Calculus 1 Solutions – 22 Feb 2023

1)

Answers:

(i) $4x^3$ (ii) $21x^2$ (iii) $-x^{-2}$ (iv) 1

2)

(i) $f'(x) = 10x + 7$

(ii) $g'(x) = 6x^2 + 8x + 1$

(iii) $\dfrac{dy}{dx} = 8$

3)

$y = 8x + 4$

this is the equation of a line

$\dfrac{dy}{dx} = 8$ this is the slope of that line

Prove it: $y = mx + c$

   m = 8

4)

Product Rule - Example

$y = (4 + 3x^2)(6x + 4x^2)$ $\dfrac{du}{dx} = 6x$ $\dfrac{dv}{dx} = 6 + 8x$

Find $\dfrac{dy}{dx}$

(hint let $u = (4 + 3x^2)$

and $v = (6x + 4x^2)$

$\dfrac{dy}{dx} = (4 + 3x^2)(6 + 8x) + (6x + 4x^2)(6x)$

$\dfrac{dy}{dx} = 24 + 32x + 18x^2 + 24x^3 + 36x^2$

$\quad\quad + 24x^3$

$= 48x^3 + 54x^2 + 32x + 24$

5) Quotient rule: $y = \dfrac{(5+6x)}{2x^2}$ $Find \ \dfrac{dy}{dx}$

Quotient Rule - Example

$y = \dfrac{(5 + 6x)}{2x^2}$ $\dfrac{du}{dx} = 6$ $\dfrac{dv}{dx} = 4x$

Find $\dfrac{dy}{dx}$

(hint let $u = (5 + 6x)$

and $v = 2x^2$

$\dfrac{dy}{dx} = \dfrac{(2x^2)(6) - (5 + 6x)(4x)}{(2x^2)^2}$

$$\frac{dy}{dx} = \frac{12x^2 - 20x - 24x^2}{4x^4}$$
$$\frac{dy}{dx} = \frac{-12x^2 - 20x}{4x^4}$$
(divide above and below by $4x$)
$$\frac{dy}{dx} = \frac{-3x - 5}{x^3}$$

6) Chain rule: $y = (2 + 3x)^4$ *Find* $\frac{dy}{dx}$

Chain Rule - Example

$$y = (2 + 3x)^4$$
$$\frac{du}{dv} = 4v^3 \quad \frac{dv}{dx} = 3$$

Find $\frac{dy}{dx}$

(hint let **v** = $(2 + 3x)$

and **u** = $(v)^4$

$$f'(x) = (4v^3)(3)$$
$$= (4(2 + 3x)^3)(3)$$
$$= (4(2 + 3x)^3)(3)$$
$$= 12(2 + 3x)^3$$

7) Find the following limits:

a) $\lim\limits_{x \to 1} \left(\frac{x^2 + x - 2}{x - 1} \right) = \lim\limits_{x \to 1} \left(\frac{(x-1)(x+2)}{x-1} \right) = \lim\limits_{x \to 1} (x + 2) = 1 + 2 = 3$

b) $\lim\limits_{n \to \infty} \left(\frac{2n^2 - 3n + 2}{6n^2 + 5n - 6} \right) = \lim\limits_{n \to \infty} \left(\frac{2 - \frac{3}{n} + \frac{2}{n^2}}{6 + \frac{5}{n} - \frac{6}{n^2}} \right) = \frac{2}{6} = \frac{1}{3}$

8) Find the derivative of $f(x) = 5 - 2x$ by first principles.

First find $f(x + h)$

$$\frac{f(x+h) - f(x)}{h} = \frac{5 - 2x - 2h - 5 + 2x}{h}$$

$$f(x + h) = 5 - 2(x + h)$$
$$= 5 - 2x - 2h$$

$$= -2$$

$$\frac{f(x + h) - f(x)}{h} = \frac{5 - 2x - 2h - 5 + 2x}{h}$$

9) Fill in the table:

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2 + x^2}$

10)

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

$$f(x) = 2x^2 - 3x - 6$$
$$f(x+h) = 2(x+h)^2 - 3(x+h) - 6 = 2x^2 + 4xh + 2h^2 - 3x - 3h - 6$$
$$f(x+h) - f(x) = 4xh + 2h^2 - 3h$$
$$\underset{h \to 0}{Limit}\left(\frac{f(x+h) - f(x)}{h} \right) = \underset{h \to 0}{Limit}\left(\frac{4xh + 2h^2 - 3h}{h} \right) = 4x - 3$$

(b) Let $f(x) = \dfrac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the

tangent to the curve $y = f(x)$ is $\dfrac{1}{4}$.

$$f(x) = \frac{2x}{x+2}$$

Let $u(x) = 2x \Rightarrow u'(x) = 2$ and $v(x) = x+2 \Rightarrow v'(x) = 1$

$$f'(x) = \frac{(x+2)(2) - 2x(1)}{(x+2)^2} = \frac{4}{(x+2)^2}$$

$$f'(x) = \frac{1}{4} \Rightarrow \frac{4}{(x+2)^2} = \frac{1}{4}$$

$$\Rightarrow 16 = (x+2)^2$$

$\Rightarrow x+2 = 4$ or $x+2 = -4$ or $x^2 + 4x - 12 = 0$

$\Rightarrow x = 2$ or $x = -6$ $(x-2)(x+6) = 0$

$\Rightarrow x - 2 = 0$ or $x + 6 = -0$

$\Rightarrow x = 2$ or $x = -6$

$$f(-6) = \frac{-12}{-6+2} = 3 \text{ and } f(2) = \frac{4}{2+2} = 1$$

Points $(-6, 3)$ and $(2, 1)$

11)

Q6	Model Solution – 25 Marks	Marking Notes
(a)	$f(x+h) - f(x) = (2x + 2h + 4)^2 - (2x + 4)^2$ $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} =$ $\lim_{h \to 0} \dfrac{(2x + 2h + 4)^2 - (2x + 4)^2}{h}$ $= \lim_{h \to 0} \left(\dfrac{[(4x^2 + 8hx + 4h^2 + 16x + 16h + 16)] - (4x^2 + 16x + 16)}{h} \right)$ $= \lim_{h \to 0} \dfrac{8hx + 4h^2 + 16h}{h}$ $= 8x + 16$ **or** $f(x) = (2x + 4)^2 = 4x^2 + 16x + 16$ $f(x+h) = 4(x+h)^2 + 16(x+h) + 16$ $= 4x^2 + 8hx + 4h^2 + 16x + 16h + 16$ $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $\lim_{h \to 0} \dfrac{8hx + 4h^2 + 16h}{h}$ $= 8x + 16$	Scale 10D (0, 2, 5, 8, 10) **Low Partial Credit** • any $f(x+h)$ **Mid Partial Credit** • limit of $\dfrac{f(x+h) - f(x)}{h}$ **High Partial Credit** • limit of $\dfrac{(2x + 2h + 4)^2 - (2x + 4)^2}{h}$ **Notes:** - omission of limit sign penalised once only - answer not from 1st Principles merits 0 marks
(b) (i)+ (ii)	$y = x . \sin\dfrac{1}{x}$ $\dfrac{dy}{dx} = \sin\dfrac{1}{x} + x\left(\cos\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)$ $\dfrac{dy}{dx} = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$ $\dfrac{dy}{dx} = \sin\dfrac{\pi}{4} - \dfrac{\pi}{4}\cos\dfrac{\pi}{4}$ $= 0 \cdot 15$	Scale 15D (0, 4, 7, 11, 15) **Low Partial Credit** • any correct differentiation **Mid Partial Credit** • product rule applied **High Partial Credit** • correct differentiation **Note:** one penalty for calculator in wrong mode

12)

Q3	Model Solution – 25 Marks	Marking Notes
(a)	$f(x + h) = \frac{1}{3}(x + h)^2 - (x + h) + 3$ $f(x) = \frac{1}{3}x^2 - x + 3$ $f(x + h) - f(x) = \frac{2xh}{3} + \frac{h^2}{3} - h$ $\frac{f(x + h) - f(x)}{h} = \frac{2x}{3} + \frac{h}{3} - 1$ $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \frac{2x}{3} - 1$	**Scale 20D (0, 5, 14, 17, 20)** *Low Partial Credit* • any $f(x + h)$ *Mid Partial Credit* • $f(x + h) - f(x)$ with some correct work *High Partial Credit* • $\frac{\frac{1}{3}(x+h)^2 - (x+h) + 3 - \left(\frac{x^2}{3} - x + 3\right)}{h}$ simplified **Notes:** • omission of limit sign penalised once only • answer not from 1st Principles merits 0 marks
(b)	$\frac{d(fg(x))}{dx} =$ $\frac{1}{(3(x + 5)^2 + 2)}(6(x + 5))$ $\frac{d(fg(\frac{1}{4}))}{dx} = \frac{6(\frac{21}{4})}{3(\frac{21}{4})^2 + 2} = \frac{504}{1355}$ $= 0.372$ **OR** $f(x) = \ln(3x^2 + 2)$ $g(x) = (x + 5)$ $f[g(x)] = \ln[3(x + 5)^2 + 2]$ $= \ln(3x^2 + 30x + 77)$ $f'(x) = \frac{6x + 30}{3x^2 + 30x + 77}$ $x = \frac{1}{4}: \ f'(x) = \frac{31 \cdot 5}{84 \cdot 6875} = 0 \cdot 3719$ $= 0 \cdot 372$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • Any correct differentiation • $fg(x)$ formulated *High Partial Credit:* • $\frac{d(fg(x))}{dx}$ found **Note:** Work with $f(x) \times g(x)$ merits low partial credit at most