Algebra Hints and Tips

Simplifying an expression. What order should you follow?

Brackets
Exponents
Divide | Multiply
Add | Subtract

To manipulate an equation, you can:

- Add or subtract the same number to/from both sides
- Multiply or divide both sides by the same number
- Square both sides, cube both sides, etc.
- Take the square root of both sides or cube root, etc

Simple Algebraic Rules

- Multiplying Terms
  $(a) \times (bc) = abc$
  $(a) \times (b+c) = ab + ac$
  $(a+b) \times (c+d) = a(c+d) + b(c+d)$

- Anything to the power of 0 is 1.

  $A^0 = 1$, $473^0 = 1$, $\pi^0 = 1$, $1{,}000{,}000^0 = 1$

- Inverse Powers
  $X^{-1} = \frac{1}{X}$, $Y^{-6} = \frac{1}{Y^6}$

- Indices

  $Y^3 + Y^2 = Y \times Y \times Y + Y \times Y$ ..... Can't be simplified

  $(Y^2)^3 = (Y \times Y)^3 = (Y \times Y) \times (Y \times Y) \times (Y \times Y) = Y^6$ ..... 2 x 3 = 6

  $Y^2 \times Y^3 = (Y \times Y) \times (Y \times Y \times Y) = Y^5$ ..... 2 + 3 = 5

  $\frac{Y^3}{Y^2} = \frac{\cancel{Y} \times \cancel{Y} \times Y}{\cancel{Y} \times \cancel{Y}} = Y^1$ ..... 3 – 2 = 1

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Cancelling terms in fractions:

$$\frac{\cancel{A}}{\cancel{A} \times B} = \frac{1}{B}$$ ✓

$$\frac{X^2+2X}{X} = \frac{\cancel{X}(X+2)}{\cancel{X}} = \frac{X+2}{1} = X + 2$$ ✓

$$\frac{\cancel{A}}{\cancel{A} + B} = \frac{1}{B}$$ ✗

$$\frac{\cancel{X}+2}{\cancel{X}} = 2$$ ✗

Indices and Logarithms (From your Log Tables – Very Important!!!)

Indices

$a^p a^q = a^{p+q}$

$a^{\frac{1}{q}} = \sqrt[q]{a}$

$\frac{a^p}{a^q} = a^{p-q}$

$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

$(a^p)^q = a^{pq}$

$(ab)^p = a^p b^p$

$a^0 = 1$

$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

$a^{-p} = \frac{1}{a^p}$

Logarithms

General rule of logs: $a = b^c \iff \log_b a = c$

$\log_a(xy) = \log_a x + \log_a y$

$\log_a\left(\frac{1}{x}\right) = -\log_a x$

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

$\log_a(a^x) = x$

$\log_a x^q = q \log_a x$

$a^{\log_a x} = x$

$\log_a 1 = 0$

$\log_b x = \frac{\log_a x}{\log_a b}$

Common mistake with Indices

$(A + B)^2 \quad = A^2 + B^2$ ✗

$(A + B)^2 \quad = (A + B) \times (A + B)$

$= A \times (A + B) + B \times (A + B)$

$= A^2 + AB + AB + B^2$

$= A^2 + 2AB + B^2$ ✓

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Common forms of ***algebraic equations you might need to factorise***:

Take out common terms	**Factorise by grouping**
$ab + ad = a(b + d)$	$ab + ad + cb + cd = a(b + d) + c(b + d) = (a + c)(b + d)$
Factorise a trinomial $a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2$	**Difference of two squares** $a^2 - b^2 = (a + b)(a - b)$
Difference of two cubes $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	**Sum of two cubes** $a^3 + b^3 = (a + b)(a^2 - 2ab + b^2)$

Solving Quadratic Equations (i.e. finding the roots of a quadratic equation)

You can either:

A. Factorise & let each factor = 0 ; OR

B. Use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; OR

C. Complete the square (write in vertex form) and set = 0

All three approaches will work. Some quadratics are easier to factorise than others, but you can always use the formula approach – even for simple quadratics.

Regarding the formula:

- If $b^2 - 4ac > 0$, the equation has two real distinct roots
- If $b^2 - 4ac = 0$, the equation has two equal real roots
- If $b^2 - 4ac < 0$, the equation has no real roots

Factor theorem

"If an algebraic expression is divided by one of its factors, then the remainder is zero. The expression (x – k) is a factor of a polynomial f(x) if the remainder when we divide f(x) by (x – k) is zero."

- If f(k) = 0, then (x – k) is a factor of f(x)
- If (x – k) is a factor of f(x), then f(k) = 0

Solving cubic equations ($ax^3 + bx^2 + cx + d = 0$)

- Find the first root, k, by trial and error
- If $x = k$ is a root, then $(x – k)$ is a factor
- Divide $f(x)$ by $(x – k)$, which always gives a quadratic expression
- Find the factors of the quadratic and then find the roots of the quadratic

Inequalities

Always remember that multiplying or dividing both sides of an inequality by a negative number reverses the direction of the inequality symbol

Quadratic inequalities

Replace $>$, $<$, $\geq$, $\leq$ with $=$ to make an equation

- Solve the equation to find the roots -> these are called the critical values of the inequality & the solution either will be between these critical values or outside these critical values
- Test a number between the critical values (often 0) in the original inequality

Two possibilities arise:

- If the inequality is true, then the solution lies between the critical values
- If the inequality is false, then the solution does not lie between the critical values

(**Note:** that if the inequality uses $<$ or $>$, the critical values are not included in the solution set, whereas if the inequality uses $\leq$ or $\geq$, the critical values are included in the solution set)

Modulus inequalities

If $|x| \leq a$, then $–a \leq x \leq a$ (**Note:** "mod x" is the same as "$|x|$")

If $|x| \geq a$, then $x \leq - a$ or $x \geq a$

Surds

Properties of surds…

$\sqrt{ab} = \sqrt{a}\sqrt{b}$
$\sqrt{\left(\frac{a}{b}\right)} = \frac{\sqrt{a}}{\sqrt{b}}$
$\sqrt{a}\sqrt{a} = a$

To simplify surds…

Find the largest possible perfect square number greater than 1 that will divide evenly into the number under the square root.

Then use the first property above. E.g. $\sqrt{63} = \sqrt{(9x7)}$ = $\sqrt{9}$ x $\sqrt{7}$ = 3 $\sqrt{7}$

Lowest Common Denominator

When you want to sum (or subtract) two fractions, you need to find a common denominator. The easiest way is usually to multiply the two denominators (this will give you a common denominator, but it won't necessarily be the lowest one).

E.g.

$$\frac{1}{7}+\frac{3}{5}$$

Lowest Common denominator = 7 x 5 = 35

Now multiply so that you have the same denominator in each fraction:

$\frac{5}{5} x \frac{1}{7}+\frac{3}{5}\ x\ \frac{7}{7}$ (effectively multiplying by 1)

$=\frac{5}{35}+\frac{21}{35}$ (you can combine once they have the same denominator)

$=\frac{26}{35}$

The same applies for algebraic fractions, e.g.

$$\frac{x}{y}+\frac{2x}{z}$$

Lowest Common denominator = yz

$\frac{z}{z}\ x\ \frac{x}{y}+\frac{2x}{z}\ x\ \frac{y}{y}$

$=\frac{zx}{zy}+\frac{2xy}{zy}$

$=\frac{zx+2xy}{zy}$