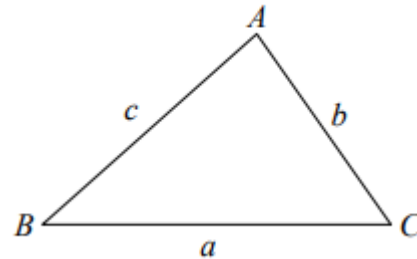**Question 1.**

a)

$$\frac{1}{2}ac\sin\angle B = \frac{1}{2}ab\sin\angle C$$

Divide by $\frac{1}{2}abc$
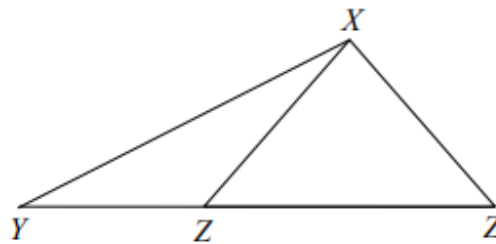
$$\frac{\sin\angle B}{b} = \frac{\sin\angle C}{c} \Rightarrow \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C}$$

b) i)

$$\frac{3}{\sin 27°} = \frac{5}{\sin\angle Z} \Rightarrow \sin\angle Z = \frac{5\sin 27°}{3} = 0\cdot 756$$

$$\Rightarrow \ |\angle Z| = 49° \quad \text{or} \quad |\angle Z| = 131°$$

ii)

c)

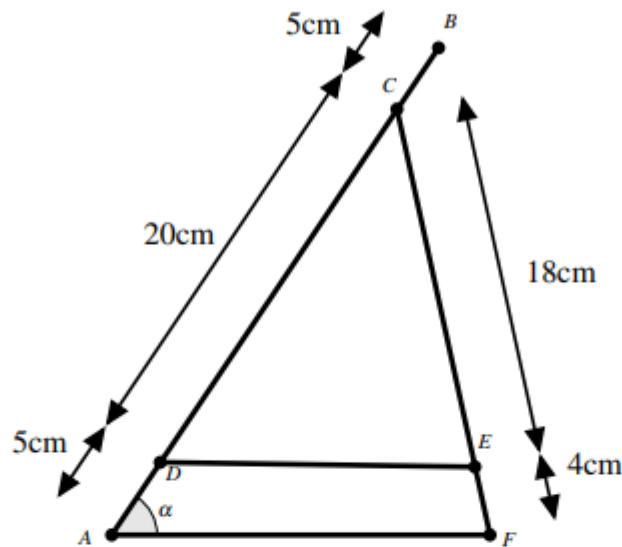$$|\angle ZXY| = 180° - (27° + 49°) = 104°$$

$$\Delta = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}(5)(3)\sin 104° = 7\cdot 27 = 7\,\text{cm}^2$$

Question 2.

a)

Consider the diagram below in which we have marked some of the lengths. Note that $|DC|$ must be 20cm since we are told that $|AB| = 30$cm. Similarly $|CE| = 18$cm since we are told that $|CF| = 22$cm.

Now suppose that $\alpha = 60°$. Then we apply the Sine Rule to the triangle $\triangle ACF$. So

$$\frac{\sin 60°}{22} = \frac{\sin(\angle F)}{25}$$

Therefore

$$\sin(\angle F) = \frac{25 \sin(60)}{22} = 0.9841$$

correct to four decimal places. Therefore $|\angle F| = \sin^{-1}(0.9841) = 79.77°$. Now we use this to calculate $\angle C$. So

$$|\angle C| = 180 - 60 - 79.77 = 40.23°.$$

Now we apply the Cosine Rule to the triangle $\triangle CDE$. Thus,

$$\begin{aligned}
|DE|^2 &= 20^2 + 18^2 - 2(20)(18)\cos(40.23°) \\
&= 400 + 324 - 549.7 \\
&= 174.3
\end{aligned}$$

Therefore

$$|DE| = \sqrt{174.3} = 13.20\text{cm}.$$

So the length of the strap when $\alpha = 60°$ is 13.20cm.

b)

The maximum possible value of α will occur when the stand is set so that CF is vertical. In that case $\triangle ACF$ is a right angled triangle with the hypotenuse $|AC| = 25$cm. The side opposite the angle α is $|CF| = 22$cm. Therefore in this case,

$$\sin \alpha = \frac{22}{25} = 0.88.$$

So

$$\alpha = \sin^{-1}(0.88) = 61.64 = 62°$$

correct to the nearest degree.

Question 3.

$$\sin 3x = \frac{\sqrt{3}}{2}$$

$\Rightarrow 3x = 60°, \quad 120°, \quad 420°, \quad 480°, \quad 780°, \quad 840°$

$\Rightarrow x = 20°, \quad 40°, \quad 140°, \quad 160°, \quad 260°, \quad 280°$

or

$3x = 60° + n(360°), n \in \mathbb{Z}$ or $3x = 120° + n(360°), n \in \mathbb{Z}$
$x = 20° + n(120°), n \in \mathbb{Z}$ or $x = 40° + n(120°), n \in \mathbb{Z}$

$n = 0 \Rightarrow x = 20°$ or $x = 40°$
$n = 1 \Rightarrow x = 140°$ or $x = 160°$
$n = 2 \Rightarrow x = 260°$ or $x = 280°$

Question 4.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

or

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A \sin B}{\cos A \cos B}} =$$

$$\frac{\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} =$$

$$\frac{\sin(A+B)}{\cos(A+B)} = \tan(A+B)$$

Question 5.   (Question 2, Paper 1, 2000)

(a) $4x-y+5z = 4$ equation(i) $6x - 2y + 6z = 2$ equation (i) x 3)
       $3x-y+3z= 1$ $x + 2y - 2z = -1$ from question

Subtracting $x+ 2z = 3$ equation (iv)   Adding $7x + 4z = 1$ (equation (v))

               $2x + 4z = 6$ (iv) x 2
               $7x + 4z = 1$ equation (v)

Subtracting $-5x = 5 \Rightarrow x = -1$

From equation (iv) $-1 + 2z= 3 \Rightarrow z = 2$
From equation (i) $-4 - y +10 = 4 \Rightarrow y= 2$

(b) Solve $x^2 - 2x - 24 = 0$

Solution $(x-6)(x+4) = 0$ $\Rightarrow$ roots are $x = 6$ and $x = -4$

Hence, find the values of x for which

if $x+ \dfrac{4}{x} = 6$ $\Rightarrow x^2 -6x + 4 = 0 \Rightarrow$ $a=1, b = -6, c = 4$

     roots are $6 +\{\sqrt{((-6)^2-4(1)(4))}\}/2(1) = [6 + \sqrt{20}]/2 = 3+2\sqrt5$
                   $6 -..... = 3-2\sqrt5$

If $x+ \dfrac{4}{x} =- 4$ $\Rightarrow x^2 +4x + 4 = 0 \Rightarrow (x+2)(x+2) = 0 \Rightarrow x=-2$

(c)   (i) $a^4 - b^4 = (a^2-b^2)(a^2+b^2) = (a+b)(a-b)(a^2+b^2)$ Difference of two squares.

       (ii) $a^5 - a^4b - ab^4 + b^5 = a^4(a-b) - b^4(a-b) = (a^4 - b^4)(a-b)$

From part (i) $= (a+b)(a-b)(a-b)(a^2+b^2) = (a^2+b^2)(a-b)^2(a+b)$

From part (ii) $a^5 - a^4b - ab^4 + b^5 = (a^2+b^2)(a-b)^2(a+b)$

So $a^5 - a^4b - ab^4 + b^5 > 0$ as all terms on the RHS are positive

      $a^5 + b^5 > a^4b + ab^4$

Question 6.　　(Question 1, Paper 2, 2000)

(a) centre =　　(0,0)

Slope of line from centre to (-7,9) is　　$(y_2-y_1)/(x_2/x_1) = (9-0)/(-7-0) = -9/7$

Slope of tangent is 7/9

(b)　　$x^2 + y^2 - 6x + 4y - 12 = 0$ is the equation of a circle.

Generic equation is $x^2 + y^2 + 2gx + 2fy + c = 0$, centre = (-g, -f), radius is $\sqrt{g^2 + f^2 - c}$

So for this circle g=-3 and f = 2, centre is (3,-2)

Radius = $\sqrt{(-3)^2 + (2)^2 + 12} = \sqrt{25} = 5$

$x^2 + y^2 + 12x - 20y + k = 0$ is another circle

Centre = (-6, 10), radius = $\sqrt{-6^2+10^2-k} = \sqrt{136-k}$

Distance between centres (3,-2) and (-6,10) is

$\sqrt{(y_2-y_1)^2 + (x_2-x_1)^2} = \sqrt{(10-(-2))^2+(-6-3)^2} = \sqrt{144+81} = \sqrt{225} = 15$
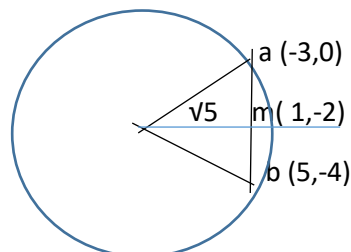
So the sum of the radii = the distance between the centres

$\sqrt{136-k} + 5 = 15 \Rightarrow \sqrt{136-k} = 10 \Rightarrow k = 36$

(c) A circle intersects a line at the points a(− 3, 0) and b(5, − 4).

Solution : m = ((-3+5)/2, (0-4)/2) = (1, -2)

The distance from the centre of the circle to m is $\sqrt{5}$.

Solution

Distance ma = $\sqrt{(-3-1)^2 + (0-(-2))^2} = \sqrt{20} = 2\sqrt{5}$

By Pythagoros   radius² = $(\sqrt{5})^2 + (2\sqrt{5})^2 = 25 \Rightarrow$ radius = 5

If generic circle is　　$x^2+y^2 + 2gx + 2fy + c = 0$

As a is on the circle $\quad (-3)^2 + 0^2 -6g+c = 0 \Rightarrow c = 6g – 9$

As b is on the circle $\quad 5^2 + (-4)^2 +10g – 8f + c = 0 \Rightarrow 25 +16 + 10g – 8f +6g – 9 = 0$

So $\quad 8f = 16g + 32 \Rightarrow f = 2g + 4$

$\sqrt{\{g^2 + [2(g+2)]^2\}-(6g-9)} = 5 \Rightarrow g^2 + 4(g^2+4g+4) – 6g + 9 =25 \Rightarrow g^2 + 4g^2+16g+16-6g+9= 25 \Rightarrow 5g^2+10g=0 \Rightarrow g^2 + 2g = 0 \Rightarrow g(g+2)= 0 \Rightarrow g=0$ (and f = 4 and c = -9) or g = -2 (and f = 0 and c= -21)

Equation Circle 1 is $x^2+y^2 +8y-9=0$ $\qquad$ Equation Circle 2 is $x^2+y^2 -4x -21 = 0$

Question 7. (Question 1, Paper 2, 2001)

(a) A circle with centre (-3, 7) passes through the point (5, -8)

Generic equation of circle is $x^2+y^2 + 2gx + 2fy + c = 0$ where centre = (-g,-f) and radius² = $\sqrt{(g^2+f^2-c)}$

So g = 3 and f = -7.

As (5,-8) is on circle $25 + 64 +6(5)-14(-8) + c = 0 \Rightarrow 119 + 112 = c = 0 \Rightarrow c = -231$

Equation is $x^2+y^2 + 6x -14y -231 = 0$

(b) The equation of a circle is $(x+1)^2 + (y-8)^2 = 160$

The line x-3y +25 = 0 intersects the circle at the points p and q.

(i) Find the co-ordinates of p and the co-ordinates of q.
(ii) Investigate if [pq] is a diameter of the circle.

X = 3y-25, replace x in equation of circle and solve for y

$(3y-24)^2 + (y-8)^2 = 160 \Rightarrow 9y^2 - 144y + 576 + y^2-16y+ 64 = 160 \Rightarrow 10y^2 -160y + 480 = 0$
$\Rightarrow y^2 -16y + 48 = 0 \Rightarrow y = 4$ (and x =-13) or y=12 (and x = 11)

Intersection is at (-13,4) and (11,12)

Centre (-1,8), radius is $\sqrt{(1 + 64 + 95)} = \sqrt{160} = 4\sqrt{10}$

If pq is a diameter, midpoint = centre of circle.
Midpoint pq =((-13+11)/2, (4+12)/2) = (-1, 8) so pq is a diameter.

(c) The circle $x^2 + y^2 + 2gx+ 2fy + c = 0$ passes through the points (3,3) and (4, 1)
The line 3x – y -6 = 0 is a tangent to the circle at (3,3)

(i) Find the real numbers g, f and c.

(3,3) => 9 + 9 + 6g + 6f + c = 0   =>   6g + 6f + c = -18
(4,1)=> 16+ 1 +8g + 6f+ c = 0   =>   8g + 2f + c = -17

             Subtract   -2g + 4f = -1   (equation (1))

Line from centre to (3,3) will have slope (-1/3) (i.e. perpendicular to tangent.)
Equation of line through centre and (3,3) is y-3 = (-1/3) (x-3) => 3y-9 = -x + 3 => x + 3y = 12
(-g,-f) on this line so –g – 3f = 12   (equation (2))   => -2g – 6f = 24 (equation (2) * 2)

Subtract Equation (2) from (1) and 10f=-25 => f = -5/2 and g = -9/2   (centre = 9/2, 5/2))
By substitution in equation (1) we can get c = 24

Equation is therefore $x^2 + y^2$ -9x -5y + 24 = 0

(ii)   Find the co-ordinates of the point on the circle at which the tangent parallel to 3x-y-6=0
          touches the circle.

          Centre is midpoint between two tangential points.
          (3,3), (9/2, 5/2), **(6, 2)**
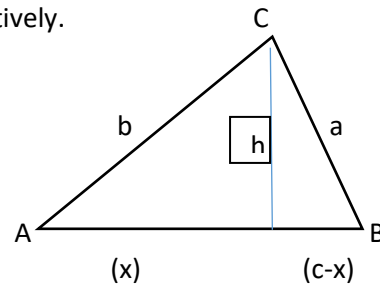
Question 8.   (Question 4, Paper 2, 2001)

(a) $\dfrac{\theta}{2\pi} = \dfrac{10}{2\pi(4)}$ $\theta = 10/4 = 2.5$ radians

1 radian = 57.296° = > 2.5 radians = 143°

(b) $\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

(ii) $\cos 2x - \sin x = 1$
          $1 - 2\sin^2 x - \sin x - 1 = 0$
          $2\sin^2 x + \sin x = 0$
          $\sin x \,(2\sin x + 1) = 0 \Rightarrow \sin x = 0$ (i.e. x = 0°, 180°, 360° or $\sin x = -1/2$ i.e. x = 210°, 330°)

(c) A triangle has sides a, b and c.
   The angles opposite a, b and c are A, B and C, respectively.

(ii)   Prove that $a^2 = b^2 + c^2 - 2bc \cos A$

$h^2 = b^2 - x^2$ but $h^2 = a^2 - (c-x)^2 = a^2 - (c^2 - 2cx + x^2)$

so $b^2 - x^2 = a^2 - c^2 + 2cx - x^2$ but $x/b = \cos A \Rightarrow x = b \cos A$

   $\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$

(iii)   Show that $c\,(b \cos A - a \cos B) = b^2 - a^2$

          $bc \cos A - ac \cos B = bc\,\dfrac{\{b^2 + c^2 - a^2\}}{2bc} - ac\,\dfrac{\{a^2 + c^2 - b^2\}}{2ac}$

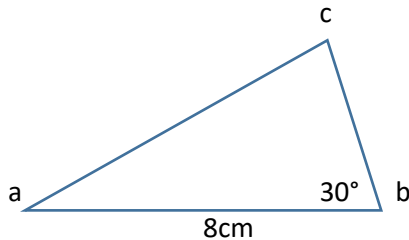 $= \dfrac{b^2 + c^2 - a^2}{2} \quad - \quad \dfrac{a^2 + c^2 - b^2}{2}$

          $= \dfrac{2b^2 - 2a^2}{2} \quad = \quad b^2 - a^2$

Question 9. (Question 5 , Paper 2, 2002)

(a) The area of triangle abc is 12 cm2. |ab | = 8 cm and |∠abc | = 30° Find |bc| .

Solution

Area of Triangle = (1/2) |ab||bc| sin 30 ° => 12cm² = (1/2) 8 |bc|(1/2) => |bc| = 6cm

(b) (i) Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$= \dfrac{(\sin A/\cos A) + (\sin B/\cos B)}{1 - (\sin A \sin B)/(\cos a \cos B)} = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(ii) Hence, or otherwise, prove that tan 22.5° = √2 -1

Let A = 22.5 ° and B = 22.5 °

So tan 45 ° = 2 tan A/ (1-tan²A) but tan 45° =1 so   1- tan²A = 2 tan A
tan²A + 2 tan A – 1 = 0

tan A = [-2 +/- √(4+4)] / 2 = -1 +/- √2
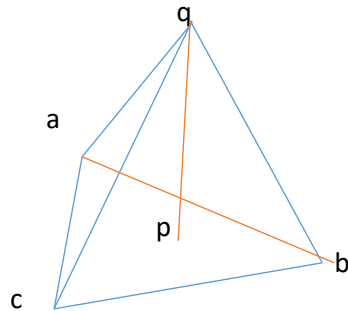
as tan 22.5 ° is positive tan 22.5 ° = -1+ √2 = √2 - 1

(c) A vertical radio mast [pq] stands on flat horizontal ground. It is supported by three cables that join the top of the mast, q, to the points a, b and c on the ground. The foot of the mast, p, lies inside the triangle abc.   Each cable is 52 m long and the mast is 48 m high.

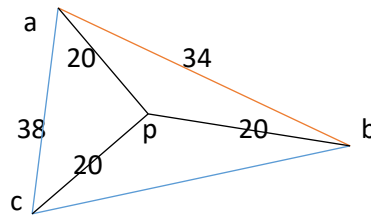(i)   Find the (common) distance from p to each of the points a, b and c.

Solution

$|pa| = |pb| = |pc|$

Triange apq is right angled at p => $|ap|^2 = |aq|^2 - |qp|^2$ => $|ap|^2 = 52^2 - 48^2 = 400$ => $|ap| = 20$

(ii)   Given that $|ac| = 38$ m and $|ab| = 34$ m, find $|bc|$ correct to one decimal place.

$\cos \angle apc = \{20^2 + 20^2 - 38^2\} / \{2(20)(20)\} = -644/800$ => apc = 143.61°

$\cos \angle apb = \{20^2 + 20^2 - 34^2\} / \{2(20)(20)\} = -356/800$ => apb = 116.42°

$|bc|^2 = 20^2 + 20^2 - 2(20)(20) \cos 99.97° = 800 + 138.506 = 938.506$ => $|bc| = 30.6$m

Question 10. (Question 4, Paper 2, 2003)

(a) The circumference of a circle is 30 π cm. The area of a sector of the circle is 75 cm² .
Find, in radians, the angle in this sector.

2 π r = 30 π = > r = 15

$\frac{75}{\pi(15^2)} = \frac{\theta}{2\pi}$ => θ = (150/225) radians = (2/3) radians

(b) Find all the solutions of the equation sin 2x + sin x = 0 in the domain 0° ≤ x ≤ 360 ° .

Solution

sin 2x + sin x = 0   =>   2 sin x cos x + sin x = 0
Sin x (2cos x + 1) = 0 = > sin x = 0 (i.e. x = 0°, 180°, 360°) or cos x = -1/2 (i.e. x = 120°, 240°)

(c)   C1 is a circle with centre a and radius r. C2 is a circle with centre b and radius r.
C1and C2 intersect at k and p. a∈ C2 and b∈ C1.

   (i)   Find, in radians, the measure of angle kap.

kab is an equilateral triangle =| <kab| = π/3 radians and | <kap| = 2 π/3 radians

   (iii)   Calculate the area of the intersection region. Give your answer in terms of r and π .
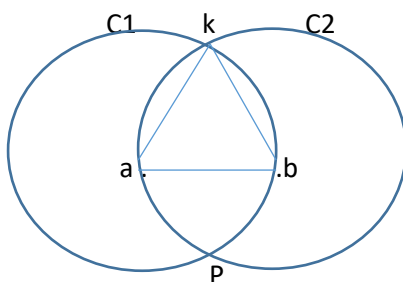
Area of triangle kap = (1/2) r² sin 2 π/3 = (√3/4) r²
Area of sector apk in C1 = (1/2) r² 2 π/3 = (π/3) r²
Area of sector apk – triangle kap = r²((π/3) - (√3/4))

Similarly in respect of circle 2 we need the area of sector bpk – triangle kpb.

Total intersection is therefore 2 r²((π/3) - (√3/4))