Society of Actuaries in Ireland

Trigonometry 2 – Solutions

Trigonometric formulae

Question 1.

We can use the formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\Rightarrow \quad \cos(2A) = \cos(A+A)$

$= \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

Question 2.

We can use the formulae $\cos(A+B) = \cos A \cos B - \sin A \sin B$

and $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\Rightarrow \quad \tan(A+B) = \sin(A+B) / \cos(A+B)$

$= (\sin A \cos B + \cos A \sin B) / (\cos A \cos B - \sin A \sin B)$

Looking at the equation we are asked to prove, we can see that the denominator should start with 1 rather than cos A cos B. If we divide every term in this expression by cos A cos B, then we have

$\tan(A+B) = (\sin A / \cos A + \sin B / \cos B) / (1 - \sin A \sin B / \cos A \cos B)$

$= (\tan A + \tan B) / (1 - \tan A \tan B)$

Question 3.

We have a mixture of functions of A and 7A on the left side of this equation, which makes it difficult to start solving this equation. We will have to use the formulae for cos (A±B) and sin (A±B) to simplify it. If we let x = 4A and y = 3A then we can express 7A and A in terms of these:

$7A = x + y$

$A = x - y$

$\Rightarrow \quad (\cos 7A + \cos A) / (\sin 7A - \sin A)$

Trigonometry 2 – Solutions

= (cos (x+y) + cos (x-y)) / (sin (x+y) - sin (x-y))

= ([cos x cos y - sin x sin y] + [cos x cos y + sin x sin y])

/ ([sin x cos y + cos x sin y] - [sin x cos y - cos x sin y])

= (2cos x cos y) / (2 cos x sin y)

= (cos y) / (sin y)

= 1 / tan y

= cot y

= cot 3A

Radians

Question 4.

Each runner runs a part of a circle (an arc) with angle θ. Let r be the radius (in metres) of the circle Kate runs on. Then the length of the arc that Kate runs is $r\theta$ metres.

Since the lanes are both 1.2 m wide, the midpoints of the lanes must be 1.2 m apart, so the radius of the circle Helen runs must be (r + 1.2) m. Then the length of the arc that Helen runs is $(r+1.2)\theta$ metres.

$\Rightarrow$ Helen runs 1.2θ metres further than Kate.

$\Rightarrow 1.2\theta = 3$

$\Rightarrow \theta = 2.5$ rad

Solving equations

Question 5.

Solution:

Let y =3x. Then sin(y) = $\sqrt{3}$ / 2.

$\sin^{-1}(\sqrt{3} / 2) = 60°$ so this is one possible value of y, but there is also a possible value in the second quadrant (between 90° and 180°) given by 180° - 60° = 120°. The solutions are in the first and second quadrant because that is where sine is positive.

Trigonometry 2 – Solutions

Normally, we restrict angles to between 0° and 360° but in this case x can have any value in this range and y = 3x so y can take y between 0° and 1,080°. We can add 360° to any value of y (which adds 120° to x) and it won't affect the value of sin y =sin. This gives 6 possible values of x:

	y	x = y / 3
First two solutions for y	60°	20°
	120°	40°
Add 360° to y	420°	140°
	480°	160°
Add another 360° to y	780°	260°
	840°	280°

Answer: x = 20°, 40°, 140°, 160°, 260°, or 280°

Question 6.

Let $x = \cos\theta$

We can use the triginometric formulae to express $\cos 2\theta$ in terms of $\cos\theta$ and $\sin\theta$, and then eliminate $\sin\theta$ so we express it purely in terms of $\cos\theta$. (Recall that $\cos^2\theta$ just means the square of $\cos\theta$.)

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1 = 2x^2 - 1$

$\Rightarrow \quad 2x^2 - 1 = 1/9$

$\Rightarrow \quad 2x^2 = 10/9$

$\Rightarrow \quad x^2 = 5/9$

$\Rightarrow \quad x = \pm\sqrt{5}/3$

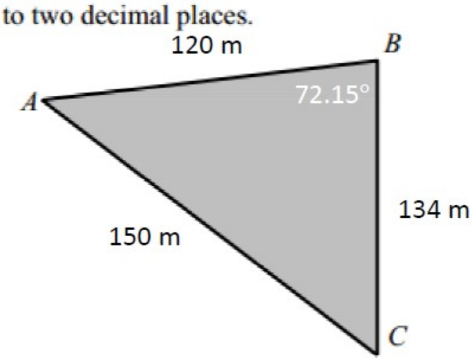
ANSWER: $\cos\theta = \pm\sqrt{5}/3$

Trigonometry 2 – Solutions

Sine rule, cosine rule, area of a triangle

Question 7.

(a) **(i)** Find $|\angle CBA|$. Give your answer, in degrees, correct to two decimal places.



$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{120^2 + 134^2 - 150^2}{2(120)(134)}$$
$$= \frac{9856}{32160}$$
$$= 0 \cdot 306468$$
$$\Rightarrow B = 72 \cdot 15°$$

(ii) Find the area of the triangle ACB correct to the nearest whole number.

$$\text{Area } \Delta ABC = \tfrac{1}{2} ac \sin B = \tfrac{1}{2}(120)(134)\sin 72 \cdot 15$$
$$= 7652 \cdot 97$$
$$\approx 7653 \text{ m}^2$$

Trigonometry 2 – Solutions

Question 8.

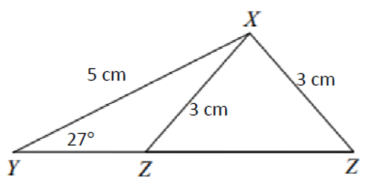
(i) Find the two possible values of $|\angle XZY|$. Give your answers correct to the nearest degree.

$$\frac{3}{\sin 27^\circ} = \frac{5}{\sin \angle Z} \Rightarrow \sin \angle Z = \frac{5\sin 27^\circ}{3} = 0\cdot 756$$

$$\Rightarrow |\angle Z| = 49^\circ \text{ or } |\angle Z| = 131^\circ$$

- **$\text{Sin}^{-1}(0.756) = 49.11°$**
- **Sin is positive in the 1st and 2nd quadrants**
- **180° – 49° = 131°**

(ii) Draw a sketch of the triangle *XYZ*, showing the two possible positions of the point *Z*.



In the case that $|\angle XZY| < 90^\circ$, write down $|\angle ZXY|$, and hence find the area of the triangle *XYZ*, correct to the nearest integer.

$$|\angle ZXY| = 180^\circ - (27^\circ + 49^\circ) = 104^\circ$$

$$\Delta = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}(5)(3)\sin 104^\circ = 7\cdot 27 = 7\text{ cm}^2$$

- Use 49° as 49°< 90°
- 27° is the angle at Y
- All angles must sum to 180° so the remaining angle must equal 104°

Trigonometry 2 – Solutions

Longer questions

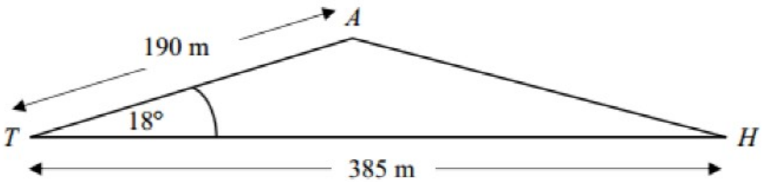
Question 9.

(a)

$$\sin \tfrac{1}{2}\alpha = \frac{15}{150} = 0\cdot 1$$
$$\Rightarrow \tfrac{1}{2}\alpha = 5\cdot 739°$$
$$\Rightarrow \alpha = 11\cdot 478°$$
$$\alpha = 11\cdot 5°$$

$$\sin^{-1} 0.1 = 5.739°$$

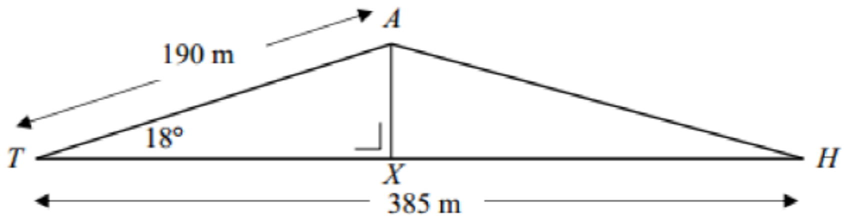
(b)



$$|AH|^2 = 190^2 + 385^2 - 2(190)(385)\cos 18°$$
$$= 36100 + 148225 - 139139\cdot 5683$$
$$= 45185\cdot 4317$$
$$|AH| = 212\cdot 57 = 213$$

Part (b) – alternative solution by splitting triangle into two right triangles:

Trigonometry 2 – Solutions



Draw AX perpendicular to TH

triangle ATX: $\sin 18° = \frac{|AX|}{190} \Rightarrow |AX| = 58\cdot71$

$\cos 18° = \frac{|TX|}{190} \Rightarrow |TX| = 180\cdot7$

$\Rightarrow |XH| = 204\cdot3$

$\Rightarrow |AH|^2 = (58\cdot71)^2 + (204\cdot3)^2$

$\Rightarrow |AH| = 212\cdot566 = 213$

(c)

(i)

The ball is at K at time t=0 so the height of K is h(0) = 8.

Answer: 8 m

(ii)

The ball reaches the point B when its height is zero. This occurs at a time t that satisfies h(t)=0.

$-6t^2 + 22t + 8 = 0$

Solve this using the quadratic formula. The solutions are 4 and -1/3. We take the positive solution t=4.

The horizontal speed is 38 m/s.

The travel time is 4 s.

$\Rightarrow$ The horizontal distance is 38 x 4 = 152 m.

Let α be the angle of elevation. Then tan (α) = 8 / 152.

$\tan^{-1}(8 / 152) = 3.01°$

Trigonometry 2 – Solutions

Answer: 3°

(d)

(i)

$\tan(\theta) = ½ \Rightarrow h/d = ½$

$\Rightarrow d = 2h$

From the diagram we can see that:

$|CD| = 25 - h$

(ii)

We have a right triangle GDC with sides of length 2h, (25-h) and 25 (the hypotenuse). By Pythagoras:

$25^2 = (2h)^2 + (25 - h)^2$

$= 4h^2 + 25^2 + h^2 - 50h$

$\Rightarrow 5h^2 = 50h$

$\Rightarrow h = 10$

Question 10.

Trigonometry 2 – Solutions

(a) (i)	$$\|EC\|^2 = 3^2 + 2{\cdot}5^2 = 15{\cdot}25$$ $$\|EC\| = \sqrt{15{\cdot}25}$$ $$\|EC\| = 3{\cdot}905$$ $$\Rightarrow \|AC\| = 1{\cdot}9525$$ $$= 1{\cdot}95$$
(a) (ii)	$$\tan 50° = \frac{\|AB\|}{1{\cdot}95}$$ $$\|AB\| = 1{\cdot}95(1{\cdot}19175) = 2{\cdot}23239$$ $$\|AB\| = 2{\cdot}3$$

Trigonometry 2 – Solutions

(a) (iii)	$\|BC\|^2 = 1{\cdot}95^2 + 2{\cdot}3^2$ $\|BC\| = 3 \cdot 015377$ $\|BC\| = 3$ Also: $\sin 40° = \frac{1{\cdot}95}{\|BC\|}$ **or** $\cos 40° = \frac{2{\cdot}3}{\|BC\|}$ **or** $\cos 50° = \frac{1{\cdot}95}{\|BC\|}$ **or** $\sin 50° = \frac{2{\cdot}3}{\|BC\|}$
(a) (iv)	$3^2 = 3^2 + 2{\cdot}5^2 - 2(3)(2{\cdot}5)\cos\alpha$ $15\cos\alpha = 6{\cdot}25$ $\alpha = 65°$ **or** $\cos\alpha = \frac{1{\cdot}25}{3}$ $\alpha = 65°$

Trigonometry 2 – Solutions

(a)

(v) $A = 2\times$ isosceles triangle $+ 2\times$ equilateral triangle

$$= 2\times\left[\frac{1}{2}(2\cdot5)(3)\sin 65°\right] + 2\times\left[\frac{1}{2}(3)(3)\sin 60°\right]$$

$$= 14\cdot59$$

$$A = 15$$

$$\tan 60° = \frac{3}{|CA|}$$

$$\Rightarrow |CA| = \sqrt{3}$$

$$|CE| = 2\sqrt{3}$$

$$x^2 + x^2 = (2\sqrt{3})^2$$

$$x = \sqrt{6}$$

Trigonometry 2 – Solutions

(b)

Qs 11 (Paper 2 – 2019 – Q2 4)

(a) Show that $\cos 2\theta = 1 - 2\sin^2\theta$

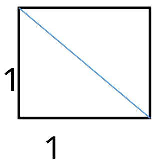
$\text{Cos } 2\ \theta = \text{Cos}\ (\theta + \theta) = \text{Cos } 2\ \theta\ - \text{Sin } 2\ \theta$ pg 14 log tables

$= 1 - \text{Sin } 2\ \theta\ - \text{Sin } 2\ \theta$ as $(\text{Cos } 2\ \theta + \text{Sin } 2\ \theta = 1)$ pg 13

$= 1 - 2\ \text{Sin } 2\ \theta$

(b) Find the cosine of the acute angle between two diagonals of a cube.

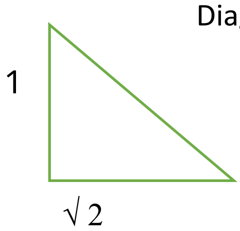
Draw the cube !!! Base of cube is a square. Assume each side is length 1. Using Pythagoros, diagonal (hypothenuse) is √2.



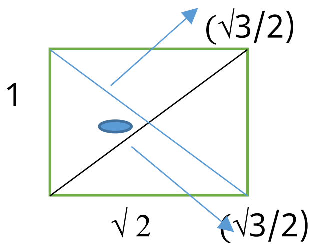
Diagonal is $\sqrt{1^2 + 1^2} = = \sqrt{2}$

Height of Cube is 1. So length of internal diagonal from corner to corner (using pythagoros) is √3.

Diagonal is $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$



Therefore cosine of acute angle is adjacent/ hypothenuse = (√2/√3)



 is the acute angle as it has the shortest base.

Trigonometry 2 – Solutions

Looking at the triangle with at the apex – from the cosine rule

$1^2 = (\sqrt{3}/2)^2 + (\sqrt{3}/2)^2 - 2(\sqrt{3}/2)(\sqrt{3}/2)\text{Cos}$

$1 = (3/4) + (3/4) - (3/2)\text{ Cos}$

⇨ Cos = [(3/2) – 1] x (2/3) = 1/3

Qs 12 (Paper 2 – 2020 – Q2 4)

(a) Find the two values of θ for which $\tan \theta/2 = -1/\sqrt{3}$, where $0 \leq \theta \leq 4\pi$.

From page 13 of your log tables Tan 30 = $-1/\sqrt{3}$

From the quadrant circle (CAST) , we know that tan A is negative in the second and fourth quadrants .

So $\tan \theta/2 = -1/\sqrt{3}$ where $\theta/2 = \pi - 30$ and θ therefore equals $2\pi - 60$ ($=5\pi/3$)

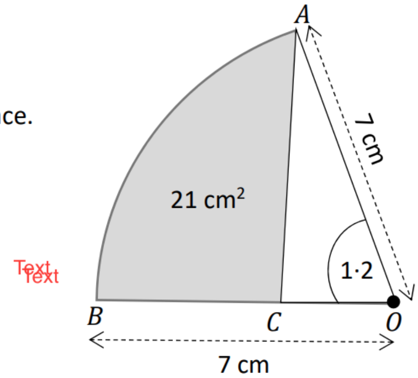
and $\tan \theta/2 = -1/\sqrt{3}$ where $\theta/2 = 2\pi - 30$ and θ there equals $4\pi - 60$ ($= 11\pi/3$)

(note that there are additional higher values for θ but these are $> 4\pi$)

Trigonometry 2 – Solutions

(b) The diagram shows OAB, a sector of a circle of radius 7 cm with centre O.
In the sector, $|\angle BOA| = 1{\cdot}2$ radians.
The area of the shaded region is 21 cm^2.

Find $|BC|$.
Give your answer correct to 1 decimal place.



Total Area of sector = ($r^2 \times 1.2$)/2 = 49* 0.6 = 29.4

The unshaded are is (1/2) CO x 7 x sin 1.2 = 3.26 CO

So – shaded area + unshaded area = total sector atrea

21 + 3.26 CO = 29.4

CO = 8.4/3.26 = 2.58

BC = 7 – 2.58 = 4.4 to 1dp