Society of Actuaries in Ireland

SAI Tutorial on Complex Numbers and Proof by Induction - Answers

Question A

(i)

$2z_1 + z_2 = 2(2+3i)+(1-2i) = 5 + 4i$

$z_1 z_2 = (2+3i)(1-2i) = 8 - i$

$z_1 - iz_2 = (2+3i)-i(1-2i) = 2i$

$|2+z_1| = |2+(2+3i)| = \sqrt{(4^2+3^2)} = 5$

- $\frac{z_1}{z_2} = \frac{-4}{5} + \frac{7i}{5}$

$\overline{2z_1+z_2} = 5 - 4i$

(ii)

$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Solve for $\cos^{-1}(2/2\sqrt{2}) = \pi/4$

$2+2i = 2\sqrt{2}(\cos \pi/4 + i\sin \pi/4)$

$-2\sqrt{2} + 2\sqrt{2}i = 4(\cos(3\pi/4) + i\sin(3\pi/4))$

Question B (Question 3 Paper 1 2022)

(a) (i)

$z - iz$ =6+2i−i(6 + 2i)

=6+2i − 6i − 2i^2

= 6 − 4i -2(-1) = 8 − 4i

Or

If z - iz = 8 - 4i => z(1 – i) = 8 - 4i

$$z = \frac{8-4i}{1-i}$$

$$z = \frac{8-4i}{1-i} \cdot \frac{1+i}{1+i} = \frac{8-4i+8i-4i^2}{1-i+i-i^2} = \frac{8+4i-4(-1)}{1-(-1)}$$

$$z = \frac{12+4i}{2}$$

z =6+2i

(a) (ii)

$|z|^2 = 6^2 + 2^2 = 40$

$|iz|^2 = 2^2 + 6^2 = 40$

$|z|^2 + |iz|^2 = 40 + 40 = 80$

$|z - iz|^2 = |8-4i|^2 = 8^2 + 4^2 = 80$

(a) (iii)

Right angled triangle so Pythagorean therefore diameter = $|z|^2 + |iz|^2$ = 80 from pervious

Radius = √80 ÷2= √20

Area = $\pi r^2 = 20\pi$ square units

OR

Calculate centre as midpoint of iz & z

Centre = 2+4i then Radius is distance to z.

$\text{Radius} = \sqrt{(6-2)^2 + (2-4)^2}$ = √20

Area = $\pi r^2 = 20\pi$ square units

(b)

De Moivre's Thoerom => [rCosθ+irSinθ]^n = r^n(Cos(nθ)+Sin(nθ))

Write complex number in polar form first i.e. $(\sqrt{3} - i)$ = rCosθ+irSinθ

$r = \sqrt{(\sqrt{3})^2 + (-1)^2}$ = √4 = 2

Cos θ = x/r => θ =Cos^{-1}(x/r) = Cos^{-1}(√3 /2) = $\pi/6$ or $11\pi/6$

Sin θ = y/r = -1/2 => θ = $11\pi/6$

$(\sqrt{3} - i) = 2(\text{Cos}(11\pi/6) + i\text{Sin}(11\pi/6))$

$(\sqrt{3} - i)^9 = 2^9\ (\text{Cos}(99\pi/6) + i\text{Sin}(99\pi/6))$

$(\sqrt{3} - i)^9 = 512\ (0 + i) = 512i$

$a = 0$, $b = 512$

Question C (Question 1 Paper 1 2021)

(a)

$$\frac{4-2i}{2+4i} = \frac{4-2i}{2+4i} \cdot \frac{2-4i}{2-4i} = \frac{8-4i-16i+8i^2}{4+8i-8i-16i^2} = \frac{8-20i+8(-1)}{4-16(-1)} = \frac{-20i}{20}$$

k= −1

(b)

Use De Moivres Theorem

Polar Form of -5+12i = rCosθ+irSinθ

r = $\sqrt{(5^2+12^2)}$= √169 = 13

Cos θ = -5/13

$-5 + 12i = 13(\cos\theta + i\ \sin\theta)$

$(-5 + 12i)^{1/2} = [13(\cos(\theta+2n\pi) + i\ \sin(\theta+2n\pi)]^{1/2}$

$= \sqrt{13}\ (\cos(\theta/2+2n\pi/2) + i\ \sin(\theta/2+2n\pi/2))$ for n=0 or 1

Solve θ in radians then calculate each one

or

Solve for n=0, Log tables $\text{Sin}^2(\theta/2) = 1/2(1-\text{Cos}\ \theta)$

$2\text{Sin}^2(\theta/2) = 1-\text{Cos}\ \theta = 1-(-5/13) = 18/13$

$\text{Sin}(\theta/2) = \sqrt{(9/13)} = 3/\sqrt{13}$

$2\cos^2(\theta/2) = 1 + \cos\theta = 1 + (-5/13) = 8/13$

$\cos(\theta/2) = \sqrt{4/13} = 2/\sqrt{13}$

$(-5 + 12i)^{1/2} = \sqrt{13}(2/\sqrt{13} + i\,3/\sqrt{13}) = 2 + 3i$

Solve second root for n=1, => $-2 - 3i$

Alternative method to

Set $(-5+12i)^{1/2} = (a+bi)$ => $(-5+12i) = (a+bi)^2$ and solve expansion

(c)

$Z^3 = -8 = r(\cos\theta + i\sin\theta)$

$|Z^3| = 8 = r$

$\cos\theta = -1 \quad \theta = \pi$

$Z = 8^{1/3}[\cos((\pi + 2n\pi)/3) + i\sin(\pi + 2n\pi)/3]$ for $n = 0,1,2$

For n=0, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/3) = 1/2$

Root is $2(1/2 + i\sqrt{3}/2) = 1 + \sqrt{3}i$

For n=1, $\sin(3\pi/3) = 0$, $\cos(3\pi/3) = -1$

Root is $2(-1 + i0) = -2$

For n=2, $\cos(5\pi/3) = 1/2$, $\sin(5\pi/3) = -\sqrt{3}/2$

Root is $2(1/2 + i(-\sqrt{3}/2)) = 1 - \sqrt{3}i$

Question D (Question 2 Paper 1 2020)

(a)

Multiply iz_1 by i to get $i^2z_1 = -z_1$ and then substitute in formulae

$$iz_1 = -4 + 3i$$
$$i(iz_1) = i(-4 + 3i)$$
$$-z_1 = -4i + 3i^2$$
$$z_1 = 3 + 4i$$
$$3z_1 - z_2 = 3(3 + 4i) - z_2 = 11 + 17i$$
$$z_2 = 9 + 12i - 11 - 17i$$
$$z_2 = -2 - 5i$$

(b) (i)

$3+2i = a_1$

$5-i = a_1r$

$r = a_1r/ a_1$

$$\frac{5-i}{3+2i} = \frac{5-i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{15-3i-10i+2i^2}{9+6i-6i-4i^2} = \frac{15-13i+2(-1)}{9-4(-1)} = \frac{13-13i}{13}$$

$r=1-i$

(b) (ii)

Write 1 – i in polar form i.e. $\sqrt{2}\,(\cos(7\pi/4) + i\sin(7\pi/4))$

$$T_9 = ar^8$$
$$T_9 = (3 + 2i)(1 - i)^8$$
$$T_9 = (3 + 2i)\left(\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\right)^8$$
$$T_9 = (3 + 2i)(\sqrt{2})^8\left(\cos\frac{7\pi(8)}{4} + i\sin\frac{7\pi(8)}{4}\right)$$
$$T_9 = (3 + 2i)(16)(\cos 14\pi + i\sin 14\pi)$$
$$T_9 = (3 + 2i)(16)(1 + 0i)$$
$$T_9 = 48 + 32i$$

Question E (Question 5 Paper 1 2019)

(a)

$$(3+2i)^2 + p(3+2i) + q = 0$$
$$5 + 12i + 3p + 2pi + q = 0$$
$$2p = -12 \Rightarrow p = -6$$
$$5 + 3p + q = 0 \;\Rightarrow q = 13$$

(b)(i)

$$|v| = \sqrt{4+12} = 4$$
$$\theta = 300°$$
$$v = 4(\cos 300° + i \sin 300°)$$

Or

$$|v| = \sqrt{4+12} = 4$$
$$\theta = \frac{5\pi}{3}$$
$$v = 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

(b)(ii)

$$w = \pm v^{\frac{1}{2}}$$

$$w = \pm 2(\cos 300 + i \sin 300)^{\frac{1}{2}}$$

$$w = \pm 2(\cos 150 + i \sin 150)$$

$$w = \pm(-\sqrt{3} + i)$$

$$w = -\sqrt{3} + i \text{ or } \sqrt{3} - i$$

Or

$$w = [4(\cos(300 + 360n) + i\sin(300 + 360n)]^{\left(\frac{1}{2}\right)}$$

$$w = 4^{\frac{1}{2}}[\cos(150 + 180n) + i\sin(150 + 180n)]$$

$\underline{n = 0}$

$$w = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$\underline{n = 1}$

$$w = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

<u>Question F (Question 1 Paper 1 2013)</u>

(a)

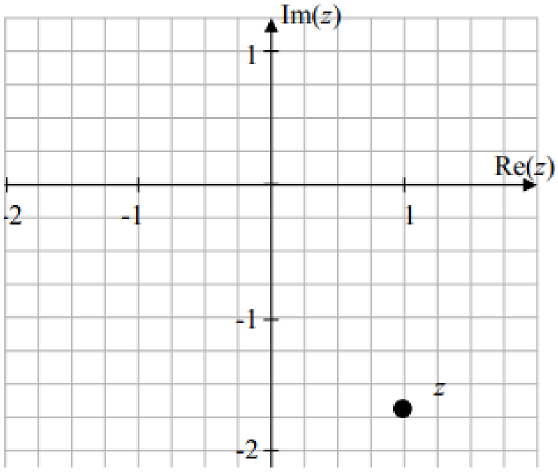
$$z = \frac{4}{1+\sqrt{3}i} = \frac{4}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{4-4\sqrt{3}i}{1+3} = 1-\sqrt{3}i$$

(b)

$$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \theta = \frac{5\pi}{3}$$

$$r = |1-\sqrt{3}i| = \sqrt{1+3} = \sqrt{4} = 2$$

$$z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$



(c)

$$z^{10} = \left[2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\right]^{10}$$

$$= 2^{10}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2^{10}\left(\cos\frac{50\pi}{3} + i\sin\frac{50\pi}{3}\right)$$

$$= 2^{10}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2^{10}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2^{9}\left(1-\sqrt{3}i\right)$$

Question G (Question 1 Paper 1 2016)

(a)

Assume other root is x+yi and expand expression using roots

(z+4-3i).(z-x-iy) and isolate iz terms to determine value the value for y (-3iz – yiz must equal 0 as no iz term in original equation)

y = -3 and then derive x using constants (3ix – 4iy must equal 0) x = -4

x + yi = -4-3i

(b)

$r = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \theta = \frac{\pi}{4}$

$$(1+i)^8 = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^8$$

$$(1+i)^8 = \{16(\cos 2\pi + i\sin 2\pi)\}$$

$(1+i)^8 = 16(1) = 16$

(c)

$$z = \frac{(2-i) \pm \sqrt{(-2+i)^2 - 4(3-i)}}{2}$$
$$= \frac{(2-i) \pm \sqrt{4 - 4i - 1 - 12 + 4i}}{2}$$
$$= \frac{2 - i \pm \sqrt{-9}}{2}$$
$$= \frac{2 - i \pm 3i}{2}$$
$$= 1 - 2i \text{ or } 1 + i$$

Question H (Question 4 Paper 1 2021)

Prove for n = 1, $2^{3.1-1}+3 = 2^2 + 3$

Assume True for n=k

Prove if true for n=k, then true for n=k+1: $2^{3.(k+1)-1}+3 = 2^{3.k+3-1}+3$

$= 2^{3k+2}+3$

$= 2^3.2^{3k-1}+3$

$= 8.2^{3k-1}+3$

$= 7.2^{3k-1}+1.\ 2^{3k-1}+3 = = 7.2^{3k-1}+(2^{3k-1}+3)$

7.2^{3k-1} is divisible by 7

If n=k true then $2^{3k-1}+3$ also divisible by 7.

Question I (Question 7 Paper 1 2020)

Prove for n = 1, [1(1+1)(2+1)]/6 = 1

Assume True for n ie : $(1 + 4 + 9 + \cdots + n^2) = [n(n+1)(2n+1)]/6$

Prove if true for n, then true for n+1: [(n+1)((n+1)+1)(2(n+1)+ 1)]/6=[(n+1)(n+2)(2n+3)]/6

$=[(n+1)(2n^2+7n+6)]/6$

$=[(n+1)(2n^2+1n+6n+6)]/6$

=[(n+1)(n(2n+1)+6(n+1))]/6

=[(n+1)(n)(2n+1)/6]+(n+1).6.(n+1))]/6

=[(n+1)(n)(2n+1)/6]+(n+1).6.(n+1))]/6

If true for n then $=(1 + 4 + 9 + \cdots + n^2)+6(n+1)^2/6$

$=(1 + 4 + 9 + \cdots + n^2)+(n+1)^2$

Therefore true for n+1 if true for n

Question J 2023 Q 4 paper 1

(a)	**Method 1** $(1+i)^2+(3-2i)(1+i)+p=0$ $1+2i+i^2+3+i-2(i)^2+p=0$ $5+3i+p=0$ $p=-5-3i$ **Method 2** Let the second root $= z_2$ Sum of roots: $1+i+z_2=-3+2i$ $z_2=-4+i$ Product of roots: $(1+i)(-4+i)=p$ $p=-5-3i$ **Method 3** $z=\frac{-(3-2i)\pm\sqrt{(3-2i)^2-4p}}{2}$ $2z=-(3-2i)\pm\sqrt{(3-2i)^2-4p}$ $2z+3-2i=\pm\sqrt{(3-2i)^2-4p}$ $[2z+3-2i]^2=(3-2i)^2-4p$ $z=1+i$ satisfies this equation

(b)	<u>Reference Angle:</u> $\alpha = \tan^{-1}\frac{\sqrt{3}}{1} = 60° \left(\frac{\pi}{3}\text{ rads}\right)$ <u>Argument:</u> $\theta = 180^0 - 60^0 = 120^0 \left(\frac{2\pi}{3}\text{ rads}\right)$ <u>Modulus:</u> $r = \sqrt{(-1)^2 + (\sqrt{3})^2}$ $= \sqrt{4}$ $= 2$ <u>General Polar Form:</u> $2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)$ $w^2 = 2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)$ $w = \left[2\left(\cos\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\left(\frac{2\pi}{3} + 2n\pi\right)\right)\right]^{\frac{1}{2}}$ <u>De Moivre:</u> $w = 2^{\frac{1}{2}}\left[\left(\cos\frac{1}{2}\left(\frac{2\pi}{3} + 2n\pi\right) + i\sin\frac{1}{2}\left(\frac{2\pi}{3} + 2n\pi\right)\right)\right]$ $= 2^{\frac{1}{2}}\left[\cos\left(\frac{\pi}{3} + n\pi\right) + i\sin\left(\frac{\pi}{3} + n\pi\right)\right]$ <u>$n = 0$:</u> $w = \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ $= \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$ <u>$n = 1$:</u> $w = \sqrt{2}\left(\cos\left(\frac{\pi}{3} + \pi\right) + i\sin\left(\frac{\pi}{3} + \pi\right)\right)$ $= \sqrt{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ $= -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$