(x) = \left(\tan\frac{\left(\frac{\pi}{2}\right)}{2}\right)\left(\frac{1}{\left(\frac{\pi}{2}\right)}\right) + (\ln\frac{\pi}{2})\left(\frac{1}{2}\sec^2\frac{\left(\frac{\pi}{2}\right)}{2}\right)$ $= 1\left(\frac{2}{\pi}\right) + \ln\frac{\pi}{2}\left(\frac{1}{2}(2)\right)$ $= \frac{2}{\pi} + \ln\frac{\pi}{2}$

Question 2 - CALCULUS

Fiona is driving on a motorway. She passes a point **A** on the motorway. Her speed is given by:

$$v(t) = \frac{2}{3}t^3 - 6t^2 + 13t + 109$$

where v is her speed in km/hour t minutes after passing the point **A**, for $0 \le t \le 5$ and $t \in \mathbb{R}$.

(a) Work out Fiona's speed when she passes the point **A**.

(a)	$v(0) = \frac{2}{3}(0)^3 - 6(0)^2 + 13(0) + 109$ $v(0) = 109$ km/hr

(b) Work out Fiona's acceleration (that is, the rate at which her speed is increasing) 5 minutes after she passes the point **A**. Give your answer in km/hour per minute.

(a)	$v(0) = \frac{2}{3}(0)^3 - 6(0)^2 + 13(0) + 109$ $v(0) = 109$ km/hr

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(c) Find the time (value of t) at which Fiona reaches her maximum speed, during the first 4 minutes after she passes the point **A**. Give your answer correct to 2 decimal places.

(c)

Maximum speed when $v'(t) = 0$

$$2t^2 - 12t + 13 = 0$$

$$t = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(13)}}{2(2)}$$

$t = 4 \cdot 58$ or $t = 1 \cdot 42$

Maximum at $t = 1 \cdot 42$

[as coefficient of $t^3 > 0$ and domain of interest is $[0, 4]$, with local min at $t = 4 \cdot 58$]

OR

[$v''(t) = 4t - 12$
$v''(1.42) < 0$
$\Rightarrow$ maximum at $t = 1.42$]

(d) Use integration to work out Fiona's average speed over the 5 minutes after she passes the point **A**. Give your answer correct to 2 decimal places.

(d)

$$\frac{1}{5-0}\left[\int_0^5 \left(\frac{2}{3}t^3 - 6t^2 + 13t + 109\right) dt\right]$$

$$= \frac{1}{5}\left[\frac{t^4}{6} - 2t^3 + \frac{13t^2}{2} + 109t\right]_0^5$$

$$= \frac{1}{5}\left[\left(\frac{(5)^4}{6} - 2(5)^3 + \frac{13(5)^2}{2} + 109(5)\right) - (0)\right]$$

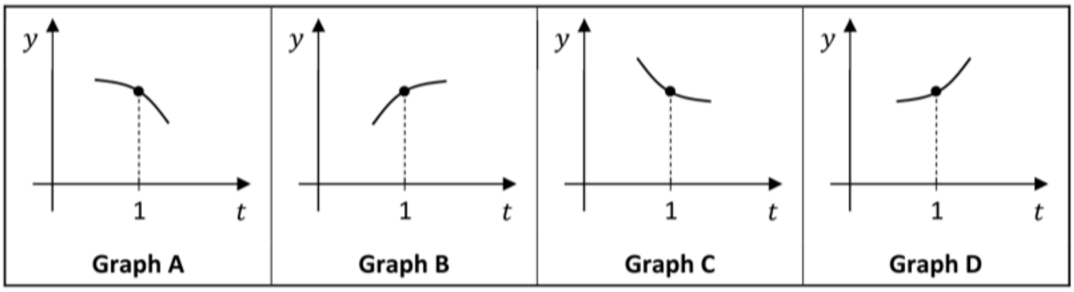
$= 112 \cdot 333 ....$ km/hr

$= 112 \cdot 33$ km/hr [2 d.p.]

(e) Taking $v'(t)$ to be the derivative of v, and $v''(t)$ to be the second derivative of v:

$$v'(1) > 0 \text{ and } v''(1) < 0$$

Four graphs, **A**, **B**, **C**, and **D**, are shown below.



Close to where $t = 1$, the graph of $y = v(t)$ must look like one of the four graphs given above. Write down which graph this is. Justify your answer, using both $v'(1)$ **and** $v''(1)$.

Answer (**A**, **B**, **C**, or **D**):

Using $v'(1) > 0$:

Using $v''(1) < 0$:

This question continues on the next page.

There is an **Average Speed Zone** on the motorway, starting at the point **A** and ending at point **B**.
The distance from **A** to **B** along the motorway is 10 km.
Cameras record the time taken for each car to travel from the point **A** to the point **B**.
Each car's average speed from **A** to **B** is then calculated.

(e) Answer: B

Justification:

$v'(1) > 0$:

Function is increasing so the slope is positive.

$v''(1) < 0$:

Rate of increase is slowing.

OR

The slope is decreasing.

(f) Work out the **minimum** time, in minutes, that a driver could get from **A** to **B**, while not driving above 100 km/hour.

(f)	Time = Distance/Speed = $\frac{10}{100}$ = 6 [minutes]

(g) Rohan drives from **A** to **B**.

He passes the point **A** driving at a constant speed of 120 km/hour. After 2 minutes driving at this speed, he starts to decelerate (reduce his speed) at a constant rate, until he reaches the point **B**. Overall, his average speed in driving from **A** to **B** is 100 km/hour.

Work out Rohan's deceleration. Give your answer in km/hour per minute.

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(g)	<u>120 km/hr for 2 minutes:</u> $\text{Distance} = 120 \times \frac{2}{60} = 4\text{km}$ $10 - 4 = 6$ km remaining to get to B <u>Average Speed:</u> $\frac{10}{\text{total time}} = 100$ $\text{total time} = \frac{1}{10}\text{hrs} = 6 \text{ minutes}$ $\Rightarrow$ 4 minutes remaining to get to B <u>Average speed for last 6 km:</u> $\text{Avg Speed} = 6 \div \left(\frac{1}{15}\right) = 90 \text{ km/hr}$ $\frac{120+v}{2} = 90$, where v is the speed at B $v = 60\ km/hr$ Decelerates from 120 to 60 over 4 minutes. So, deceleration = 15 km/hr per minute **OR** Average speed for last 6 km: $\frac{120+v}{2}$, where v is the speed at B $\text{Distance} = \left(\frac{120+v}{2}\right) \times \text{time}$ $6 = \left(\frac{(120+v)}{2}\right) \times \frac{4}{60}$ $v = 60$ Decelerates from 120 to 60 over 4 minutes. Deceleration = 15 km/hr per minute **OR** Average speed for the last 6km: $\frac{1}{4}\int_0^4 (120 - at)dt = 90$ $\frac{1}{4}\left[120t - \frac{1}{2}at^2\right]_0^4 = 90$ $\frac{1}{4}\left[120(4) - \frac{1}{2}a(4)^2\right] = 90$ $120 - 2a = 90$ $a = 15\text{kmh}^{-1}$ per minute

Question 3 – Financial Maths

James invests €5,000. He gets 4%, 5% and 5.5% per annum respectively in the first three years.

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(i) Find the value of his investment at the end of the third year.

(ii) At the end of the third year, James withdraws €X and allows the rest of his investment to grow for one further year at 6% per annum.

If the value of his investment at the end of the fourth year is €5,257.92, find the value of €X, correct to the nearest euro.

ANSWER

(i)

Year 1:

Beginning: 5000

End: 5000(1 + 0.04)

5000 (1.04) = 5200

Year 2:

Beginning 5200

End: 5200 (1 + 0.05)

5200 (1.05) = 5460

Year 3:

Beginning: 5460

End: 5460 (1.055) = 5760.30

James has €5760.30 at the end of year 3

(ii)

Year 4:

Beginning: 5760.30 – X

End: (5760.30 – X)(1.06) = (6105.92 – 1.06X)

(6105.92 – 1.06X) = 5257.92 = 6105.92 – 1.06X

1.06X = 6105.92 – 5257.92

1.06X = 848

X = $\frac{848}{1.06}$

X = 800

Question 4 – Financial Maths

Ellen and Mike get a loan of €200,000 to be repaid at the end of each month in a series of equal payments over 25 years. The interest rate for the loan is 8.00% APR.

Calculate:

(i) the rate of interest per month, that would if paid and compounded monthly, be equivalent to an effective annual rate of 8.00%

(ii) the amount of each monthly repayment

(iii) the total amount to be repaid

(iv) the total interest to be paid

(v) If they had paid fortnightly, what would the repayment amount be and what would the total interest paid be in this case?

ANSWER

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$$F = P(1+i)^t$$
$$1.08 = 1(1+i)^{12}$$

$$1+i = (1.08)^{\frac{1}{12}}$$
$$1+i = 1.006434030$$
$$i = 0.006434030$$

Using the amortisation and loans formula on page 31 of the formula and tables booklet

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1} = 200000\frac{0.006434030(1.006434030)^{300}}{(1.006434030)^{300} - 1} = €1506.83$$

(Alternatively use S_n of a geometric series.)

(iii) The total amount to be repaid $= 300(€1506.83) = €452,049.00$

(iv) Interest paid $= €252,049.00$

Question 5 – Statistics

In a food packaging company, they are investigating the size of potatoes in 5kg bags. They are investigating the hypothesis that the median mass of potatoes in a bag correlates with the number of potatoes in it. They weigh some potatoes and tabulate the results.

Median mass of potatoes (grams)	72	87	96	105	110	125	136	142	147	159	174	192
Number of potatoes per bag	50	51	36	40	45	35	35	36	40	28	32	25

i) Draw a scatter plot to illustrate the data

ii) Which does the scatter plot show? Explain your answer
   a. No correlation
   b. Positive correlation
   c. Negative correlation

iii) Would you accept or reject the hypothesis from your observations of the scatter plot?

iv) Use your calculator to evaluate r, the correlation coefficient of the data. Write down your value for r.

v) Would you accept or reject the hypothesis based on your calculation of r? Explain your answer.

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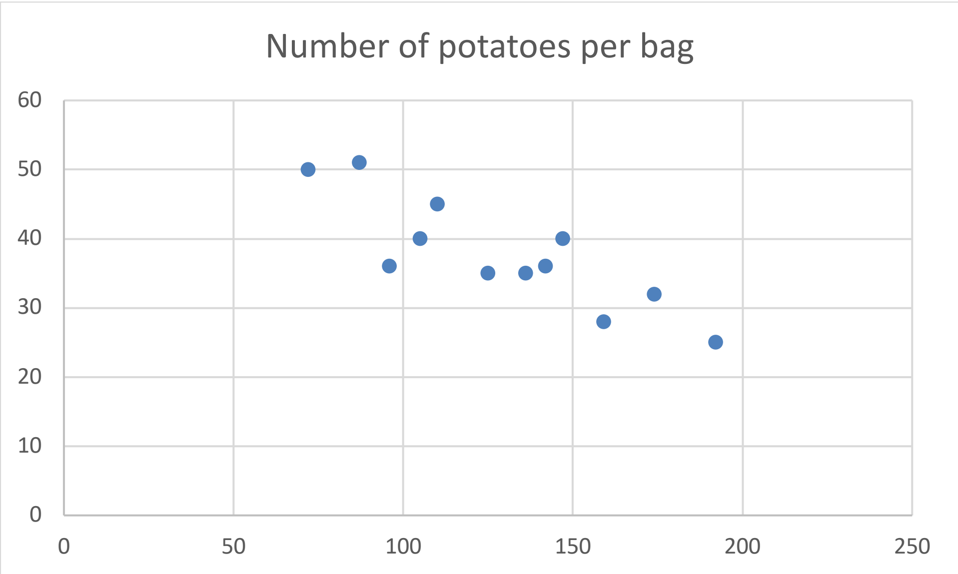
ANSWER



ii) c -negative correlation as line of best fit is downward sloping

iii) Accept the null hyptothesis as linear result from graph

iv) Correlation co-efficient is -0.85

v) This is close to 1 so can accept the null hypothesis

Question 6 – Probability

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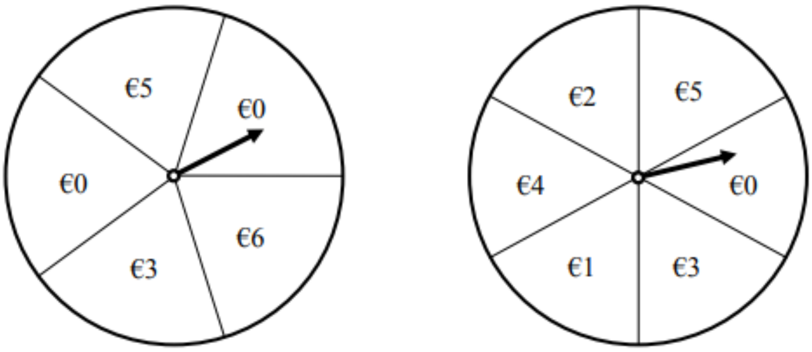
Two different games of chance, shown below, can be played at a charity fundraiser. In each game, the player spins an arrow on a wheel and wins the amount shown on the sector that the arrow stops in. Each game is fair in that the arrow is just as likely to stop in one sector as in any other sector on that wheel.



Game *A* Game *B*

(a) John played Game *A* four times and tells us that he has won a total of €8. In how many different ways could he have done this?

5, 3, 0, 0 or 3, 0, 5, 0 etc

$\frac{4!}{2!} = \frac{24}{2} = 12$ ways

(b) To spin either arrow once, the player pays €3. Which game of chance would you expect to be more successful in raising funds for the charity? Give a reason for your answer.

$E(A) = 0\left(\frac{2}{5}\right) + 3\left(\frac{1}{5}\right) + 5\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right)$

$= 0 + 3/5 + 1 + 6/5 = 1\frac{9}{5} = \frac{14}{5} = 2\frac{4}{5} = 2.8$ euro

$E(B) = 0\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right)$

$= 0 + \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} = \frac{15}{6} = 2\frac{3}{6} = 2.5$ euro

GAME B IS MORE SUCCESSFUL. DIFFERENCE = 50 cent per game

(c) Mary plays Game *B* six times. Find the probability that the arrow stops in the €4 sector exactly twice.

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$P(4) = \frac{1}{6} = p \qquad q = P(\text{Not } 4) = \frac{5}{6}$

$n = 6$ Trials

$p = \frac{1}{6} \qquad q = \frac{5}{6}$

$r = 2$

$P(X=r) = \binom{n}{r} p^r q^{n-r}$

$= \binom{6}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 = 0.2009$

Question 7 – Probability

Question 2 **(25 marks)**

(a) A random variable X follows a normal distribution with mean 60 and standard deviation 5.

(i) Find $P(X \le 68)$.

$$P(X \le 68) = P\left(Z \le \frac{68-60}{5}\right) = P(Z \le 1\cdot 6) = 0\cdot 9452$$

(ii) Find $P(52 \le X \le 68)$.

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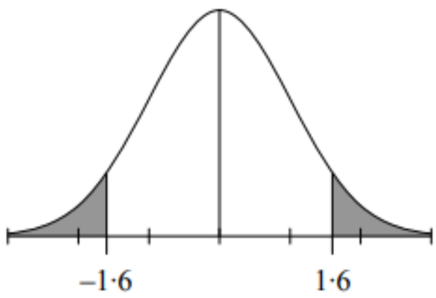
$$P(52 \le X \le 68) = P\left(\frac{52-60}{5} \le Z \le \frac{68-60}{5}\right)$$
$$= P(-1 \cdot 6 \le Z \le 1 \cdot 6)$$



$$P(Z \le -1 \cdot 6) = P(Z \ge 1 \cdot 6)$$
$$= 1 - P(Z \le 1 \cdot 6)$$
$$= 1 - 0 \cdot 9452 = 0 \cdot 0548$$

$$P(-1 \cdot 6 \le Z \le 1 \cdot 6) = P(Z \le 1 \cdot 6) - P(Z \le -1 \cdot 6)$$
$$= 0 \cdot 9452 - 0 \cdot 0548 = 0 \cdot 8904$$

OR

$$P(52 \le X \le 68) = P\left(\frac{52-60}{5} \le Z \le \frac{68-60}{5}\right)$$
$$= P(-1 \cdot 6 \le Z \le 1 \cdot 6)$$
$$= 1 - 2P(Z \ge 1{\cdot}6)$$
$$= 1 - 2(1 - P(Z \le 1{\cdot}6))$$
$$= 1 - 2(1 - 0{\cdot}9452) = 1 - 2(0{\cdot}0548) = 1 - 0{\cdot}1096 = 0{\cdot}8904$$

(b) The heights of a certain type of plant, when ready to harvest, are known to be normally distributed, with a mean of μ. A company tests the effects of three different growth hormones on this type of plant. The three hormones were used on a different large sample of the crop. After applying each hormone, it was found that the heights of the plants in the samples were still normally distributed at harvest time.

The diagrams A, B and C, on the next page, show the expected distribution of the heights of the plants, at harvest time, without the use of the hormones.

The effect, on plant growth, of each of the hormones is described on the next page. Sketch, on each diagram, a new distribution to show the effect of the hormone.

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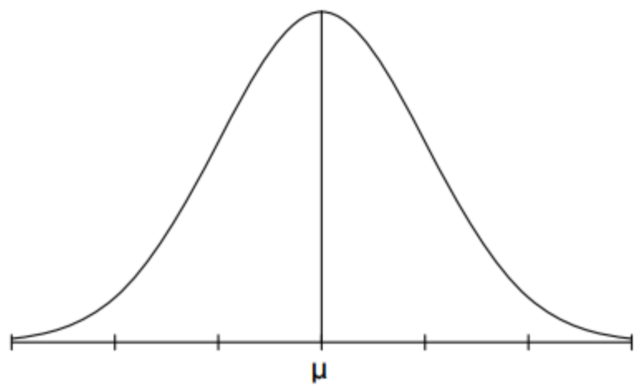
Hormone A

The effect of hormone A was to increase the height of all of the plants.

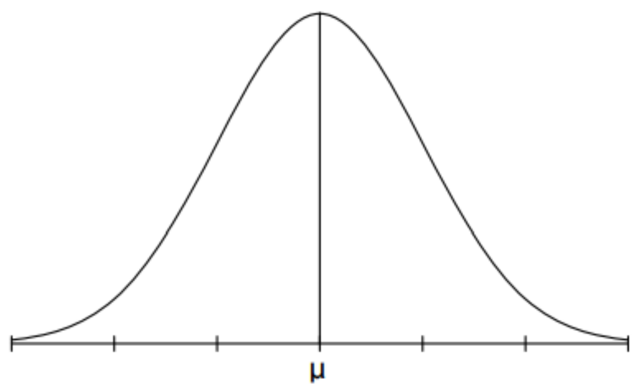
Diagram A



Hormone B

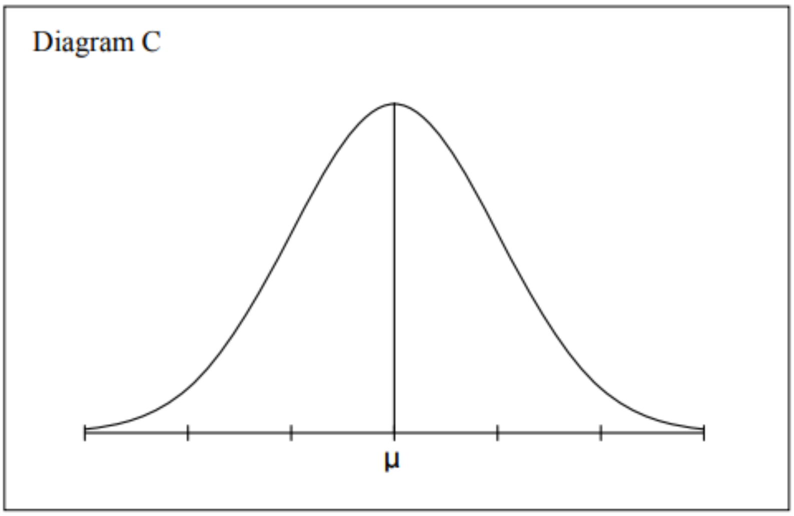
The effect of hormone B was to reduce the number of really small plants and the number of really tall plants. The mean was unchanged.

Diagram B



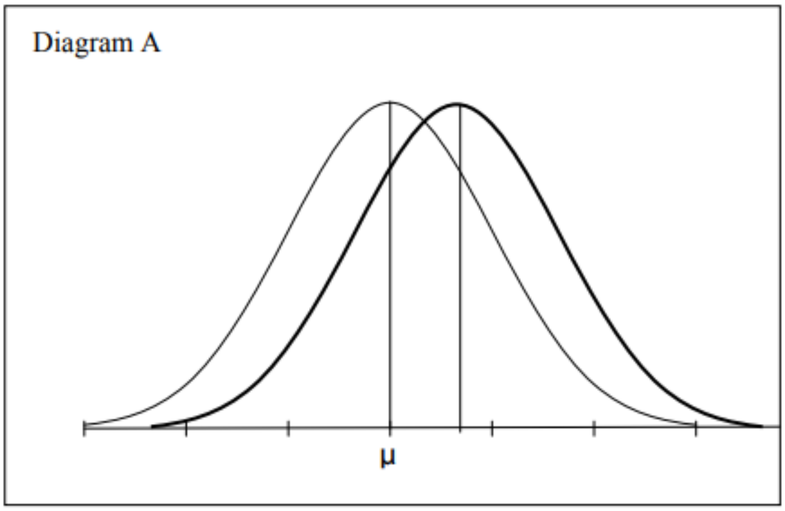
Hormone C

The effect of hormone C was to increase the number of small plants and the number of tall plants. The mean was unchanged.



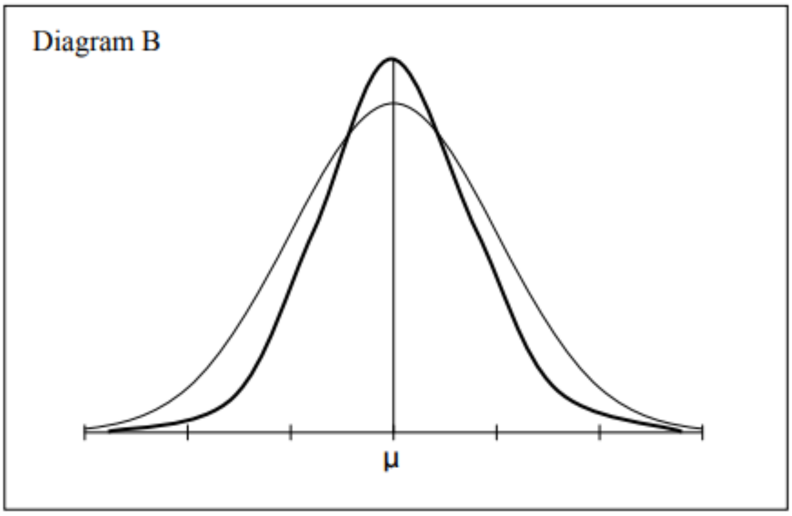
Hormone A

The effect of hormone A was to increase the height of all of the plants.



Hormone B

The effect of hormone B was to reduce the number of really small plants and the number of really tall plants. The mean was unchanged.



Hormone C

The effect of hormone C was to increase the number of small plants and the number of tall plants. The mean was unchanged.