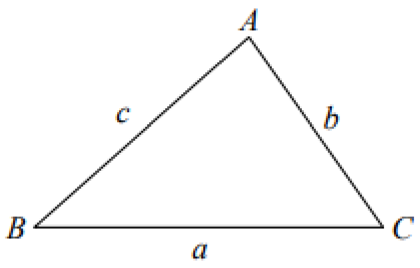
Question 1.

a)

$$\frac{1}{2}ac\sin\angle B = \frac{1}{2}ab\sin\angle C$$

Divide by $\frac{1}{2}abc$

$$\frac{\sin\angle B}{b} = \frac{\sin\angle C}{c} \Rightarrow \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}$$

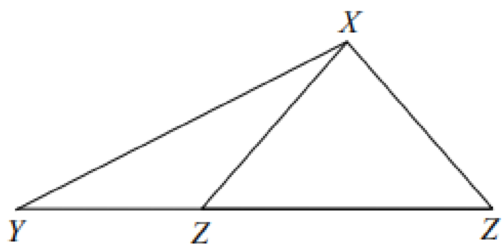


b) i)

$$\frac{3}{\sin 27^\circ} = \frac{5}{\sin\angle Z} \quad \Rightarrow \quad \sin\angle Z = \frac{5\sin 27^\circ}{3} = 0\cdot756$$

$$\Rightarrow \quad |\angle Z| = 49^\circ \quad \text{or} \quad |\angle Z| = 131^\circ$$

ii)



c)

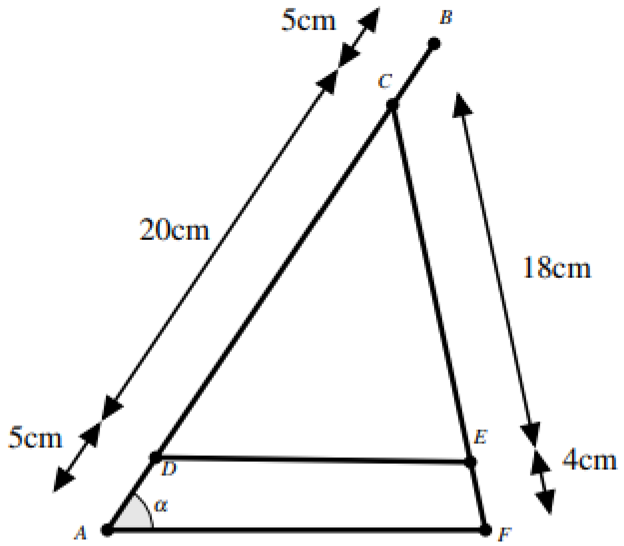
$$|\angle ZXY| = 180^\circ - (27^\circ + 49^\circ) = 104^\circ$$

$$\Delta = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}(5)(3)\sin 104^\circ = 7\cdot 27 = 7\text{ cm}^2$$

Question 2.

a)

Consider the diagram below in which we have marked some of the lengths. Note that $|DC|$ must be 20cm since we are told that $|AB| = 30$cm. Similarly $|CE| = 18$cm since we are told that $|CF| = 22$cm.



Now suppose that $\alpha = 60^\circ$. Then we apply the Sine Rule to the triangle $\triangle ACF$. So

$$\frac{\sin 60^\circ}{22} = \frac{\sin(\angle F)}{25}$$

Therefore

$$\sin(\angle F) = \frac{25\sin(60)}{22} = 0.9841$$

correct to four decimal places. Therefore $|\angle F| = \sin^{-1}(0.9841) = 79.77^\circ$. Now we use this to calculate $\angle C$. So

$$|\angle C| = 180 - 60 - 79.77 = 40.23^\circ.$$

Now we apply the Cosine Rule to the triangle $\triangle CDE$. Thus,

$$\begin{aligned} |DE|^2 &= 20^2 + 18^2 - 2(20)(18)\cos(40.23^\circ) \\ &= 400 + 324 - 549.7 \\ &= 174.3 \end{aligned}$$

Therefore

$$|DE| = \sqrt{174.3} = 13.20\text{cm}.$$

So the length of the strap when $\alpha = 60^\circ$ is 13.20cm.

b)

The maximum possible value of α will occur when the stand is set so that CF is vertical. In that case $\triangle ACF$ is a right angled triangle with the hypotenuse $|AC| = 25$cm. The side opposite the angle α is $|CF| = 22$cm. Therefore in this case,

$$\sin\alpha = \frac{22}{25} = 0.88.$$

So

$$\alpha = \sin^{-1}(0.88) = 61.64 = 62^{\circ}$$

correct to the nearest degree.

Question 3.

$$\sin 3x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow 3x = 60°,\quad 120°,\quad 420°,\quad 480°,\quad 780°,\quad 840°$$

$$\Rightarrow x = 20°,\quad 40°,\quad 140°,\quad 160°,\quad 260°,\quad 280°$$

or

$3x=60° + n(360°),\ n \in \mathbb{Z}$ or $3x=120° + n(360°),\ n \in \mathbb{Z}$

$x=20° + n(120°),\ n \in \mathbb{Z}$ or $x=40° + n(120°),\ n \in \mathbb{Z}$

$n=0 \Rightarrow x= 20°$ or $x= 40°$

$n=1 \Rightarrow x= 140°$ or $x= 160°$

$n=2 \Rightarrow x= 260°$ or $x= 280°$

Question 4.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

$$=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

or

$$\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A\sin B}{\cos A\cos B}}=$$

$$\frac{\dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B-\sin A\sin B}{\cos A\cos B}}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}=$$

$$\frac{\sin(A+B)}{\cos(A+B)}=\tan(A+B)$$

Question 5. (Question 2, Paper 1, 2000)

(a) $4x-y+5z = 4$ equation(i) $6x - 2y + 6z = 2$ equation (i) x 3)
$3x-y+3z= 1$ $x + 2y - 2z = -1$ from question

Subtracting $x+ 2z = 3$ equation (iv) Adding $7x + 4z = 1$ (equation (v))

$2x + 4z = 6$ (iv) x 2
$7x + 4z = 1$ equation (v)

Subtracting $-5x = 5 \Rightarrow x = -1$

From equation (iv) $-1 + 2z= 3 \Rightarrow z = 2$
From equation (i) $-4 - y +10 = 4 \Rightarrow y= 2$

(b) Solve $x^2 - 2x - 24 = 0$

Solution $(x-6)(x+4) = 0 \Rightarrow$ roots are $x = 6$ and $x = -4$

Hence, find the values of x for which

if $x+ \frac{4}{x} = 6$ $\Rightarrow x^2 -6x + 4 = 0 \Rightarrow a=1, b = -6, c = 4$

roots are $6 +\{ \sqrt{((-6)^2-4(1)(4))}\}/2(1) = [6 + \sqrt{20}\}/2 = 3+2\sqrt{5}$
$6 - \ldots\ldots = 3-2\sqrt{5}$

If $x+ \frac{4}{x} =- 4$ $\Rightarrow x^2 +4x + 4 = 0 \Rightarrow (x+2)(x+2) = 0 \Rightarrow x=-2$

(c) (i) $a^4 - b^4 = (a^2-b^2)(a^2+b^2) = (a+b)(a-b)(a^2+b^2)$ Difference of two squares.

(ii) $a^5 - a^4b - ab^4 + b^5 = a^4(a-b) - b^4(a-b) = (a^4 - b^4)(a-b)$

From part (i) $= (a+b)(a-b)(a-b)(a^2+b^2) = (a^2+b^2)(a-b)^2(a+b)$

From part (ii) $a^5 - a^4b - ab^4 + b^5 = (a^2+b^2)(a-b)^2(a+b)$

So $a^5 - a^4b - ab^4 + b^5 > 0$ as all terms on the RHS are positive

$a^5 + b^5 > a^4b + ab^4$

Question 6. (Question 1, Paper 2, 2000)

(a) centre = (0,0)

Slope of line from centre to (-7,9) is (y2-y1)/ (x2/x1) = (9-0)/(-7- 0) = -9/7

Slope of tangent is 7/9

(b) $x^2 + y^2 - 6x + 4y - 12 = 0$ is the equation of a circle.

Generic equation is $x^2 + y^2 + 2gx + 2fy + c = 0$, centre = (-g, -f), radius is $\sqrt{g^2 + f^2 - c}$

So for this circle g=-3 and f = 2, centre is (3,-2)

Radius = $\sqrt{(-3)^2 + (2)^2 + 12} = \sqrt{25} = 5$

$x^2 + y^2 + 12x - 20y + k = 0$ is another circle

Centre = (-6, 10), radius = $\sqrt{-6^2+10^2-k} = \sqrt{136-k}$

Distance between centres (3,-2) and (-6,10) is

$\sqrt{(y2-y1)^2 + (x2-x1)^2} = \sqrt{(10-(-2))^2+(-6-3)^2} = \sqrt{144+81} = \sqrt{225} = 15$

So the sum of the radii = the distance between the centres

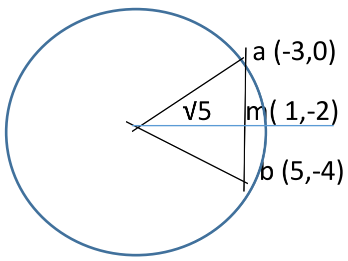
$\sqrt{136-k} + 5 = 15 \Rightarrow \sqrt{136-k} = 10 \Rightarrow k = 36$

(c) A circle intersects a line at the points a(– 3, 0) and b(5, – 4).

Solution : m = ((-3+5)/2, (0-4)/2) = (1, -2)

The distance from the centre of the circle to m is $\sqrt{5}$.

Solution



Distance ma = $\sqrt{(-3-1)^2 + (0-(-2))^2} = \sqrt{20} = 2\sqrt{5}$

By Pythagoros radius^2 = $(\sqrt{5})^2 + (2\sqrt{5})^2 = 25 \Rightarrow$ radius = 5

If generic circle is $x^2+y^2 + 2gx + 2fy + c = 0$

As a is on the circle $(-3)^2 + 0^2 -6g+c = 0$ => $c = 6g – 9$

As b is on the circle $5^2 + (-4)^2 +10g – 8 f + c = 0$ = > $25 +16 + 10g – 8f +6g – 9 = 0$

So $8f = 16g + 32$ => $f = 2g + 4$

$\sqrt{g^2 + [2 (g+2)]^2-(6g-9)} = 5$ = > $g^2 + 4(g^2+4g+4) – 6g + 9 =25$ => $g^2 + 4g^2+16g+16-6g+9= 25$ => $5g^2+10g=0$ => $g^2 + 2g = 0$ = >$g(g+2)= 0$ => $g=0$ (and $f = 4$ and $c = -9$) or $g = -2$ (and $f = 0$ and $c= -21$)

Equation Circle 1 is $x^2+y^2 +8y-9=0$ Equation Circle 2 is $x^2+y^2 -4x -21 = 0$

Question 7. (Question 1, Paper 2, 2001)

(a) A circle with centre (-3, 7) passes through the point (5, -8)

Generic equation of circle is $x^2+y^2 + 2gx + 2fy + c = 0$ where centre = $(-g,-f)$ and radius^2 = $\sqrt{(g^2+f^2-c)}$

So $g = 3$ and $f = -7$.

As (5,-8) is on circle $25 + 64 +6(5)-14(-8) + c = 0$ => $119 + 112 = c = 0$ => $c = - 231$

Equation is $x^2+y^2 + 6x -14y -231 = 0$

(b) The equation of a circle is $(x+1)^2 + (y-8)^2 = 160$

The line $x-3y +25 = 0$ intersects the circle at the points p and q.

(i) Find the co-ordinates of p and the co-ordinates of q.
(ii) Investigate if [pq] is a diameter of the circle.

$X = 3y-25$, replace x in equation of circle and solve for y

$(3y-24)^2 + (y-8)^2 = 160$ = > $9y^2 - 144y + 576 + y^2-16y+ 64 = 160$ => $10y^2 -160y + 480 = 0$

⇨ $y^2 -16y + 48 = 0$ => $y = 4$ (and $x =-13$) or $y=12$ (and $x = 11$)

Intersection is at (-13,4) and (11,12)

Centre (-1,8), radius is $\sqrt{(1 + 64 + 95)} = \sqrt{160} = 4\sqrt{10}$

If pq is a diameter, midpoint = centre of circle.
Midpoint pq =((-13+11)/2, (4+12)/2) = (-1, 8) so pq is a diameter.

(c) The circle $x^2 + y^2 + 2gx+ 2 fy + c = 0$ passes through the points (3,3) and (4, 1)
The line $3x – y -6 = 0$ is a tangent to the circle at (3,3)

(i) Find the real numbers g, f and c.

(3,3) => 9 + 9 + 6g + 6f + c = 0 => 6g + 6f + c = -18
(4,1)=> 16+ 1 +8g + 6f+ c = 0 => 8g + 2f + c = -17

Subtract -2g + 4f = -1 (equation (1))

Line from centre to (3,3) will have slope (-1/3) (i.e. perpendicular to tangent.)
Equation of line through centre and (3,3) is y-3 = (-1/3) (x-3) => 3y-9 = -x + 3 => x + 3y = 12
(-g,-f) on this line so –g – 3f = 12 (equation (2)) => -2g – 6f = 24 (equation (2) * 2)

Subtract Equation (2) from (1) and 10f=-25 => f = -5/2 and g = -9/2 (centre = 9/2, 5/2))
By substitution in equation (1) we can get c = 24

Equation is therefore $x^2 + y^2 -9x -5y + 24 = 0$

(ii) Find the co-ordinates of the point on the circle at which the tangent parallel to 3x-y-6=0 touches the circle.

Centre is midpoint between two tangential points.
(3,3), (9/2, 5/2), **(6, 2)**

Question 8. (Question 4, Paper 2, 2001)

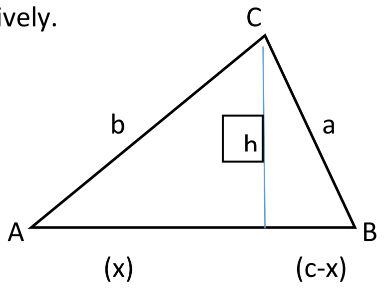
(a) $\frac{\theta}{2\pi} = \frac{10}{2\pi(4)}$ $\theta = 10/4 = 2.5$ radians

1 radian = 57.296° = > 2.5 radians = 143°

(b) $\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

(ii) $\cos 2x - \sin x = 1$
$1 - 2\sin^2 x - \sin x - 1 = 0$
$2\sin^2 x + \sin x = 0$
$\sin x\,(2\sin x + 1) = 0 \Rightarrow \sin x = 0$ (i.e. x = 0°, 180°, 360° or $\sin x = -1/2$ i.e. x = 210°, 330°)

(c) A triangle has sides a, b and c.
The angles opposite a, b and c are A, B and C, respectively.



(ii) Prove that $a^2 = b^2 + c^2 - 2bc\cos A$

$h^2 = b^2 - x^2$ but $h^2 = a^2 - (c-x)^2 = a^2 - (c^2 - 2cx + x^2)$

so $b^2 - x^2 = a^2 - c^2 + 2cx - x^2$ but $x/b = \cos A \Rightarrow x = b\cos A$

⇨ $a^2 = b^2 + c^2 - 2bc\cos A$

(iii) Show that $c(b\cos A - a\cos B) = b^2 - a^2$

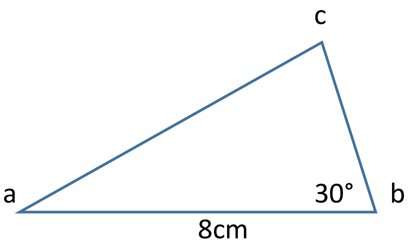
$bc\cos A - ac\cos B = bc\frac{\{b^2 + c^2 - a^2\}}{2bc} - ac\frac{\{a^2 + c^2 - b^2\}}{2ac}$

$= \frac{b^2 + c^2 - a^2}{2} - \frac{a^2 + c^2 - b^2}{2}$

$= \frac{2b^2 - 2a^2}{2} = b^2 - a^2$

Question 9. (Question 5 , Paper 2, 2002)

(a) The area of triangle abc is 12 cm2. |ab | = 8 cm and |∠abc | = 30° Find |bc| .



Solution

Area of Triangle = (1/2) |ab||bc| sin 30 ° => $12cm^2$ = (1/2) 8 |bc|(1/2) => |bc| = 6cm

(b) (i) Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{(\sin A/\cos A) + (\sin B/\cos B)}{1 - (\sin A \sin B)/(\cos a \cos B)} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) Hence, or otherwise, prove that tan 22.5° = √2 -1

Let A = 22.5 ° and B = 22.5 °

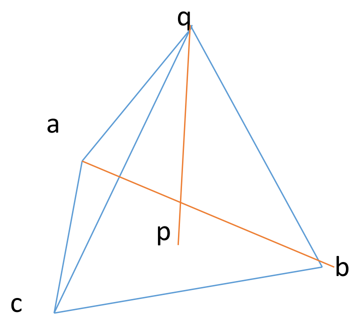
So tan 45 ° = 2 tan A/ (1-$\tan^2$A) but tan 45° =1 so 1- $\tan^2$A = 2 tan A
$\tan^2$A + 2 tan A – 1 = 0

tan A = [-2 +/- √(4+4)] / 2 = -1 +/- √2

as tan 22.5 ° is positive tan 22.5 ° = -1+ √2 = √2 - 1

(c) A vertical radio mast [pq] stands on flat horizontal ground. It is supported by three cables that join the top of the mast, q, to the points a, b and c on the ground. The foot of the mast, p, lies inside the triangle abc. Each cable is 52 m long and the mast is 48 m high.

(i) Find the (common) distance from p to each of the points a, b and c.

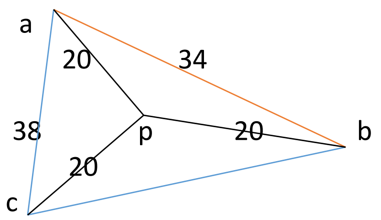


Solution

|pa| = |pb| = |pc|

Triange apq is right angled at p => $|ap|^2 = |aq|^2 - |qp|^2$ => $|ap|^2 = 52^2 - 48^2 = 400$ => |ap| = 20

(ii) Given that |ac| = 38 m and |ab| = 34 m, find |bc| correct to one decimal place.



cos <apc = $\{20^2 + 20^2 - 38^2\} / \{2(20)(20)\} = -644/800$ => apc = 143.61°

cos <apb = $\{20^2 + 20^2 - 34^2\} / \{2(20)(20)\} = -356/800$ => apb = 116.42°

$|bc|^2 = 20^2 + 20^2 - 2(20)(20)\cos 99.97° = 800 + 138.506 = 938.506$ = > |bc| = 30.6m

Question 10. (Question 4, Paper 2, 2003)

(a) The circumference of a circle is 30 π cm. The area of a sector of the circle is 75 cm^2 .
Find, in radians, the angle in this sector.

$2 \pi r = 30 \pi \Rightarrow r = 15$

$\frac{75}{\pi (15^2)} = \frac{\theta}{2 \pi}$ => $\theta = (150/225)$ radians = (2/3) radians

(b) Find all the solutions of the equation sin 2x + sin x = 0 in the domain 0° ≤ x ≤ 360 ° .

Solution

sin 2x + sin x = 0 => 2 sin x cos x + sin x = 0

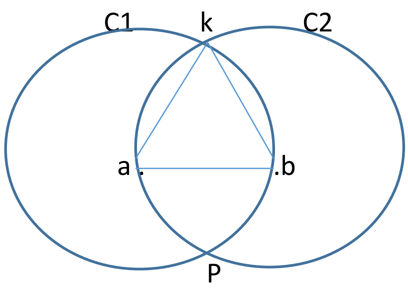
Sin x (2cos x + 1) = 0 = > sin x = 0 (i.e. x = 0°, 180°, 360°) or cos x = -1/2 (i.e. x = 120°, 240°)

(c) C1 is a circle with centre a and radius r. C2 is a circle with centre b and radius r.
C1and C2 intersect at k and p. a∈ C2 and b∈ C1.

(i) Find, in radians, the measure of angle kap.

kab is an equilateral triangle =| <kab| = π/3 radians and | <kap| = 2 π/3 radians

(iii) Calculate the area of the intersection region. Give your answer in terms of r and π .



Area of triangle kap = (1/2) r^2 sin 2 π/3 = (√3/4) r^2

Area of sector apk in C1 = $(1/2)\, r^2\, 2\pi/3 = (\pi/3)\, r^2$
Area of sector apk – triangle kap = $r^2((\pi/3) - (\sqrt{3}/4))$

Similarly in respect of circle 2 we need the area of sector bpk – triangle kpb.

Total intersection is therefore $2\, r^2((\pi/3) - (\sqrt{3}/4))$