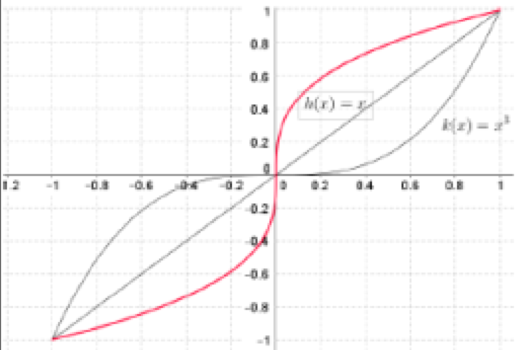
Society of Actuaries in Ireland

Calculus

Question 1.

Q6	**Model Solution – 25 Marks**	**Marking Notes**
(a)	$x^3 = x$ $\Rightarrow x^3 - x = 0$ $\Rightarrow x(x^2 - 1) = 0$ $x(x-1)(x+1) = 0$ $x = 0$ or $x = \pm 1$ $(-1, -1), (0, 0), (1, 1)$	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* Equation written One correct solution from the graph Solution of the form (a, a) where $a \neq 0, 1$ *High Partial Credit:* Equation factorised (3 factors) 2 correct points x values only
(b) (i)	$2\int_0^1 x - x^3 dx$ $= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right] = 2\left[\frac{1}{2} - \frac{1}{4} - 0\right] =$ $\frac{1}{2}$ unit^2	**Scale 10C (0, 4, 8, 10)** *Low Partial Credit:* Integral indicated One relevant area found *High Partial Credit:* Integral evaluated at $x = 1$ (upper limit) $\int_{-1}^{1} x - x^3 dx = 0$
(b) (ii)		**Scale 5B (0, 2, 5)** *Partial Credit:* Incomplete image 2 correct image points $k^{-1}(x) = x^{\frac{1}{3}}$

Question 2.

(a) What is the height of the surface at time $t = 0$?

$h(0) = 10^2 = 100$ cm.

(b) After how many seconds will the height of the surface be 64 cm?

$$\left(10 - \frac{t}{200}\right)^2 = 64$$

$$10 - \frac{t}{200} = 8$$

$$t = 400$$

Answer: 400 seconds.

Note: (10 – t/200) could also be equal to -8 here.
However, if you solve this, you will get t = 3600 seconds i.e. 60 minutes.
The question tells us that the cylinder is only leaking for 20 minutes, so t = 3600 seconds cannot be the solution.

(c) Find the rate at which the **volume** of water in the tank is decreasing at the instant when the height is 64 cm.
Give your answer correct to the nearest cm^3 per second.

$V = \pi r^2 h = \pi(52)^2 h = 2704\pi h$.

$$\frac{dV}{dt} = 2704\pi \frac{dh}{dt}$$

$$= 2704\pi\left(\tfrac{-2}{25}\right) = -216{\cdot}32\pi$$

$\therefore$ Volume is decreasing at $216{\cdot}32\pi$ cm^3 s^{-1} ≈ 680 cm^3 s^{-1}.

$$\frac{dh}{dt} = 2\left(10 - \frac{t}{200}\right)\frac{-1}{200}$$

$$\left.\frac{dh}{dt}\right|_{t=400} = \frac{-2}{25}$$

(d) The rate at which the volume of water in the tank is decreasing is equal to the speed of the water coming out of the hole, multiplied by the area of the hole. Find the speed at which the water is coming out of the hole at the instant when the height is 64 cm.

$$\frac{dV}{dt} = Av$$
$$216{\cdot}32\pi = \pi 1^2 v$$
$$v = 216{\cdot}32 \text{ cm s}^{-1}$$

(e) Show that, as t varies, the speed of the water coming out of the hole is a constant multiple of $\sqrt{h}$.

$$v = -\frac{1}{\pi}\frac{dV}{dt}$$
$$= -\frac{1}{\pi}2704\pi\frac{dh}{dt}$$
$$= 27{\cdot}04\left(10 - \frac{t}{200}\right)$$
$$= 27{\cdot}04\sqrt{h}$$

which is a constant multiple of $\sqrt{h}$

(f) The speed, in centimetres per second, of water coming out of a hole like this is known to be given by the formula

$$v = c\sqrt{1962h}$$

where c is a constant that depends on certain features of the hole.
Find, correct to one decimal place, the value of c for this hole.

$$c\sqrt{1962} = 27{\cdot}04$$
$$c \approx 0{\cdot}6$$

Question 3.

Q8	Model Solution – 40 Marks	Marking Notes
(a)	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ At $x = 0$: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)^2}$ $= \frac{1}{\sqrt{2\pi}}(1)$ $\therefore (0, \frac{1}{\sqrt{2\pi}})$ is the y intercept	Scale 10C (0, 4, 8, 10) *Low Partial Credit:* $x = 0$ Value for x substituted into $f(x)$ *High Partial Credit:* $\frac{1}{\sqrt{2\pi}}$ *Full credit* – 1: (0, 0·3989)
(b)	$\text{Area} = \left[(2)\left(\frac{1}{\sqrt{2\pi e}}\right)\right] = 0{\cdot}4839$ $= 0{\cdot}484$ Units²	Scale 10C (0, 4, 8, 10) *Low Partial Credit:* length = 2 Width = [y co-ordinate] *High Partial Credit:* $\left[(1)\left(\frac{1}{\sqrt{2\pi e}}\right)\right]$ *Full credit* –1: Area = −0·484 *Zero Credit:* Integrating original function
(c)	$C(1, \frac{1}{\sqrt{2\pi e}})$ due to symmetry $f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(-x)$ At $x = 1$: $f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1)^2}(-1) < 0$ $\left[= -\frac{1}{\sqrt{2\pi e}} \quad (-0{\cdot}24197) \quad < 0\right]$ $\Rightarrow$ Decreasing	Scale 10C (0, 4, 8, 10) *Low Partial Credit:* $x = 1$ identified Some correct differentiation Indicates significance of $\frac{dy}{dx} < 0$ *High Partial Credit:* Derivative found

(d)	$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(-x)$ $f''(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(-1) + (-x)\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(-x)$ $= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}(x^2 - 1)$ $f''(-1) = 0$ as $1^2 - 1 = 0$ $\Rightarrow$ point of inflection at $x = -1$	**Scale 10D (0, 3, 5, 8, 10)** *Low Partial Credit:* $f'(x)$ transferred or found Mention of $f''(x)$ Identifies $x = -1$ *Mid Partial Credit:* $f''(x)$ identified and some correct differentiation *High Partial Credit:* $f''(x)$ found Note: if the product rule and chain rule are not applied in finding $f''(x)$ then the candidate can be awarded mid partial credit at most

Probability

Question 4.

(d) A property is said to be in "negative equity" if the person owes more on the mortgage than the property is worth. A report about mortgaged properties in Ireland in December 2010 has the following information:

- Of the 475 136 properties examined, 145 414 of them were in negative equity.
- Of the ones in negative equity, 11 644 were in arrears.
- There were 317 355 properties that were neither in arrears nor in negative equity.

(i) What is the probability that a property selected at random (from all those examined) will be in negative equity?
Give your answer correct to two decimal places.

$$\frac{145414}{475136} = 0{\cdot}30604711 = 0{\cdot}31 \text{ (to two decimal places)}$$

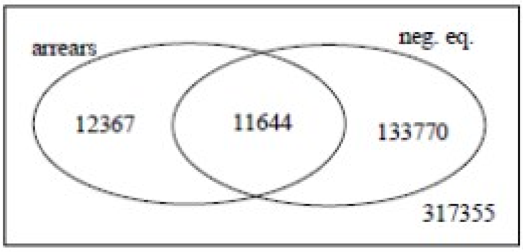
(ii) What is the probability that a property selected at random from all those in negative equity will also be in arrears?
Give your answer correct to two decimal places.

$$\frac{11644}{145414} = 0{\cdot}08007482 = 0{\cdot}08 \text{ (to two decimal places)}$$

(iii) Find the probability that a property selected at random from all those in arrears will also be in negative equity.
Give your answer correct to two decimal places.

	arrears	¬arrears	total
neg. eq.	11644	133770	145414
¬neg. eq.	12367	317355	329722
total	24011	451125	475136

$$\frac{11644}{24011} = 0{\cdot}4849 = 0{\cdot}48 \text{ (to two decimal places)}$$



OR

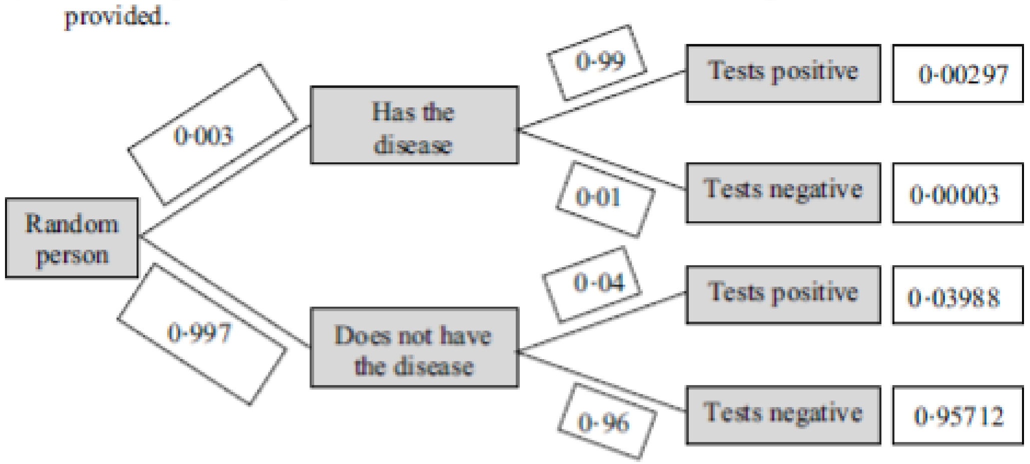
$$P(A|N) = \frac{P(A \cap N)}{P(N)} \Rightarrow 0{\cdot}08007 = \frac{P(A \cap N)}{0{\cdot}30604} \Rightarrow P(A \cap N) = 0{\cdot}0245$$

$$\text{But } P(A) = \frac{24011}{475136} = 0{\cdot}05053$$

$$P(N|A) = \frac{P(N \cap A)}{P(A)} = \frac{0{\cdot}0245}{0{\cdot}05053} = 0{\cdot}4848 = 0{\cdot}48 \text{ (to two decimal places)}$$

Question 5.

(a) (i) Write the probability associated with each branch of the tree diagram in the blank boxes provided.



(ii) Hence, or otherwise, calculate the probability that a person selected at random from the population tests positive for the disease.

P(Positive test) $= 0 \cdot 00297 + 0 \cdot 03988 = 0 \cdot 04285$

(iii) A person tests positive for the disease. What is the probability that the person actually has the disease. Give your answer correct to three significant figures.

$$P(\text{Has disease}|\text{positive test}) = \frac{0 \cdot 00297}{0 \cdot 04285} = 0 \cdot 0693$$

(iv) The health authority is considering using a test on the general population with a view to treatment of the disease. Based on your results, do you think that the above test would be an effective way to do this? Give a reason for your answer.

Test is not very useful.
A person who tests positive has the disease only 7% of the time.

Statistics

Question 6.

(a) Complete the table below, indicating whether the statement is correct (✓) or incorrect (✗) with respect to each data set.

Solution:

	A	B	C	D
The data are skewed to the left			✓	
The data are skewed to the right	✓			
The mean is equal to the median		✓		✓
The mean is greater than the median	✓			
There is a single mode	✓	✓	✓	

(b) Assume that the four histograms are drawn on the same scale. State which of them has the largest standard deviation, and justify your answer.

Solution:

Histogram D has the largest standard deviation.

Justification: The standard deviation measures how far the data are spread out from the mean. Since histogram D is symmetric, the mean is in the middle.

But we can also see that in histogram D, most of the data are spread out to the far right and left of the diagram, generally far from the mean.

Therefore, of the four histograms, D has the largest standard deviation.

Question 7.

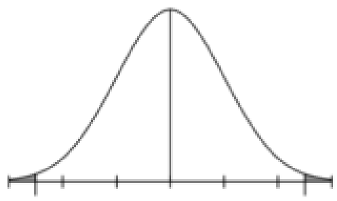
A company produces calculator batteries. The diameter of the batteries is supposed to be 20 mm. The tolerance is 0·25 mm. Any batteries outside this tolerance are rejected. You may assume that this is the only reason for rejecting the batteries.

(a) The company has a machine that produces batteries with diameters that are normally distributed with mean 20 mm and standard deviation 0·1 mm. Out of every 10 000 batteries produced by this machine, how many, on average, are rejected?

Solution:

$$Z=\frac{20{\cdot}25-20}{0{\cdot}1}=2{\cdot}5$$

$$\begin{aligned} P(|X-20|>0{\cdot}25) &= P(|Z|>2{\cdot}5) \\ &= 2(1-P(Z\le 2.5)) \\ &= 2(1-0{\cdot}9938) \\ &= 0{\cdot}0124 \end{aligned}$$

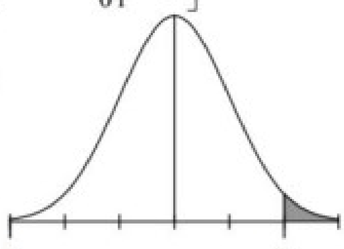


Answer = 10 000 × 0·0124 = 124.

(b) A setting on the machine slips, so that the mean diameter of the batteries increases to 20·05 mm, while the standard deviation remains unchanged. Find the percentage increase in the rejection rate for batteries from this machine.

Solution:

$$\begin{aligned} P(X\le 1{\cdot}975)+P(X\ge 20{\cdot}25) &= P\left[Z\le\frac{19{\cdot}75-20{\cdot}25}{0{\cdot}1}\right]+P\left[Z\ge\frac{20{\cdot}25-20{\cdot}05}{0{\cdot}1}\right] \\ &= P(Z\le -3)+P(Z\ge 2) \\ &= 1-P(Z\le 3)+1-P(Z\le 2) \\ &= 1-0{\cdot}9987+1-0{\cdot}9772 \\ &= 2-1{\cdot}9759 \\ &= 0{\cdot}0241 \end{aligned}$$



$$\frac{0{\cdot}0241}{0{\cdot}0124}=1{\cdot}9435\ldots \;\Rightarrow 94{\cdot}35\%\text{ increase}$$

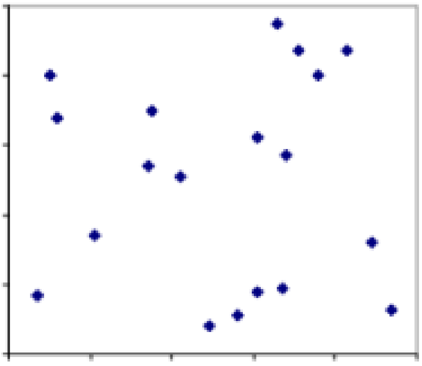
or increase: $0{\cdot}0241-0{\cdot}0124=0{\cdot}0117$

% Increase: $\left(\frac{0{\cdot}0117}{0{\cdot}0124}\right)100=94{\cdot}35\%$

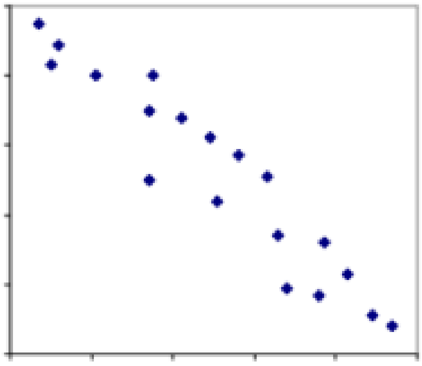
Question 8.

(a) For each of the four scatter plots below, estimate the correlation coefficient.

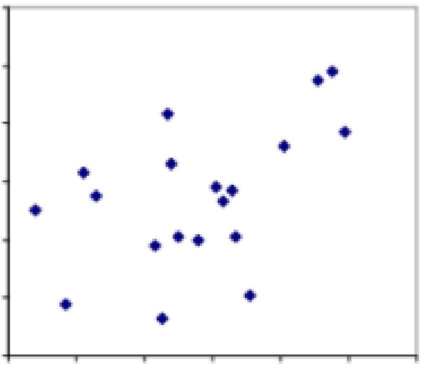
15 C



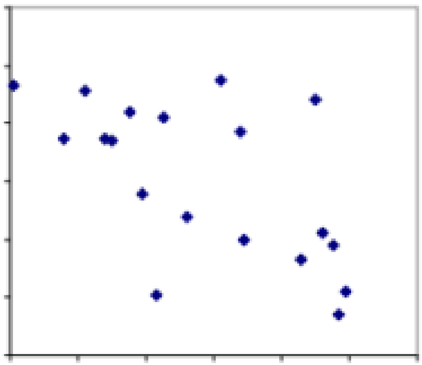
Correlation ≈ 0



Correlation ≈ -0.9



Correlation ≈ +0.5



Correlation ≈ –0.6

(b) Answer: 0.7582 => 0.76

Financial Maths

Question 9.

(a) Write down the present value of a future payment of €20,000 in one year's time.
Solution:

We have

$$P = \frac{F}{1+i} = \frac{20000}{1.03} = 19417.48$$

So the present value is €19417.48 to the nearest cent.

(b) Write down, in terms of t, the present value of a future payment of €20,000 in t years' time.
Solution:

$$P = \frac{F}{(1+i)^t} = \frac{20000}{(1.03)^t}.$$

(c) Pádraig wants to have a fund that could, from the date of his retirement, give him a payment of €20,000 at the start of each year for 25 years. Show how to use the sum of a geometric series to calculate the value, on the date of retirement, of the final fund required.
Solution:

Using the solution to part (b), we see that the amount required on the date of requirement is given by

$$A = 20000 + \frac{20000}{1.03} + \cdots \frac{20000}{(1.03)^{24}}$$

Using the notation of the formula on page 22 of the Formula and Tables book, we have a geometric series with $a = 20000$, $r = \frac{1}{1.03}$ and $n = 25$. Therefore

$$A = \frac{20000\left(1 - \left(\frac{1}{1.03}\right)^{25}\right)}{1 - \frac{1}{1.03}}$$

Using a calculator we obtain

$$A = €358711$$

to the nearest euro.

(d) Pádraig plans to invest a fixed amount of money every month in order to generate the fund calculated in part (c). His retirement is 40×12 = 480 months away.

i. Find, correct to four significant places, the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate of 3%.

Solution:

We must solve $(1+i)^{12} = 1.03$. So $(1+i) = \sqrt[12]{1.03}$. Therefore

$$i = \sqrt[12]{1.03} - 1 = 0.002466$$

correct to 4 significant places. So the answer is

$$0.2466\%.$$

ii. Write down, in terms of n and P, the value on the retirement date of a payment of €P made n months before the retirement date.

Solution:

Using the formula on page 30 of the Formula and Tables booklet we obtain

$$P(1.002466)^n.$$

iii. If Pádraig makes 480 equal payments of €P from now until his retirement, what value of P will give him the fund he requires?

Solution:

We must solve

$$P(1.002466)^{480} + P(1.002466)^{479} + \cdots + P(1.002466) = 358711$$

or

$$P(1.002466 + (1.002466)^2 + \cdots + (1.002466)^{480}) = 358711$$

Using the formula for the sum of a geometric series, we obtain

$$P\left(\frac{1.002466(1-(1.002466)^{480})}{1-1.002466}\right) = 358711$$

or

$$P(919.38) = 358711.$$

Therefore $P = \frac{358711}{919.38} =$ €390.17 to the nearest cent.

(e) If Pádraig waits for ten years before starting his pension fund, how much will he then have to pay each month in order to generate the same pension fund?

Solution:

Now the number of months until his retirement date is $30 \times 12 = 360$. So as above we must solve

$$P(1.002466)^{360} + P(1.002466)^{359} + \cdots + P(1.002466) = 358711$$

or

$$P(1.002466 + (1.002466)^2 + \cdots + (1.002466)^{360}) = 358711$$

Using the formula for the sum of a geometric series, we obtain

$$P\left(\frac{1.002466(1-(1.002466)^{360})}{1-1.002466}\right) = 358711$$

or

$$P(580.11) = 358711.$$

Therefore, in this case, $P = \frac{358711}{580.11} =$ €618.35 to the nearest cent.

Other Requested Topics

Question 10. Logs

Remember your logarithm rules

Logarithm product rule	$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
Logarithm quotient rule	$\log_b(x / y) = \log_b(x) - \log_b(y)$
Logarithm power rule	$\log_b(x^y) = y \cdot \log_b(x)$
Logarithm base switch rule	$\log_b(c) = 1 / \log_c(b)$

So,

$\log_c \sqrt{x} + \log_c (cx)$

$\log_c \sqrt{x}$

$\log_c \sqrt{x} = \log_c (x^{1/2}) = (1/2).\log_c x = (1/2).p$

$\log_c (cx)$

$\log_c (cx) = \log_c (c) + \log_c (x) = 1 + p$

Therefore,

$\log_c \sqrt{x} + \log_c (cx) = (1/2).p + 1 + p = (3/2).p + 1$

Question 11. Functions and Inequalities

Solve

$2x + 8y – 3z = -1$

$2x – 3y + 2z = 2$

$2x + y + z = 5$

Consider the first two:

$2x + 8y – 3z = -1$

$2x – 3y + 2z = 2$

And the last two:

$2x – 3y + 2z = 2$

$2x + y + z = 5$

Eliminate the x’s to get equations in terms of y & z only

$2x + 8y – 3z = -1$

$\underline{2x – 3y + 2z = 2}$

$11y – 5z = -3$ Call this equation A

$2x – 3y + 2z = 2$

$\underline{2x + y + z = 5}$

$- 4y + z = -3$ Call this equation B

Use A & B to solve for y & z

$11y – 5z = - 3$ This is equation A

$\underline{-20y + 5z = -15}$ This is 5 times equation B

$- 9y = -18$

$\Rightarrow y = 2$

Substituting this into A gives:

$11 * (2) - 5z = -3$

$-5z = -25 \Rightarrow z = 5$

Now we have y = 2 & z = 5

Fill these into one of the original equations to find x

Picking the first (2x + 8y – 3z = -1) gives:

$2x + 8 * (2) – 3 * (5) = -1$

$2x +16 – 15 = -1$

$2x = -2 \Rightarrow x = -1$

So solution is

$x = -1, y = 2, z = 5$

Let's check if this works.

Pick the second equation ($2x – 3y + 2z = 2$)

and see if the solutions work

$2 * (-1) – 3 *(2) + 2 * (5) = 2$

$– 1 – 6 + 10 = 2$

This is true, therefore solutions are correct

Note: You don't need to do this but it can be a handy check if the numbers look odd

(b) (i)

$f(x) = |x-3|$

$g(x) = 2$

Consider the points:

What's happening at A & B?

$f(x)$ & $g(x)$ are intersecting, so $f(x) = g(x)$

or $|x – 3| = 2$

We know $y = 2$ but we need to solve for x, this can be done by:

- Removing modulus to give 2 function
- Squaring both side

Removing the modulus

If $|x-3| = 2$, then

$x – 3 = 2$, or

$x – 3 = -2$

Solving both

$x – 3 = 2$ $x – 3 = -2$

$x = 5$ $x = 1$

Since A comes before B:

$A = (1, 2)$ $B = (5, 2)$

Squaring both sides

Squaring both sides of: $|x – 3| = 2$

gives

$x^2 – 6x + 9 = 4$

$x^2 - 6x + 5 = 0$

This is a quadratic which solves as:

$(x - 5)(x - 1) = 0$

So roots are x= 5 & x = 1

Remainder is as above!

Consider C next

What's happening here?

f(x) = 0, so |x – 3| = 0

We know y (= 0) but need x

Modulus is redundant (zero is always zero!), so just solve:

x – 3 = 0

x = 3

Note: Looking at the graph this seems reasonable as C lies about half way between A & B. This can be a useful check on the reasonability of the answer!

So, C = (3, 0)

Finally, consider D

Here x = 0 so we need to find y

So find f(0)

f(0)= |0 – 3|

= |-3| = 3

So, D = (0, 3)

(ii)

Says "hence, or otherwise" so try to use what we've already done!

Looking back to (i), this question is essentially asking when f(x) < g(x)

Lets look at the graph again:

This occurs between A & B

So, |x – 3| < 2 when it lies between x = 1 & x = 5

(Stripping out what we know about A & B from part (i))

Or, in maths "language"

|x – 3| < 2 when

1 < x < 5

Note: It is possible to solve this by squaring both sides but this is overkill here because you've done most of the work in part (i)!