



**SAI Tutorial on Complex Numbers and Proof by Induction - Answers**

**Question A**

$$2z_1 + z_2 = 2(2+3i) + (1-2i) = 5 + 4i$$

$$\overline{2z_1 + z_2} = 5 - 4i$$

$$z_1 - iz_2 = (2+3i) - i(1-2i) = 2i$$

$$z_1 z_2 = (2+3i)(1-2i) = 8 - i$$

$$|2 + z_1| = |2 + (2+3i)| = \sqrt{4^2 + 3^2} = 5$$

$$\bullet \frac{z_1}{z_2} = \frac{-4}{5} + \frac{7i}{5}$$

(ii)

$$r = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\text{Solve for } \cos^{-1}(2/2\sqrt{2}) = \pi/4$$

$$2 + 2i = 2\sqrt{2}(\cos \pi/4 + i \sin \pi/4)$$

$$-2\sqrt{2} + 2\sqrt{2}i = 4(\cos(3\pi/4) + i \sin(3\pi/4))$$

**Question B (Question 3 Paper 1 2022)**

(a) (i)

$$z - iz = 6 + 2i - i(6 + 2i)$$

$$= 6 + 2i - 6i - 2i^2$$

$$= 6 - 4i - 2(-1) = 8 - 4i$$

Or

$$\text{If } z - iz = 8 - 4i \Rightarrow z(1 - i) = 8 - 4i$$

$$z = \frac{8-4i}{1-i}$$

$$z = \frac{8-4i}{1-i} \cdot \frac{1+i}{1+i} = \frac{8-4i+8i-4i^2}{1-i+i-i^2} = \frac{8+4i-4(-1)}{1-(-1)}$$

$$z = \frac{12+4i}{2}$$

$$z = 6+2i$$

(a) (ii)

$$|z|^2 = 6^2 + 2^2 = 40$$

$$|iz|^2 = 2^2 + 6^2 = 40$$

$$|z|^2 + |iz|^2 = 40 + 40 = 80$$

$$|z - iz|^2 = |8-4i|^2 = 8^2 + 4^2 = 80$$

(a) (iii)

Right angled triangle so Pythagorean therefore diameter =  $|z|^2 + |iz|^2 = 80$  from pervious

$$\text{Radius} = \sqrt{80} \div 2 = \sqrt{20}$$

$$\text{Area} = \pi r^2 = 20\pi \text{ square units}$$

OR

Calculate centre as midpoint of  $iz$  &  $z$

Centre =  $2+4i$  then Radius is distance to  $z$ .

$$\text{Radius} = \sqrt{((6-2)^2 + (2-4)^2)} = \sqrt{20}$$

$$\text{Area} = \pi r^2 = 20\pi \text{ square units}$$

(b)

De Moivre's Theorem  $\Rightarrow [r\cos\theta + ir\sin\theta]^n = r^n(\cos(n\theta) + i\sin(n\theta))$

Write complex number in polar form first i.e.  $(\sqrt{3} - i) = r\cos\theta + ir\sin\theta$

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

$$\cos \theta = x/r \Rightarrow \theta = \cos^{-1}(x/r) = \cos^{-1}(\sqrt{3}/2) = \pi/6 \text{ or } 11\pi/6$$

$$\sin \theta = y/r = -1/2 \Rightarrow \theta = 11\pi/6$$

$$(\sqrt{3} - i) = 2(\cos(11\pi/6) + i\sin(11\pi/6))$$

$$(\sqrt{3} - i)^9 = 2^9 (\cos(99\pi/6) + i\sin(99\pi/6))$$

$$(\sqrt{3} - i)^9 = 512 (0 + i) = 512i$$

$$a = 0, b = 512$$

### Question C (Question 1 Paper 1 2021)

(a)

$$\frac{4-2i}{2+4i} = \frac{4-2i}{2+4i} \cdot \frac{2-4i}{2-4i} = \frac{8-4i-16i+8i^2}{4+8i-8i-16i^2} = \frac{8-20i+8(-1)}{4-16(-1)} = \frac{-20i}{20}$$

$$k = -1$$

(b)

Use De Moivre's Theorem

Polar Form of  $-5+12i = r\cos\theta + ir\sin\theta$

$$r = \sqrt{5^2+12^2} = \sqrt{169} = 13$$

$$\cos \theta = -5/13$$

$$-5 + 12i = 13(\cos\theta + i \sin\theta)$$

$$(-5 + 12i)^{1/2} = [13(\cos(\theta+2n\pi) + i \sin(\theta+2n\pi))]^{1/2}$$

$$= \sqrt{13} (\cos(\theta/2+2n\pi/2) + i \sin(\theta/2+2n\pi/2)) \text{ for } n=0 \text{ or } 1$$

Solve for  $n=0$ , Log tables  $\sin^2(\theta/2) = 1/2(1-\cos\theta)$

$$2\sin^2(\theta/2) = 1-\cos\theta = 1-(-5/13) = 18/13$$

$$\sin(\theta/2) = \sqrt{9/13} = 3/\sqrt{13}$$

$$2\cos^2(\theta/2) = 1+\cos\theta = 1+(-5/13) = 8/13$$

$$\cos(\theta/2) = \sqrt{4/13} = 2/\sqrt{13}$$

$$(-5 + 12i)^{1/2} = \sqrt{13}(2/\sqrt{13} + i 3/\sqrt{13}) = 2 + 3i$$

Solve second root for  $n=1$ ,  $\Rightarrow -2 - 3i$

Alternative method to

Set  $(-5+12i)^{1/2} = (a+bi) \Rightarrow (-5+12i)^{1/2} = (a+bi)^2$  and solve expansion

(c)

$$Z^3 = -8 = r(\cos\theta + i\sin\theta)$$

$$|Z^3| = 8 = r$$

$$\cos\theta = -1 \quad \theta = \pi$$

$$Z = 8^{1/3}[\cos((\pi + 2n\pi)/3) + i\sin((\pi + 2n\pi)/3)] \text{ for } n = 0, 1, 2$$

$$\text{For } n=0, \sin(\pi/3) = \sqrt{3}/2, \cos(\pi/3) = 1/2$$

$$\text{Root is } 2(1/2 + i\sqrt{3}/2) = 1 + \sqrt{3}i$$

$$\text{For } n=1, \sin(3\pi/3) = 0, \cos(3\pi/3) = -1$$

$$\text{Root is } 2(-1 + i0) = -2$$

$$\text{For } n=2, \cos(5\pi/3) = 1/2, \sin(5\pi/3) = -\sqrt{3}/2$$

$$\text{Root is } 2(1/2 + i(-\sqrt{3}/2)) = 1 - \sqrt{3}i$$

**Question D (Question 2 Paper 1 2020)**

(a)

Multiply  $iz_1$  by  $i$  to get  $i^2z_1 = -z_1$  and then substitute in formulae

$$iz_1 = -4 + 3i$$

$$i(iz_1) = i(-4 + 3i)$$

$$-z_1 = -4i + 3i^2$$

$$z_1 = 3 + 4i$$

$$3z_1 - z_2 = 3(3 + 4i) - z_2 = 11 + 17i$$

$$z_2 = 9 + 12i - 11 - 17i$$

$$z_2 = -2 - 5i$$

(b) (i)

$$3+2i = a_1$$

$$5-i = a_1r$$

$$r = a_1r / a_1$$

$$\frac{5-i}{3+2i} = \frac{5-i \cdot 3-2i}{3+2i \cdot 3-2i} = \frac{15-3i-10i+2i^2}{9+6i-6i-4i^2} = \frac{15-13i+2(-1)}{9-4(-1)} = \frac{13-13i}{13}$$

$$r=1-i$$

(b) (ii)

Write  $1 - i$  in polar form i.e.  $\sqrt{2} (\cos (7\pi/4) + i \sin (7\pi/4))$

$$T_9 = ar^8$$

$$T_9 = (3 + 2i)(1 - i)^8$$

$$T_9 = (3 + 2i) \left( \sqrt{2} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) \right)^8$$

$$T_9 = (3 + 2i)(\sqrt{2})^8 \left( \cos \frac{7\pi(8)}{4} + i \sin \frac{7\pi(8)}{4} \right)$$

$$T_9 = (3 + 2i)(16)(\cos 14\pi + i \sin 14\pi)$$

$$T_9 = (3 + 2i)(16)(1 + 0i)$$

$$T_9 = 48 + 32i$$

**Question E (Question 5 Paper 1 2019)**

(a)

$$(3 + 2i)^2 + p(3 + 2i) + q = 0$$

$$5 + 12i + 3p + 2pi + q = 0$$

$$2p = -12 \Rightarrow p = -6$$

$$5 + 3p + q = 0 \Rightarrow q = 13$$

(b)(i)

$$|v| = \sqrt{4 + 12} = 4$$

$$\theta = 300^\circ$$

$$v = 4(\cos 300^\circ + i \sin 300^\circ)$$

Or

$$|v| = \sqrt{4 + 12} = 4$$

$$\theta = \frac{5\pi}{3}$$

$$v = 4\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$$

(b)(ii)

$$w = \pm v^{\frac{1}{2}}$$

$$w = \pm 2(\cos 300 + i \sin 300)^{\frac{1}{2}}$$

$$w = \pm 2(\cos 150 + i \sin 150)$$

$$w = \pm(-\sqrt{3} + i)$$

$$w = -\sqrt{3} + i \text{ or } \sqrt{3} - i$$

Or

$$w = [4(\cos(300 + 360n) + i \sin(300 + 360n))]^{\frac{1}{2}}$$

$$w = 4^{\frac{1}{2}}[\cos(150 + 180n) + i \sin(150 + 180n)]$$

$$\underline{n = 0}$$

$$w = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$$

$$\underline{n = 1}$$

$$w = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

**Question F (Question 1 Paper 1 2013)**

(a)

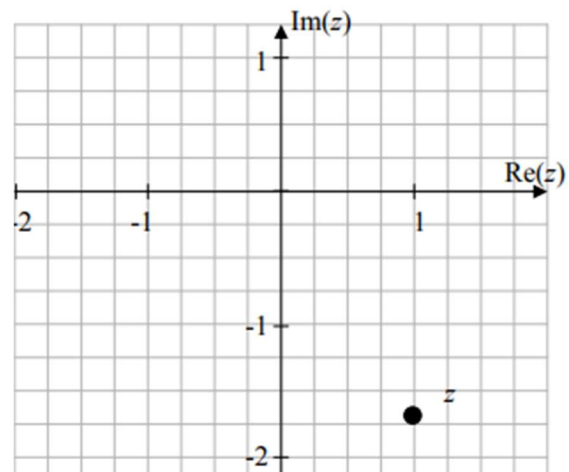
$$z = \frac{4}{1 + \sqrt{3}i} = \frac{4}{1 + \sqrt{3}i} \times \frac{1 - \sqrt{3}i}{1 - \sqrt{3}i} = \frac{4 - 4\sqrt{3}i}{1 + 3} = 1 - \sqrt{3}i$$

(b)

$$\tan \alpha = \frac{\sqrt{3}}{1} \Rightarrow \alpha = \frac{\pi}{3} \Rightarrow \theta = \frac{5\pi}{3}$$

$$r = |1 - \sqrt{3}i| = \sqrt{1+3} = \sqrt{4} = 2$$

$$z = 2 \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right)$$



(c)

$$\begin{aligned} z^{10} &= \left[ 2 \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) \right]^{10} \\ &= 2^{10} \left( \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right) = 2^{10} \left( \cos \frac{50\pi}{3} + i \sin \frac{50\pi}{3} \right) \\ &= 2^{10} \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) = 2^{10} \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) = -2^9 (1 - \sqrt{3}i) \end{aligned}$$



**Question G (Question 1 Paper 1 2016)**

(a)  $-4-3i$

(b)

$$r = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \theta = \frac{\pi}{4}$$
$$(1 + i)^8 = \left\{ \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right\}^8$$

$$(1 + i)^8 = \{16(\cos 2\pi + i \sin 2\pi)\}$$

$$(1 + i)^8 = 16(1) = 16$$

(c)

$$z = \frac{(2 - i) \pm \sqrt{(-2 + i)^2 - 4(3 - i)}}{2}$$
$$= \frac{(2 - i) \pm \sqrt{4 - 4i - 1 - 12 + 4i}}{2}$$
$$= \frac{2 - i \pm \sqrt{-9}}{2}$$
$$= \frac{2 - i \pm 3i}{2}$$
$$= 1 - 2i \text{ or } 1 + i$$

**Question H (Question 4 Paper 1 2021)**

Prove for  $n = 1$ ,  $2^{3 \cdot 1 - 1} + 3 = 2^2 + 3$

Assume True for  $n=k$

Prove if true for  $n=k$ , then true for  $n=k+1$ :  $2^{3 \cdot (k+1) - 1} + 3 = 2^{3 \cdot k + 3 - 1} + 3$

$$= 2^{3k+2} + 3$$

$$= 2^3 \cdot 2^{3k-1} + 3$$

$$= 8 \cdot 2^{3k-1} + 3$$

$$= 7 \cdot 2^{3k-1} + 1 \cdot 2^{3k-1} + 3 = 7 \cdot 2^{3k-1} + (2^{3k-1} + 3)$$

$7 \cdot 2^{3k-1}$  is divisible by 7

If  $n=k$  true then  $2^{3k-1} + 3$  also divisible by 7.

**Question I (Question 7 Paper 1 2020)**

Prove for  $n = 1$ ,  $[1(1+1)(2+1)]/6 = 1$

Assume True for  $n$  ie :  $(1 + 4 + 9 + \dots + n^2) = [n(n+1)(2n+1)]/6$

Prove if true for  $n$ , then true for  $n+1$ :  $[(n+1)((n+1)+1)(2(n+1)+1)]/6 = [(n+1)(n+2)(2n+3)]/6$

$$= [(n+1)(2n^2+7n+6)]/6$$

$$= [(n+1)(2n^2+1n+6n+6)]/6$$

$$= [(n+1)(n(2n+1)+6(n+1))]/6$$

$$= [(n+1)(n)(2n+1)/6 + (n+1).6.(n+1)]/6$$

$$= [(n+1)(n)(2n+1)/6 + (n+1).6.(n+1)]/6$$

If true for  $n$  then  $= (1 + 4 + 9 + \dots + n^2) + 6(n+1)^2/6$

$$= (1 + 4 + 9 + \dots + n^2) + (n+1)^2$$

Therefore true for  $n+1$  if true for  $n$