Society of Actuaries in Ireland

Calculus 1 Solutions

1) (i) $4x^3$ (ii) $21x^2$ (iii) $-x^{-2}$ (iv) 1

2) (i) $f'(x) = 10x + 7$ (ii) $g'(x) = 6x^2 + 8x + 1$ (iii) $\frac{dy}{dx} = 8$

3) y = 8x +4 is the equation of the line. The slope of the line is the derivative.

⇨ $\frac{dy}{dx} = 8$. 8 is the slope of the line.

4) Product Rule: $y = (4 + 3x^2)(6x + 4x^2)$

Let $u = (4 + 3x^2)$ and $v = (6x + 4x^2)$

$\frac{dy}{dx} = u * \frac{dv}{dx} + v * \frac{du}{dx}$

$\frac{du}{dx} = 6x$ $\frac{dv}{dx} = 6 + 8x$

$\frac{dy}{dx} = (4 + 3x^2)(6 + 8x) + (6x + 4x^2)(6x)$

⇨ $= 24 + 32x + 18x^2 + 24x^3 + 36x^2 + 24x^3$

⇨ $= 48x^3 + 54x^2 + 32x + 24$

5) Quotient Rule: $y = \frac{(5+6x)}{2x^2}$

Let $u = (5 + 6x)$ and $v = (2x^2)$

$\frac{du}{dx} = 6$ $\frac{dv}{dx} = 4x$

$\frac{dy}{dx} = \frac{(v*du)-(u*dv)}{v^2}$

⇨ $\frac{dy}{dx} = \frac{(2x^2*6)-((5+6x)*4x)}{(2x^2)^2}$

⇨ $\frac{dy}{dx} = \frac{12x^2-20x-24x^2}{4x^4}$

⇨ $\frac{dy}{dx} = \frac{-12x^2-20x}{4x^4}$

⇨ $\frac{dy}{dx} = \frac{-3x-5}{x^3}$ (Divide above and below by 4x)

6) Chain rule: $y = (2 + 3x)^4$ $Find$ $\frac{dy}{dx}$

$u = v^4$ $v = (2 + 3x)$

$\frac{du}{dv} = 4v^3$ $\frac{dv}{dx} = 3$

$\frac{dy}{dx} = \frac{du}{dv} * \frac{dv}{dx}$ -> $\frac{dy}{dx} = 4v^3 * 3$

$\frac{dy}{dx} = 12(2+3x)^3$

7) Chain rule: y = Cos(3x + 5). Find $\frac{dy}{dx}$

u = Cos(v) v = (3x + 5)

$\frac{du}{dv} = -Sin(v)$ $\frac{dv}{dx} = 3$

$\frac{dy}{dx} = \frac{du}{dv} * \frac{dv}{dx}$ -> $\frac{dy}{dx} = -Sin(v) * 3$

$\frac{dy}{dx} = -3Sin(3x + 5)$

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8) Find the following limits:

a) $\lim_{x\to1}\left(\frac{x^2+x-2}{x-1}\right)=\lim_{x\to1}\left(\frac{(x-1)(x+2)}{x-1}\right)=\lim_{x\to1}(x+2)$ =1+2=3

b) $\lim_{n\to\infty}\left(\frac{2n^2-3n+2}{6n^2+5n-6}\right)=\lim_{n\to\infty}\left(\frac{2-\frac{3}{n}+\frac{2}{n^2}}{6+\frac{5}{n}-\frac{6}{n^2}}\right)=\frac{2}{6}=\frac{1}{3}$

9) Find the derivative of $f(x)=5-2x$ by first principles.

⇨ f(x+h) = 5 – 2(x + h) = 5 – 2x – 2h. (Subbing in x+h for x)

⇨ $\frac{f(x+h)-f(x)}{h}=\frac{5-2x-2h-(5-2x)}{h}=\frac{-2h}{h}$ = -2

10)

(a) Differentiate the function $2x^2-3x-6$ with respect to x from first principles.

$$f(x)=2x^2-3x-6$$
$$f(x+h)=2(x+h)^2-3(x+h)-6=2x^2+4xh+2h^2-3x-3h-6$$
$$f(x+h)-f(x)=4xh+2h^2-3h$$
$$\underset{h\to0}{Limit}\left(\frac{f(x+h)-f(x)}{h}\right)=\underset{h\to0}{Limit}\left(\frac{4xh+2h^2-3h}{h}\right)=4x-3$$

(b)

$$f(x)=\frac{2x}{x+2}$$

Let $u(x)=2x\Rightarrow u'(x)=2$ and $v(x)=x+2\Rightarrow v'(x)=1$

$$f'(x)=\frac{(x+2)(2)-2x(1)}{(x+2)^2}=\frac{4}{(x+2)^2}$$
$$f'(x)=\frac{1}{4}\Rightarrow\frac{4}{(x+2)^2}=\frac{1}{4}$$
$$\Rightarrow16=(x+2)^2$$

$\Rightarrow x+2=4$ or $x+2=-4$
$\Rightarrow x=2$ or $x=-6$

or

$x^2+4x-12=0$
$(x-2)(x+6)=0$
$\Rightarrow x-2=0$ or $x+6=-0$
$\Rightarrow x=2$ or $x=-6$

$f(-6)=\frac{-12}{-6+2}=3$ and $f(2)=\frac{4}{2+2}=1$

Points $(-6,3)$ and $(2,1)$

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11)

(a)	$f(x+h)-f(x)=(2x+2h+4)^2-(2x+4)^2$ $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=$ $\lim_{h\to 0}\frac{(2x+2h+4)^2-(2x+4)^2}{h}$ $=\lim_{h\to 0}\left(\frac{\left[\begin{array}{c}(4x^2+8hx+4h^2+16x+16h+16)\\-(4x^2+16x+16)\end{array}\right]}{h}\right)$ $=\lim_{h\to 0}\frac{8hx+4h^2+16h}{h}$ $=8x+16$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • any $f(x+h)$ *Mid Partial Credit* • limit of $\frac{f(x+h)-f(x)}{h}$ *High Partial Credit* • limit of $\frac{(2x+2h+4)^2-(2x+4)^2}{h}$ **Notes:** - *omission of limit sign penalised once only* - *answer not from 1st Principles merits 0 marks*

(b) (i)+ (ii)	$y=x.\sin\frac{1}{x}$ $\frac{dy}{dx}=\sin\frac{1}{x}+x\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$ $\frac{dy}{dx}=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ $\frac{dy}{dx}=\sin\frac{\pi}{4}-\frac{\pi}{4}\cos\frac{\pi}{4}$ $=0{\cdot}15$	Scale 15D (0, 4, 7, 11, 15) *Low Partial Credit* • any correct differentiation *Mid Partial Credit* • product rule applied *High Partial Credit* • correct differentiation **Note:** *one penalty for calculator in wrong mode*

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12)

(b)

$$\frac{d(fg(x))}{dx} = \frac{1}{(3(x+5)^2+2)}(6(x+5))$$

$$\frac{d(fg\left(\frac{1}{4}\right))}{dx} = \frac{6(\frac{21}{4})}{3(\frac{21}{4})^2+2} = \frac{504}{1355}$$

$$= 0{\cdot}372$$

OR

$$f(x) = \ln(3x^2+2)$$

$$g(x) = (x+5)$$

$$f[g(x)] = \ln[3(x+5)^2+2]$$

$$= \ln(3x^2+30x+77)$$

$$f'(x) = \frac{6x+30}{3x^2+30x+77}$$

$$x = \frac{1}{4}: \ f'(x) = \frac{31{\cdot}5}{84{\cdot}6875} = 0{\cdot}3719$$

$$= 0{\cdot}372$$

Scale 5C (0, 3, 4, 5)

Low Partial Credit:
- Any correct differentiation
- $fg(x)$ formulated

High Partial Credit:
- $\frac{d(fg(x))}{dx}$ found

Note:
Work with $f(x) \times g(x)$ merits low partial credit at most

13)

(a) (i)

$h(10)$

$= 0{\cdot}001(10)^3 - 0{\cdot}12(10)^2 + p(10) + 5$

$= 30$

$10p = 36$

$p = 3{\cdot}6$

Scale 5C(0, 2, 3, 5)

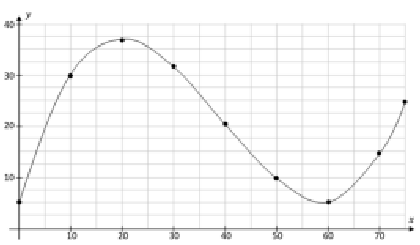
Low Partial Credit:
- $h(10)$ with some relevant substitution

High Partial Credit:
- Equation in p

(a) (ii)

x	0	10	20	30	40
$h(x)$	**5**	30	**37**	**32**	21

x	50	60	70	75
$h(x)$	**10**	5	**12**	21·875



Scale 15D (0, 4, 8, 12, 15)
14 items are required: 5 table entries and 9 plots (which also need to be joined appropriately for *Full Credit*)

Low Partial Credit:
- Any one item correct

Mid Partial Credit:
- any 7 items correct

High Partial Credit:
- any 11 items correct

Full Credit –1:
- All items correct but points not joined or joined incorrectly
- All items but one correct, and points appropriately joined

Calculus 1 Solutions

(b) (i)	$h'(x) = 0{\cdot}003x^2 - 0{\cdot}24x + 3{\cdot}6$	**Scale 5C (0, 2, 3, 5)** *Low Partial Credit:* • correct differentiation of 1 term *High Partial Credit:* • correct differentiation of 2 terms

(b) (ii)	$h'(x) = 0{\cdot}003x^2 - 0{\cdot}24x + 3{\cdot}6$ $h'(20) = 0{\cdot}003(20)^2 - 0{\cdot}24(20) + 3{\cdot}6$ $= 0$, so [local] max/min at $x = 20$ $h''(20) = 0{\cdot}006(20) - 0{\cdot}24 < 0$, so local max Also $h(20) > h(0)$ and $h(20) > h(75)$ **OR** $h'(x) = 0{\cdot}003x^2 - 0{\cdot}24x + 3{\cdot}6$ $h'(20) = 0{\cdot}003(20)^2 - 0{\cdot}24(20) + 3{\cdot}6$ $= 0$, so [local] max/min at $x = 20$ From graph, turning point at $x = 20$ is a [local] max, and it is above the two endpoints [0 and 75]	**Scale 10C (0, 3, 7, 10)** *Low Partial Credit:* • $0{\cdot}003x^2 - 0{\cdot}24x + 3{\cdot}6$ • States $h'(x) = 0$, or similar *High Partial Credit:* • Shows that $h'(20) = 0$, but no further justification that it is the max in the range $[0, 75]$.
(b) (iii)	$h''(x) = 0{\cdot}006x - 0{\cdot}24 = 0$ $0{\cdot}006x = 0{\cdot}24$ $x = 40$ $h(40) = 0{\cdot}001(40)^3 - 0{\cdot}12(40)^2$ $+ 3{\cdot}6(40) + 5$ $= 21$ [m]	**Scale 5C (0, 2, 3, 5)** *Note:* work presented in (b)(iii) must involve calculus, or be based on calculus from (b)(ii), to be awarded any credit. *Low Partial Credit:* • Some correct differentiation of $h'(x)$ • $h''(x)$ indicated *High Partial Credit:* • $x = 40$

