Society of Actuaries in Ireland

Question A

(i) If $z_1 = 2 + 3i$ *and* $z_2 = 1 - 2i$ find :

- $2z_1 + z_2$
- $\overline{2z_1 + z_2}$
- $z_1 - iz_2$
- $z_1 z_2$
- $|2 + z_1|$
- $\frac{z_1}{z_2}$

(ii) Write the following in polar format

$2+2i$

$-2\sqrt{2} + 2\sqrt{2}i$

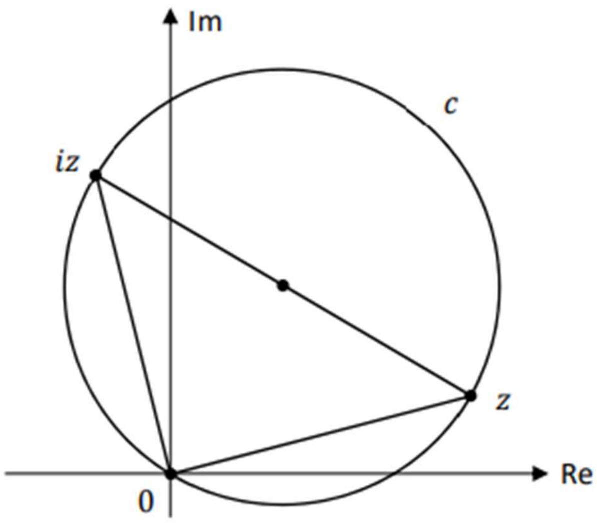
Question B

(a) $z = 6 + 2i$, where $i^2 = -1$.

(i) Show that $z - iz = 8 - 4i$.

(ii) Show that $|z|^2 + |iz|^2 = |z - iz|^2$

(iii) The circle c passes through the points z, iz, and 0, as shown in the diagram below (not to scale). z and iz are endpoints of a diameter of the circle. Find the area of the circle c in terms of π.



(b) $(\sqrt{3} - i)^9$ can be written in the form $a + ib$, where $a, b \in \mathbb{Z}$ and $i^2 = -1$. Use de Moivre's Theorem to find the value of a and the value of b.

Question C

(a) $\frac{(4-2i)}{(2+4i)} = 0 + ki$, where $k \in \mathbb{Z}$, and $i^2 = -1$. Find the value of k.

(b) Find $\sqrt{-5 + 12i}$.
Give both of your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

(c) Use De Moivre's theorem to find the **three** roots of $z^3 = -8$.
Give each of your answers in the form $a + bi$, where $a, b \in \mathbb{R}$, and $i^2 = -1$.

Question D

(a) Find the two complex numbers z_1 and z_2 that satisfy the following simultaneous equations, where $i^2 = -1$:

$$iz_1 = -4 + 3i$$
$$3z_1 - z_2 = 11 + 17i.$$

Write your answers in the form $a + bi$ where $a, b \in \mathbb{Z}$.

(b) The complex numbers $3 + 2i$ and $5 - i$ are the first two terms of a **geometric** sequence.

(i) Find r, the common ratio of the sequence.
Write your answer in the form $a + bi$ where $a, b \in \mathbb{Z}$.

Hint for Geometric sequence $a_n = a_1 r^{n-1}$

(ii) Use de Moivre's Theorem to find T_9, the ninth term of the sequence.
Write your answer in the form $a + bi$, where $a, b \in \mathbb{Z}$.

Question E

(a) $3 + 2i$ is a root of $z^2 + pz + q = 0$, where $p, q \in \mathbb{R}$, and $i^2 = -1$.
Find the value of p and the value of q.

(b) **(i)** $v = 2 - 2\sqrt{3}i$. Write v in the form $r(\cos\theta + i\sin\theta)$, where $r \in \mathbb{R}$ and $0 \leq \theta \leq 2\pi$.

(ii) Use your answer to **part (b)(i)** to find the **two** possible values of w, where $w^2 = v$. Give your answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

Question F

$z = \dfrac{4}{1+\sqrt{3}i}$ is a complex number, where $i^2 = -1$.

(a) Verify that z can be written as $1-\sqrt{3}i$.

(b) Plot z on an Argand diagram and write z in polar form.

(c) Use De Moivre's theorem to show that $z^{10} = -2^9\left(1-\sqrt{3}i\right)$.

Question G

(a) $(-4 + 3i)$ is one root of the equation $az^2 + bz + c = 0$, where $a, b, c \in \mathbb{R}$, and $i^2 = -1$. Write the other root.

(b) Use De Moivre's Theorem to express $(1 + i)^8$ in its simplest form.

(c) $(1 + i)$ is a root of the equation $z^2 + (-2 + i)z + 3 - i = 0$.
Find its other root in the form $m + ni$, where $m, n \in \mathbb{R}$, and $i^2 = -1$.

Question H

Prove using induction that $2^{3n-1} + 3$ is divisible by 7 for all $n \in \mathbb{N}$.

Question I

Prove using **induction** that, for all $n \in \mathbb{N}$, the sum of the first n square numbers can be found using the formula:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$