Society of Actuaries in Ireland

Sequences and Series - Answers

Q4 2021 Paper 1

(i) $T_n = a + (n-1).d$ for arithmetic sequence

$T_1 = a + (0-1).d$ => $T_1 = p$ => a=p

$T_2 = p + (2-1).d$ => $T_2 = p + 7$, => d= 7

$T_n = p + (n-1).7$

(ii) 2021/7 = 288 mod (5) => p = 5 $T_{288} = 5 + (288-1).\ 7$

Q4 Paper 1 2022

(a) $u_3 = \sqrt{\frac{u_2}{U_1}} = \sqrt{\frac{64}{2}} = \sqrt{32} = \sqrt{2^5} = (2^5)^{1/2} = 2^{5/2}$

(b)

(i) $T_n = a + (n-1).d$

$T_1 = a + (1-1).d = 5e^{-k}$ => $a = 5e^{-k}$

$T_2 = 5e^{-k} + (2-1).d = 13$ => $5e^{-k} + d = 13$ => $d = 13 - 5e^{-k}$

$T_3 = 5e^{-k} + (3-1).d = 5e^{k}$ => $5e^{-k} + 2(13 - 5e^{-k}) = 5e^{k}$ => $26 - 5e^{-k} = 5e^{k}$

$5e^{k} - 5e^{-k} + 26 = 0$ $y = e^{k}$ & $e^{-k} = \frac{1}{e^k} = \frac{1}{y}$

$5y - \frac{5}{y} + 26 = 0$ ---- multiply by y => $5y^2 + 26y - 5 = 0$

(ii) Factors are (5y-1)(y-5)

$(5y-1) = 0$ => $5y=1$ => $y=\frac{1}{5} = e^{k}$

$\text{Log}_e\left(\frac{1}{5}\right) = \text{Log}_e\, e^{k} = k$ => $k = \text{Log}_e\left(\frac{1}{5}\right) = \text{Log}_e\,(5^{-1}) = -\text{Log}_e\,(5)$

Or

$(y-5) = 0$ => $y = 5$ => $e^{k} = 5$

$\text{Log}_e\,(5) = \text{Log}_e\, e^{k} = k$

$k = \text{Ln}\,(5)$

Q1 2015 Paper 1

(a)

Bounce	**0**	**1**	**2**	**3**	**4**
Height	**2**	$\frac{3}{2}$	$\frac{9}{8}$	$\frac{27}{32}$	$\frac{81}{128}$

(b)

Sum of 2 series – 5 downward falls and 4 upward bounces. After first drop of 2m, height of bounce = height of fall. Start series from when hits ground first to capture sum of upward bounces:

$S_n = \frac{a(1-r^n)}{1-r}$ =>

a=$\frac{3}{2}$ **r = ¾** **n = 4** $S_4 = \frac{\frac{3}{2}(1-(\frac{3}{4})^4)}{1-\frac{3}{4}}$ $S_4 = \frac{4}{1}.\frac{3}{2}\left(1-\left(\frac{3}{4}\right)^4\right) = 6\left(1-\frac{81}{256}\right) = \frac{1050}{256} = \frac{525}{128}$

Vertical movement is upward plus downward movement so travelled 2 x S4 after first bounce.

Total travel including initial drop = 2 + 2 x S_4 = 2 + 2. $\frac{525}{128} = \frac{653}{64}$

Or use numbers in table [Bounce 0 + 2 x (Bounces 1 – 4)]

(c) Travel indefinitely = 2 + 2 x S_∞ $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{\frac{3}{2}}{1-\frac{3}{4}}$ = $\frac{4}{1} x \frac{3}{2}$ = 6

Total vertical distance travelled 2 + 2 x 6 = 14m

Q9 2016 Paper 1

(i) First Term = 4

$S_n = \frac{a(1-r^n)}{1-r}$ = 7.9375 a = 4 r = 0.5

$S_n = \frac{4(1-0.5^n)}{1-0.5}$ = 7.9375 => $4(1-0.5^n)$ = $\frac{1}{2}$ x 7.9375 => 1- 0.5^n = 3.96875/4

$0.5^n = 0.0078125$ $\log 0.5^n = \log(0.0078125)$

nlog0.5 = -2.1072 **n** = $\frac{-2.1072}{-.301029}$

n = 7

(ii)

$S_\infty = \frac{4}{1-\frac{1}{2}}$ = 8 units

(iii) Treat x and y as two separate series and look at each one

For x coordinate $S_\infty = \frac{4}{1-(-\frac{1}{4})} = \frac{4}{\frac{5}{4}} = \frac{16}{5}$

For y coordinate $S_\infty = \frac{2}{1-(-\frac{1}{4})} = \frac{2}{\frac{5}{4}} = \frac{8}{5}$

The target co-ordinate is $(\frac{16}{5}, \frac{8}{5})$

Stage	1st	2nd	3rd	4th	5th	6th	7th	8th	9th
Change in x	+4	0	-1	0	$\frac{1}{4}$	0	$-\frac{1}{16}$	0	$\frac{1}{64}$
Change in y	0	2	0	$-\frac{1}{2}$	0	$\frac{1}{8}$	0	$-\frac{1}{32}$	0

Q2 2023 Paper 1

$$\lim_{n\to\infty}\left[\frac{n}{n+1} + \frac{n+1000}{n} + \left(\frac{1}{3}\right)^n\right] =$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right) + \lim_{n\to\infty}\left(\frac{n+1000}{n}\right) + \lim_{n\to\infty}\left(\frac{1}{3}\right)^n =$$

$$\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) + \lim_{n\to\infty}\left(\frac{1+\frac{1000}{n}}{1}\right) + \lim_{n\to\infty}\left(\frac{1}{3}\right)^n =$$

$$\frac{1}{1+0} + \frac{1+0}{1} + 0 = 2$$

Q2 2018 Paper 1

(a) $T_n = ar^{n-1}$ $\quad\quad T_1 = x^2 = a$

$T_2 = 5x - 8 = x^2r$

$T_3 = x + 8 = x^2r^2$

$r = \frac{T3}{T2} = \frac{x+8}{5x-8}$

Substitute r into T_2

$5x - 8 = x^2\frac{(x+8)}{(5x-8)}$ => $(5x-8).(5x-8) = x^3 + 8x^2$

$25x^2 - 80x + 64 = x^3 + 8x^2$ => $x^3 - 17x^2 + 80x - 64$

(b) $1^3 - 17.1^2 + 80.1 - 64 = 0$

$$\frac{(x^3 - 17x^2 + 80x - 64)}{(\boldsymbol{x-1})} = x^2 - 16x + 64 = (x-8).(x-8)$$

Other factor is x = 8

(c) x= 1 then a = 1 and $r = \frac{1+8}{5-8} = -3$

$$S_\infty = \frac{a}{1-r} \; only\ works\ where -1 < r < 1$$

x= 8 then a = 64 and r = $\frac{8+8}{40-8} = 1/2$

$$S_\infty = \frac{64}{1-\frac{1}{4}} = \frac{264}{3} = 88$$

Q2 2020 Paper 1

$T_n = ar^{n-1}$ $T_1 = 3 + 2i = a$

$T_2 = 5 - i = (3 + 2i)r$

$\frac{5-i}{(3+2i)} = r$ multiply by conjugate to remove denominator

$$r = \frac{5-i}{(3+2i)} x \frac{3-2i}{(3-2i)} = \frac{(15-3i-10i+2i^2)}{(9-4i^2)} = \frac{13-13i}{13}$$

r = 1-i

Question 10 Paper 1 2023

(a) Area is area of triangle plus parts of rectangles outside of the triangle.

Triangular portion of rectangles outside large triangle has a height of 8/3 + base of 1/6 (3 rectangles)

Area of triangle = ½ x 2 x 8 = 8

Area of other portions of 3 rectangles outside the large triangle = 6 x 8/3 x 1/6 = 8/3

Total area = 8 + 8/3 = 32/3 units

(b) Area of rectangles = length x breadth i.e. for
T_1 = triangle base x triangle height,
T_2 = triangle base x (triangle height/2) + (triangle base /2 x (triangle height/2)
= (triangle base x triangle height/2) x (1+1/2)

$T_n =$ triangle base x triangle height $. \frac{1}{n}$ x $(\frac{1 + (n-1) + ...+1}{n})$
$T_n = 2 \text{ x} \frac{8}{n} \text{x} (n + (n\text{-}1) + ...1). \frac{1}{n} \quad = \frac{16}{n^2} \text{ x } S_n$ where a= 1 and d= 1
where $S_n = \frac{n}{2}.(2.a + (n-1).d) = \frac{n}{2}.(2.1 + (n-1).1) = \frac{n.n+1}{2}$
$T_n = \frac{16}{n^2} \text{ x } \frac{n.n+1}{2} \quad = \frac{8(n+1)}{n}$

(c) Area of triangle = 0.5 x 2 x 8 = 8 => 95% Area = 7.6 so A_n must be > 7.6
$\frac{8(n-1)}{n} > 7.6$ => 8n-8 > 7.6n => 0.4n > 8
n > 20 so n=21 is lowest value of n

Question 9 Paper 1 2018

Step	0	1	2	3
Number of black triangles	1	3	9	27
Fraction of the original triangle remaining	1	3/4	$\frac{9}{16}$	$\frac{27}{64}$

(b) (i) $T_n = 3^n$ derived from table

(ii) $T_k > 1 \, x \, 10^9$ => $3^k > 1 \, x \, 10^9$ => $\text{Log}_{10}3^k > \text{Log}_{10}(1 \, x \, 10^9)$
$k\text{Log}_{10}3 > 9$ => $k > \frac{9}{\text{Log}_{10}3}$ => $k > 18.86$ i.e. $k = 19$

(c) (i) $T_h = (\frac{3}{4})^h$ derived from the fact adding an equilateral triangle to the centre removes 1/4th of original triangle so leaves ¾ of area in the previous term

$T_h < \frac{1}{100}$ => $\frac{1}{100} > (\frac{3}{4})^h$ => $\text{Log}_{10}\frac{1}{100} > \text{Log}_{10}(\frac{3}{4})^h$

-2 > $h\text{Log}_{10}\frac{3}{4}$ => $\frac{-2}{-.12493} < h$ $16.007 < h$

h= 17

(ii) $\lim_{n\to\infty}(\frac{3}{4})^n = 0$ as $(\frac{3}{4})^\infty = 0$

(d) (i)

Step	0	1	2	3	4
Perimeter	3	9/2	27/4	81/8	243/16

Logic is that a new triangle placed in centre creates 3 equilateral triangles each one is half the perimeter to the original triangle so perimeter increased x 50% so $T_n = 3 \text{ x} (\frac{3}{2})^n$

(ii) $T_{35} = 3 \text{ x } (\frac{3}{2})^{35}$ = 4,368,329 to nearest unit

(e) As the number of steps increases the area remaining of the black triangles decreases to a negligible amount but the perimeter on the increased number of black triangles gor to infinity so they are inversely related.

Q7 Paper 1 2021

Swing	1	2	3	4	5
Length of Arc (cm)	**45**	$\frac{81}{2}$	$\frac{729}{20}$	$\frac{6561}{200}$	$\frac{59049}{2000}$

(ii) $T_n = 45(0\cdot9)^{25-1}$ = 3.589 i.e 3.6 to one decimal place

(iii) $T_n = 45(0\cdot9)^{n-1}$ Geometric series where a = 45 and r = 0.9

$S_n = \frac{a.(1-r^n)}{1-r}$ $S_{40} = \frac{45.(1-0.9^{40})}{1-0.9} = \frac{44.334}{0.1}$ = 443.3 i.e. 443 cm (nearest)

(iv) $T_n < 2$ => $45(0\cdot9)^{n-1} < 2$ => $(0\cdot9)^{n-1} < \frac{2}{45}$

$Log_{10}(0\cdot9)^{n-1} < Log_{10}(\frac{2}{45})$ => $(n-1)\,Log_{10}0\cdot9 < -1.3521$

(n-1) (-.04575) < -1.3521 => $(n-1) > (\frac{-1.3521}{-.04575})$

(n-1) > 29.55 => n > 30.55 i.e. swing 31

(b) (i) length of arc = $2\pi r\frac{\theta}{360°}$ l = 45cm r = 100cm

$45 = 2\pi.100\frac{\theta}{360°}$ $\theta = \frac{45\,x\,360°}{200\pi}$ = 25.78 nearest is 26°

(ii) $S_\infty = \frac{a}{1-r} = \frac{26}{1-0.9}$ = 260°

(iii) When $S_n = \frac{a.(1-r^n)}{1-r} = 130°$

$\frac{26.(1-0.9^n)}{1-0.9} = 130°$ => $(1-0.9^n) = \frac{130\,x\,0.1}{26}$

$-0.9^n = 0.5 - 1$ => $-0.9^n = -0.5$=> $0.9^n = 0.5$

$nLog_{10}(0\cdot9) = Log_{10}(0.5)$ => $n = \frac{-0.301029}{-0.0457574}$ = 6.5788

$S_{6.5788} = \frac{45.(1-0.9^{6.5788})}{1-0.9}$ = 224.99 or 225 to nearest cm

Q9 Paper 1 2022

(a) $15(0\cdot6)^{2.5}$ = 4.18mg

(b) $15(0\cdot6)^n$ = 1mg => $Log_{10}(0\cdot6)^n = Log_{10}(\frac{1}{15})$

n $Log_{10}(0\cdot6) = -1.17609$ => $n = \frac{-1.17609}{-0.22184}$ = 5.3 days to one decimal place

(c) 15: amount from injection just given

$15(0 \cdot 6)$: amount from injection 1 day ago

$15(0 \cdot 6^2)$: amount from injection 2 days ago

$15(0 \cdot 6^3)$: amount from injection 3 days ago

(d) $S_n = \frac{a.(1-r^n)}{1-r}$ a = 15 r = 0.6 n = 10

$S_{10} = \frac{15\ (1-0.6^{10})}{1-0.6}$ = 37.273 ie 37.27mg rounded to 2 decimal place

(e) $S_\infty = \frac{a}{1-r} = \frac{15}{1-0.6} = 37.5mg$

(f) (i) $T_n = d(0{\cdot}85)^t$

$S_n = \frac{d.(1-0.85^n)}{1-0.85} = \frac{d.(1-0.85^n)}{0.15} = \frac{d.(1-0.85^n)}{\frac{3}{20}} = \frac{20d.(1-0.85^n)}{3}$

(ii) $S_7 = 50$ and

$S_7 = \frac{20d.(1-0.85^7)}{3} = \frac{20d(0.67942)}{3} = 4.5294(d) = 50$

so d= 11.038mg or 11mg to nearest