Society of Actuaries in Ireland

Algebra 1 – Solutions

Question 1

(i) Solve for x:

$$2(4 - 3x) + 12 = 7x - 5(2x - 7).$$

$$8 - 6x + 12 = 7x - 10x + 35$$
$$-15 = 3x$$
$$x = -5$$

(ii) Verify your answer to **(i)** above.

$$x = -5$$

$2(4-(-15))+12$	$7(-5)-5(-10-7)$
$38+12$	$-35+85$
50	50

$$[50 = 50]$$

Question 2

Solve the simultaneous equations:

$$x + y = 7$$
$$x^2 + y^2 = 25.$$

$$x = 7 - y$$
$$(7-y)^2 + y^2 = 25$$
$$y^2 - 7y + 12 = 0$$
$$(y-4)(y-3) = 0$$

$y = 4$	$y = 3$
$x = 7 - 4$	$x = 7 - 3$
$x = 3$	$x = 4$
$(3, 4)$	$(4, 3)$

Question 3

Simplify $\frac{x^2 - xy}{x^2 - y^2}$.

Factorise the numerator (the top line of the equation) and denominator (bottom line of the equation- difference of two squares)

$$\frac{x(x-y)}{(x+y)(x-y)}$$

$$\frac{x(x-y)}{(x+y)(x-y)}$$

$$= \frac{x}{(x+y)}$$

Question 4

Express the following as a single fraction in its simplest form:

$$\frac{6y}{x(x+4y)} - \frac{3}{2x}$$

Step 1:
Find the common denominator by multiplying the bottom lines:
2x * x(x + 4y) = $2x^2$(x+4y)

So $2x^2$(x+4y) is our **common denominator**

Step 2:
Find the numerator by cross multiplying (top lines by bottom lines):

6y*2x – 3*x(x+4y) = 12xy - 3*(x^2+4xy)
=12xy - $3x^2$- 12xy
= $-3x^2$

So $-3x^2$ is our **numerator**

Step 3:
The answer is the numerator divided by the denominator:

$$\frac{6y}{x(x+4y)} - \frac{3}{2x} = \frac{Numerator}{Denominator}$$

$$= \frac{-3x^2}{2x^2(x+4y)}$$

$$= \frac{-3}{2(x+4y)}$$

$$= \frac{-3}{2x+8y}$$

Question 5

Solve the simultaneous equations:

$$x^2 + xy + 2y^2 = 4 \qquad \textit{Equation (1)}$$
$$2x + 3y = -1. \qquad \textit{Equation (2) Line}$$

Find x in terms of y using the linear equation; Equation (2) *2x + 3y = -1*

$$x = \frac{-3y - 1}{2}$$

Substitute $x = \frac{-3y-1}{2}$ *into Equation (1)*

$$\left(\frac{-3y - 1}{2}\right)^2 + \left(\frac{-3y - 1}{2}\right)y + 2y^2 = 4$$

Multiply across by 4

$$(-3y - 1)^2 + (-3y - 1)2y + 8y^2 = 16$$

Expand the bracket and take the 16 over to the left hand side

$$9y^2 + 6y + 1 - 6y^2 - 2y + 8y^2 - 16 = 0$$

Group terms together

$$11y^2 + 4y - 15 = 0$$

$$(11y + 15)(y - 1) = 0$$

$$y = \frac{-15}{11} \, or \; y = 1$$

Substitute $y = \frac{-15}{11}$ $or \; y = 1$ *into Equation (2) to solve for x*

$$2x + 3\left(\frac{-15}{11}\right) = -1$$

$$2x + \left(\frac{-45}{11}\right) = -1$$

$$2x = -1 + \frac{45}{11}$$

$$x = \frac{17}{11}$$

OR

$$2x + 3(1) = -1$$

$$2x + 3 = -1$$

$$2x = -4$$

$$x = -2$$

Give the answer matching the appropriate x and y values:

Answer = $\left(\frac{17}{11}, \frac{-15}{11}\right)$ and $(-2,1)$

Question 6

Express the following as a single fraction in its simplest form:

$$\frac{x^2+4}{x^2-4} - \frac{x}{x+2}$$

Hint: x^2 − 4 is the difference between two squares i.e. $(x)^2 - (2)^2 = (x + 2)(x - 2)$

Step 1:

Find the common denominator by multiplying the bottom lines:

So $(x^2 - 4)(x+2)$ is our **common denominator**

Step 2:

Find the numerator by cross multiplying (top lines by bottom lines):

$(x^2 + 4)(x+2) - x(x^2 - 4)$ = $(x+2) * [(x^2 + 4) - x(x-2)]$ …. because : $x^2 - 4 = (x + 2)(x - 2)$

So $(x+2) * [(x^2 + 4) - x(x-2)]$ is our **numerator**

Step 3:

The answer is the numerator divided by the denominator:

$$\frac{x^2+4}{x^2-4} - \frac{x}{x+2} = \frac{\textbf{Numerator}}{\textbf{Denominator}}$$

$$= \frac{(x+2) * [(x^2+4) - x(x-2)]}{(x^2-4)(x+2)}$$

$$= \frac{(x^2+4) - x(x-2)}{(x^2-4)}$$ …(x+2) cancels

$$= \frac{(x^2+4) - x^2 + 2x)}{(x^2-4)}$$

$$= \frac{2x+4}{(x^2-4)}$$

$$= \frac{2(x+2)}{(x+2)(x-2)}$$ …difference of two squares

$$= \frac{\mathbf{2}}{\mathbf{(x-2)}}$$

Question 7

Find the range of values of x for which $|x-4| \geq 2$, where $x \in \mathbb{R}$.

Method 1:

Expand the bracket $x^2 - 8x + 16 \geq 4$

Take the 4 to the left hand side $x^2 - 8x + 12 \geq 0$

Solve for the roots of the equation $(x-2)(x-6) \geq 0$

$$x = 2$$

$$x = 6$$

Answer $x \leq 2 \text{ or } x \geq 6$

Method 2:

Split into 2 separate equations: $+(x-4) \geq 2 \ \ or \ \ -(x-4) \geq 2$

Solve each equation separately:

$$x - 4 \geq 2$$

$$x - 4 + 4 \geq 2 + 4$$

$$x \geq 6$$

OR

$$-(x-4) \geq 2$$

$$+(x-4) \leq -2$$

$$x - 4 + 4 \leq -2 + 4$$

$$x \leq 2$$

Question 8

Find the set of all real values of *x* for which $2x^2 + x - 15 \geq 0$.
Step 1:
For inequalities, first set the equation = 0 and solve.
$2x^2 + x - 15 = 0$
(2x-5) (x+3) = 0
2x-5=0 x+3=0
x =2.5 x= -3

Step 2:
Then look at the sign in the inequality in the question.
If the sign is ≤ 0 we are "between the posts" which means the answer will be in the format of number ≤ x ≤ number e.g. -3 ≤ x ≤ 2.5
If the sign is ≥ 0 we are "outside the posts" which means the answer will be in the format of

x ≤ number and x ≥ number   e.g. x≤-3 and x ≥ 2.5

Step 3:
Therefore in this case the answer is
x≤-3 and x ≥ 2.5   *(You can sub in values to the original question to check if your answer is correct!)*

Question 9

$$x=\sqrt{x+6}$$
$$\Rightarrow x^2=x+6$$
$$\Rightarrow x^2-x-6=0$$
$$\Rightarrow (x+2)(x-3)=0$$
$$\Rightarrow x=-2,\quad x=3$$
$$x=-2:\ \ -2\neq\sqrt{-2+6}=\sqrt{4}=2\ \times$$
$$x=3:\ \ 3=\sqrt{3+6}=\sqrt{9}=3\ \checkmark$$

Question 10

Solve the following for x, y and z.

x + 2y – z = 1
2x + y + z = 4
x + 2y + z = 2

Solution:

Step 1:
Number each equation

x + 2y – z = 1 .... (1)
2x + y + z = 4 .... (2)
x + 2y + z = 2 .... (3)

Step 2:
Add two equations together to find equations (4) and (5):

x + 2y – z = 1 .... (1)
x + 2y + z = 2 .... (3)
2x+4y = 3 .... (4)

x + 2y – z = 1 .... (1)
2x + y + z = 4 .... (2)
3x + 3y = 5 .... (5)

Step 3:
Solve equations (4) and (5):

2x+4y = 3 .... (4)
3x + 3y = 5 .... (5)

6x + 12y = 9 .... (4) (x3)
6x + 6y = 10 .... (5) (x2)

6x + 12y = 9 .... (4)
-6x - 6y = -10 .... (5) (x-1)
6y = -1

y = -1/6

2x+4y = 3 (equation 4)

2x +4(-1/6) = 3

x = 11/6

x + 2y – z = 1 (Equation 1)

(11/6) + 2(-1/6) – z = 1

z = 1/2

(You can sub in values to the original question to check if your answer is correct!)

Question 11

Solve the equation

$$|4x – 3| > 5$$

<u>Solution 2:</u>

<u>Step 1:</u>

Set the equation = instead of less than/greater than and solve

|4x - 3| = 5

This could mean that

4x - 3= 5	or	4x - 3= -5
4x = 5+3		4x = -5 + 3
4x = 8		4x = -2
x = 2		x = -½

<u>Step 2:</u>

Is the answer “between the roots” or “outside the roots”?

In this question the sign is > so the answer is outside the roots. So the answer is:

x > 2 and x < -½

Question 12

<u>Step 1:</u>

Set the equation = instead of less than/greater than and solve

$$|3x + 2| < 4$$

This could mean that

3x + 2= 4	or	3x + 2= -4
3x = 4-2		3x = -4 -2
3x = 2		3x = -6
x = ⅔		x = -2

<u>Step 2:</u>

Is the answer “between the roots” or “outside the roots”?

In this question the sign is < so the answer is between the roots. So the answer is:

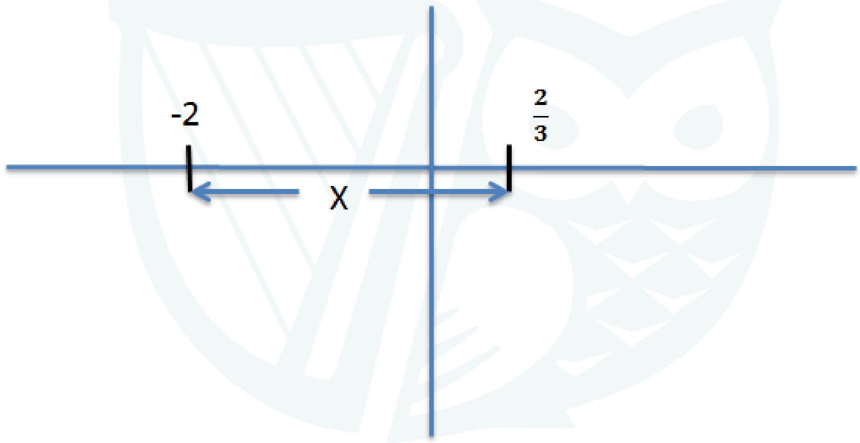
-2 < x < ⅔

<u>Step 3:</u>

Is the answer between the roots or outside the roots?

(Note: that x = -2 and x = ⅔ should not be included on the number in the shaded region as inequality does not include equals, it’s just less than)

This answer is between the roots so on a graph this looks like:



Question 13

Step 1:
Set the equation = 0

$$f(x) = 2x^3 - 4x^2 - 22x + 24 = 0$$

Step 2:
To find x, try a few values of x and see if they give you zero.
Try x=0 first:

$$f(0) = 2*0 - 4*0 - 22*0 + 24$$
$$= 24 \neq 0$$

This is not equal to zero so x=0 is not a solution (or "root").
Try x=1:

$$f(1) = 2*1 - 4*1 - 22*1 + 24$$
$$= 0\,!$$

This is equal to zero so **x=1** is a solution (or "**root**") which means that **(x-1)** is a **factor**.

Step 3:
Divide the equation by the factor you just found.

$$2x^2 - 2x - 24$$
$$(x-1)\sqrt{2x^3 - 4x^2 - 22x + 24}$$
$$2x^3 - 2x^2$$
$$-2x^2 - 22x + 24$$
$$-2x^2 + 2x$$
$$-24x + 24$$
$$-24x + 24$$

Step 4:

Factorise fully.

$2x^3 - 4x^2 - 22x + 24$ = (x-1)($2x^2$ $-2x-24$)

= (x-1)(2x+6)(x-4)

<u>Step 5:</u>

Pull out the final answers, also called “roots”.

x-1 = 0

x=1

2x+6 = 0

2x = -6

x = -3

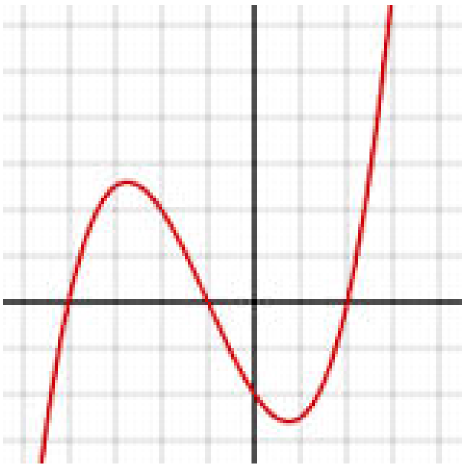
x-4 = 0

x = 4

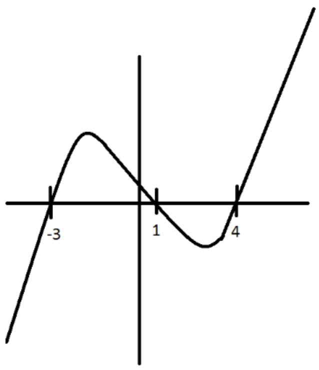
x = -3 , 1 ,4 are the solutions

<u>Step 6:</u>

The graph always looks something like this:



So just make sure it crosses the x axis at your solutions x = -3 , 1 ,4:



Question 14

(a)

$x = -3, \quad x = -1, \quad x = 2$

$f(x) = (x+3)(x+1)(x-2) = x^3 + 2x^2 - 5x - 6$

OR

$f(x) = x^3 + 2x^2 - 5x - 6$

$f(-3) = -27 + 18 + 15 - 6 = 0 \Rightarrow (x+3)$ is a factor

$f(-1) = -1 + 2 + 5 - 6 = 0 \Rightarrow (x+1)$ is a factor

$f(2) = 8 + 8 - 10 - 6 = 0 \Rightarrow (x-2)$ is a factor

$f(x) = (x+3)(x+1)(x-2) = x^3 + 2x^2 - 5x - 6$

(b) (i)

$f(x) = g(x)$

$x^3 + 2x^2 - 5x - 6 = -2x - 6$

$\Rightarrow x^3 + 2x^2 - 3x = 0$

$\Rightarrow x(x^2 + 2x - 3) = 0$

$\Rightarrow x(x-1)(x+3) = 0$

$\Rightarrow x = 0, \quad x = 1, \quad x = -3$

$\Rightarrow y = -6, \quad y = -8, \quad y = 0$

Points: $(-3, 0), (0, -6), (1, -8)$

(ii)

$g(x) = -2x - 6$

$g(-3) = -2(-3) - 6 = 6 - 6 = 0 \Rightarrow (-3, 0)$

$g(0) = -2(0) - 6 = -6 \Rightarrow (0, -6)$

Question 15

a (i)

$$f(x) = x^3 + kx^2 - 4x - 12$$

$(x+3)$ is factor $\Rightarrow f(-3) = 0$

$$f(-3) = (-3)^3 + k(-3)^2 - 4(-3) - 12 = 0$$

$$-27 + 9k + 12 - 12 = 0$$

$$9k = 27 \Rightarrow k = 3$$

(ii)

$$\frac{3}{1+x^p}+\frac{3}{1+x^{-p}}=\frac{3\left(1+x^{-p}\right)+3\left(1+x^p\right)}{\left(1+x^p\right)\left(1+x^{-p}\right)}$$
$$=\frac{3\left(1+x^{-p}+1+x^p\right)}{1+x^p+x^{-p}+x^0}$$
$$=\frac{3\left(2+x^{-p}+x^p\right)}{\left(2+x^{-p}+x^p\right)}$$
$$=3$$

Question 16

(i) Let x = Stage number.

There are 4 times as many blue tiles than the Stage number

Blue tiles = 4x

There are 4 white tiles in every stage. This is a constant and remains 4 no matter what stage number we use.

The total number of green tiles is the square of the stage number.

Number of Green tiles = x^2

The total number of tiles (T) must be the green tiles + blue tiles + white tiles

$T = x^2 + 4x + 4$

(ii) $x^2 + 4x + 4 = 324$

Factorise $(x+2)(x+2)=324$

$(x+2)^2=324$

$x+2=18$

$x=16$

There are x^2 green tiles therefore $16^2 = 256$ green tiles.

(iii) Mary's kitchen is square. Therefore the length of each side = $\sqrt{6.76} = 2.6$m = 260 cm.
Each tile has sides of 20 cm each and 13 x 20 = 260. Therefore there are 13 tiles on each side or in each row.
In the first row there are two white tiles and the rest (13-2=11) are blue. Therefore this must be stage 11.

Green = x^2=121

Blue =4x =44

White = 4

Check: The total number of tiles = 121+44+4=169.

The area of each tile = 0.20 x 0.20 = 0.04 m^2. The total number of tiles needed = $6.76 \div 0.04 = 169$.