1972 - 2022
Society of Actuaries in Ireland

Coordinate Geometry 1

23 November 2022

Welcome!

- Introduction
- Coordinate Geometry 1
- Question Focused Approach
- 2 hours - 10mins break @7pm
- Any questions? – Just Ask your tutor!!

Society of Actuaries in Ireland

Please note: All attempts have been made to ensure the accuracy and reliability of the information provided in this document.

Coordinate Geometry: The Line – Hints & Tips

General Hints and Tips

1 **Always draw diagrams**. This is useful in every question, but it is particularly helpful with questions relating to the circle or more difficult questions.

2 Make sure you **know which formulae are in the tables,** and where in the tables they are. Formulae in the tables:

- **Slope of a line**
  $\frac{(y2 - y1)}{(x2 - x1)}$
- **Distance between 2 points**
  $\sqrt{(x2 - x1)^2 + (y2 - y1)^2}$
- **Midpoint formula**
  $\left(\frac{x1 + x2}{2}, \frac{y1 + y2}{2}\right)$
- **Equation of a line (2 different formats)**
  $(y - y1) = m(x - x1)$
- **Area of a triangle with one point at the origin**
  $(½|x1y2 - x2y1|)$
- **Point dividing a line segment in the ratio a:b**
-
- **To find the angle between 2 lines:**
  $Tan\theta = \pm(m1 - m2)/(1 + m1.m2)$
- **Perpendicular distance from a point to a line**
  $\frac{|ax1+by1+c|}{\sqrt{a^2+b^2}}$

3 **Learn other formulae** off by heart.

The Line

1 To get an **equation of a line** you always need 2 things:

- **A point**
- **A slope**

Once you have these, use the formula $y - y_1 = m(x - x_1)$

2 To check if a point is on a line, substitute it into the equation.
If the answer = 0, then the point is on the line, otherwise it is not.

3 To plot a line, you need two points on the line.
An easy way to find points on a line is:
Let x = 0, solve for y. This will give you a point (0,y)
Let y = 0, solve for x. This will give you a point (x,0)
Use these two points to plot the line.

4 If a line **intersects the x-axis**, then **y = 0** at that point.
If a line **intersects the y-axis**, then **x = 0** at that point.

5 Use simultaneous equations to find the point of intersection between 2 lines.

6 If lines are **parallel**, their **slopes are equal**.
If lines are **perpendicular**, then multiplying their slopes together equals -1 ($\mathbf{m_1.m_2 = -1}$)
An example - you want the slope of a line and are told it is perpendicular to another line with slope 2/3
Turn it upside down and change the sign of it. So in this case, the slope of the line you want is -3/2

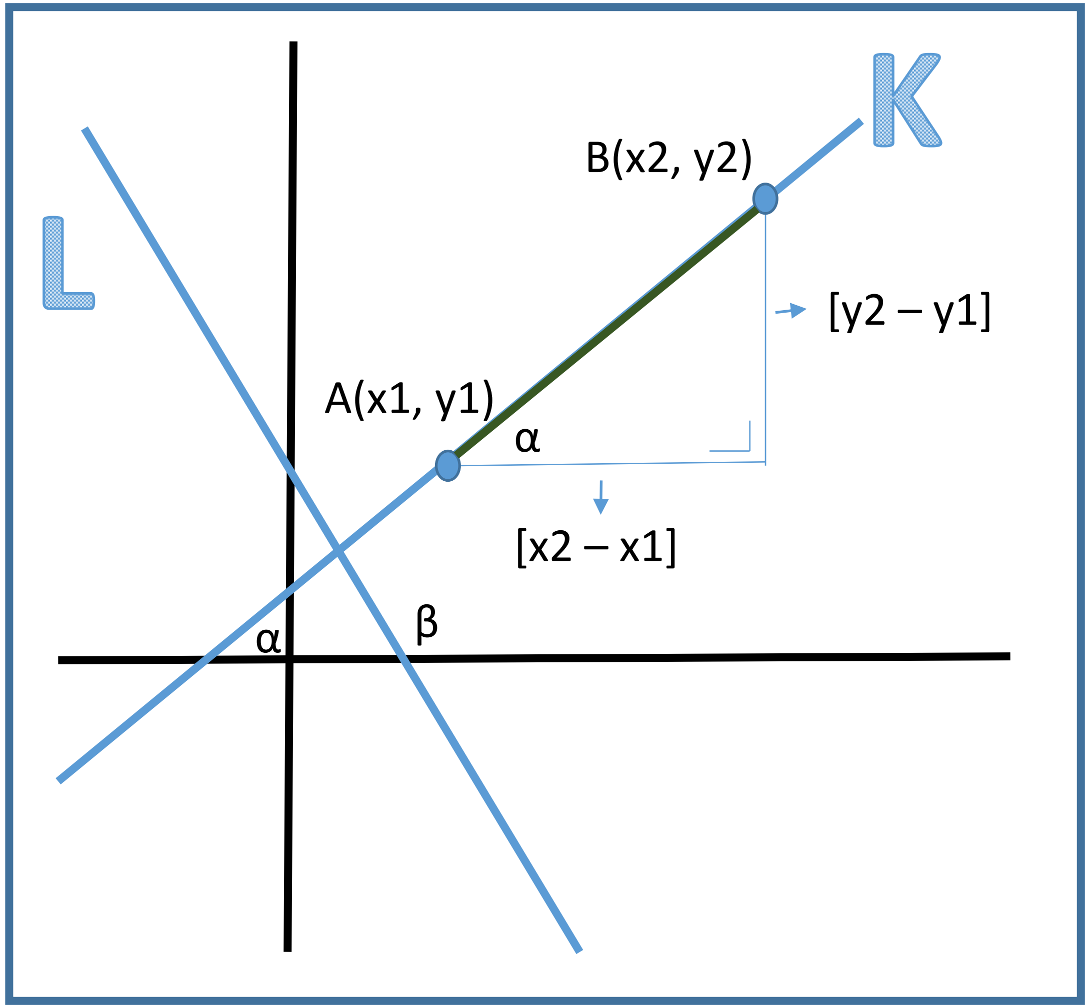
8 To use the area of a triangle formula ($\frac{1}{2}|x_1y_2 - x_2y_1|$) one of the points needs to be (0,0).
If you are looking for the area of a triangle, where no points are at the origin (0,0),
use translations to bring one of the points to (0,0) and then use the formula as normal.
Alternatively, you can use the area = ½ base x perpendicular height formula.

9 If 3 or more points lie on the same line, they are said to be collinear.
To check if 3 points (e.g. a, b, c) are collinear, see what the slopes of |ab| and |bc| are.
If they are the same, then the points are collinear, otherwise they are not.
An alternative way of doing this is to calculate the area of the triangle using the 3 points.
If the area = 0, then the points are collinear, otherwise they are not.

The Basics – Pages 18 & 19 of "Formulae and Tables"

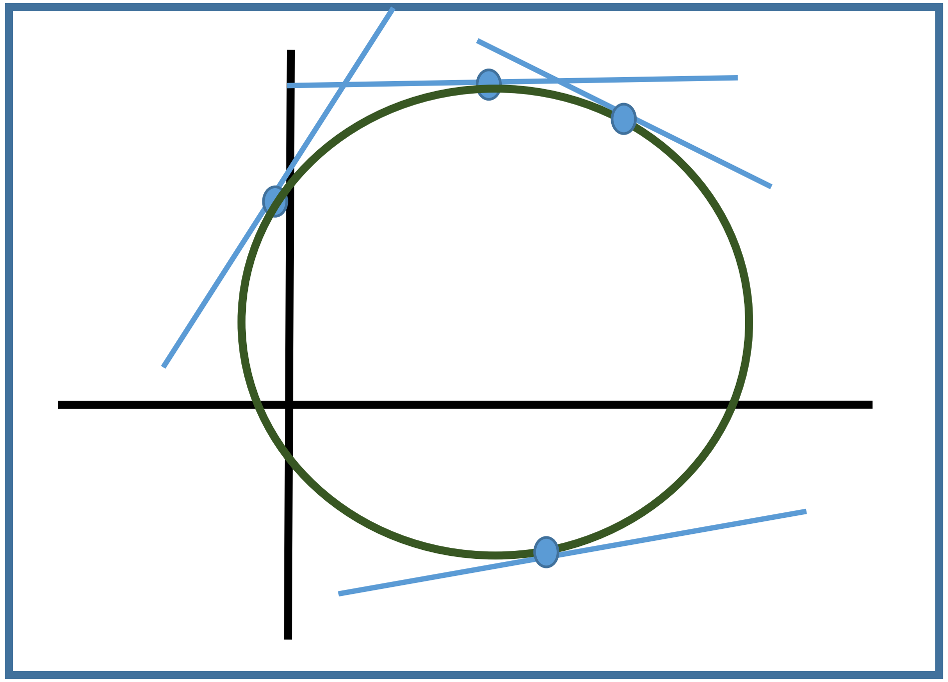
Definitions of slope (m)
$m = \frac{dy}{dx}$
$m = \frac{(y2 - y1)}{(x2 - x1)}$
m = tan α $= \frac{opposite}{adjacent}$
m = "the rise over the run"
m, when line written y = mx + c L1: 4x – 2y + 7 = 0 2y = 4x + 7 y = 2x + 7/2 m = 2

When you have a point (x1, y1) and the slope m find the equation of the line using:
(y – y1) = m (x – x1)



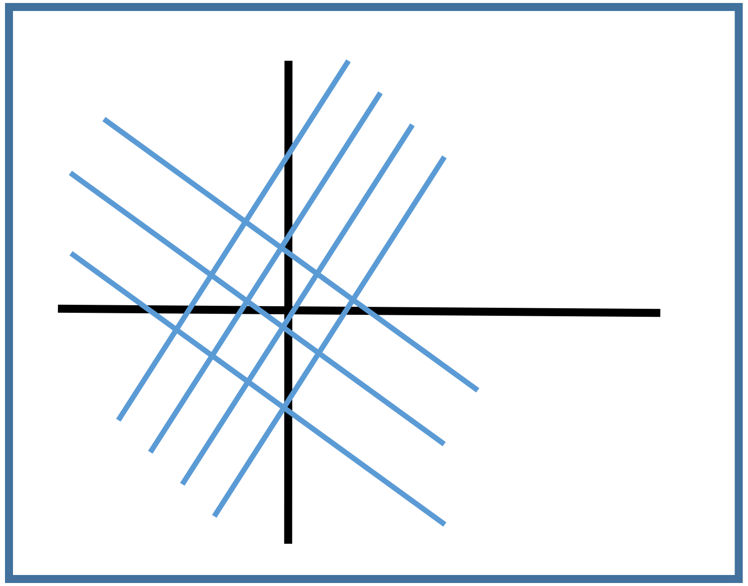
Line K has a positive slope (α < 90°)
Line L has a negative slope (β > 90°)

The slope continued



A line has a constant slope, more complex functions, like a circle or a quadratic, don't

But the slope at any point on a curve has the same slope as the tangent to the curve at that point



Lines with the same slope are parallel (and vice versa!)

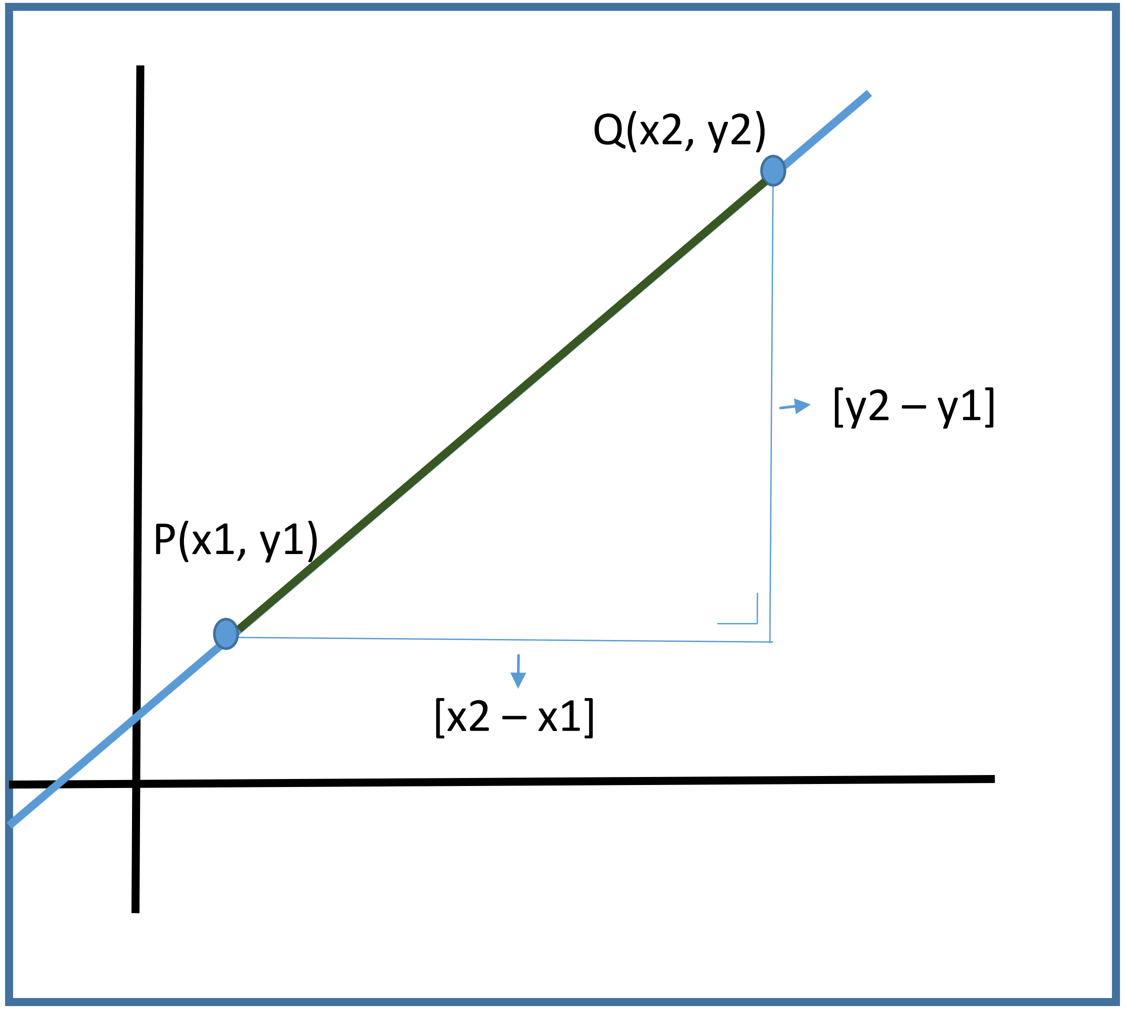
Lines with the slopes **m** and **-1/m** are perpendicular. For example if a line has the slope -2/3 then all lines perpendicular to this line will have a slope 3/2

The Basics – Pages 18 & 19 of “Formulae and Tables”

Distance:

$$|PQ| = \sqrt{(x2 - x1)^2 + (y2 - y1)^2}$$

(Pythagoras’s Theorem with the line segment as the hypotenuse)



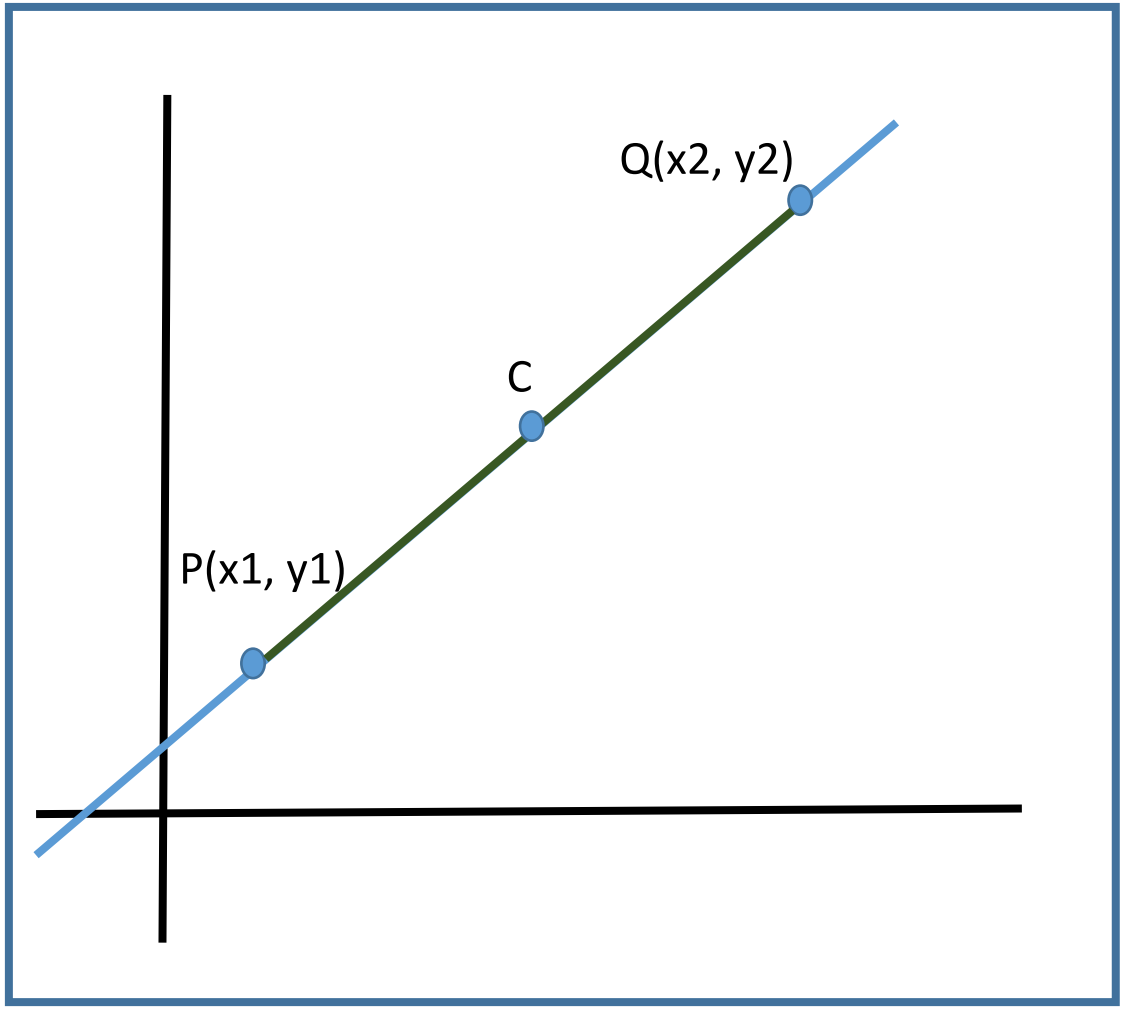
The Basics – Pages 18 & 19 of “Formulae and Tables”

Point dividing [PQ] in the ratio a:b

$$= \left(\frac{bx1+ax2}{a+b}, \frac{by1+ay2}{a+b}\right)$$

Midpoint:

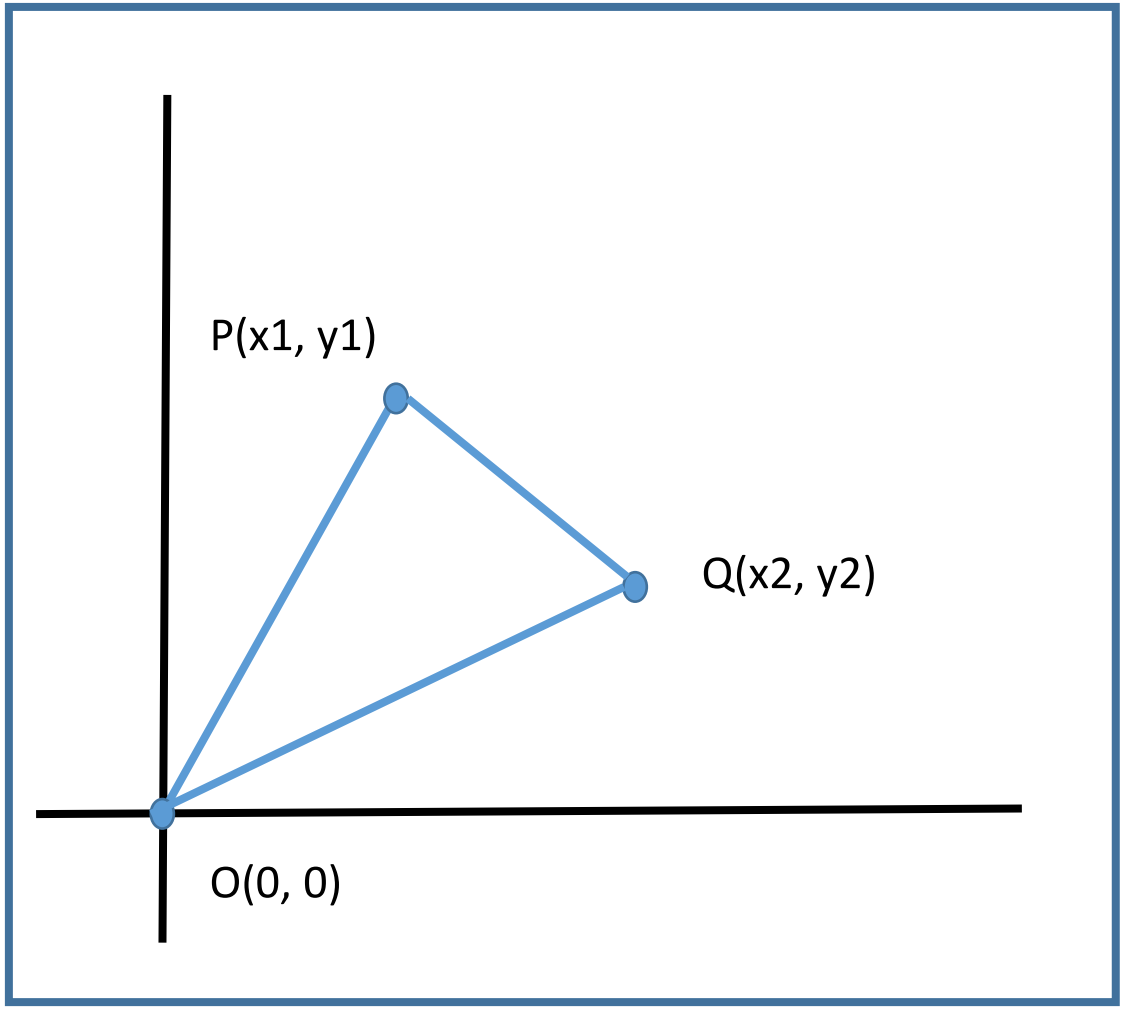
$$C = \left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$$



The Basics – Pages 18 & 19 of “Formulae and Tables”

Area of Triangle: OPQ

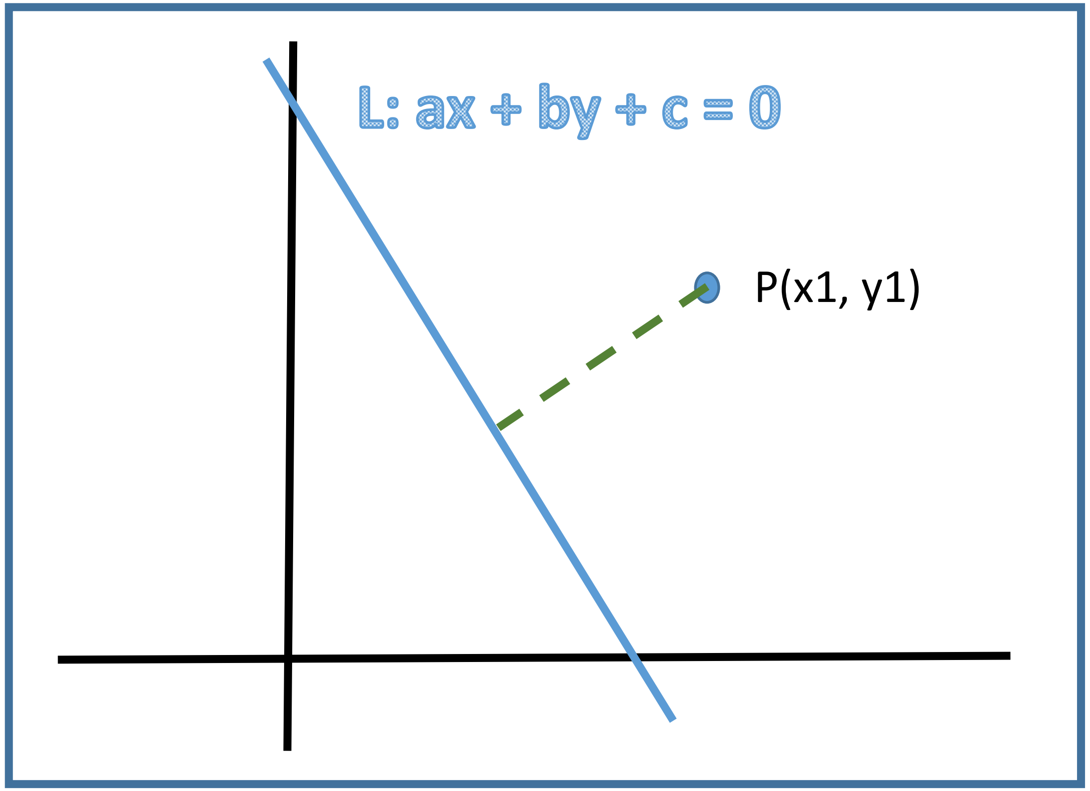
Area = $\frac{1}{2}\ |x1y2\ \ -x2y1|$



The Basics – Pages 18 & 19 of "Formulae and Tables"

Perpendicular distance from point $(x1,\ y1)$ to line ax +by + c = 0

$$\frac{|ax1 + by1 + c|}{\sqrt{a^2 + b^2}}$$



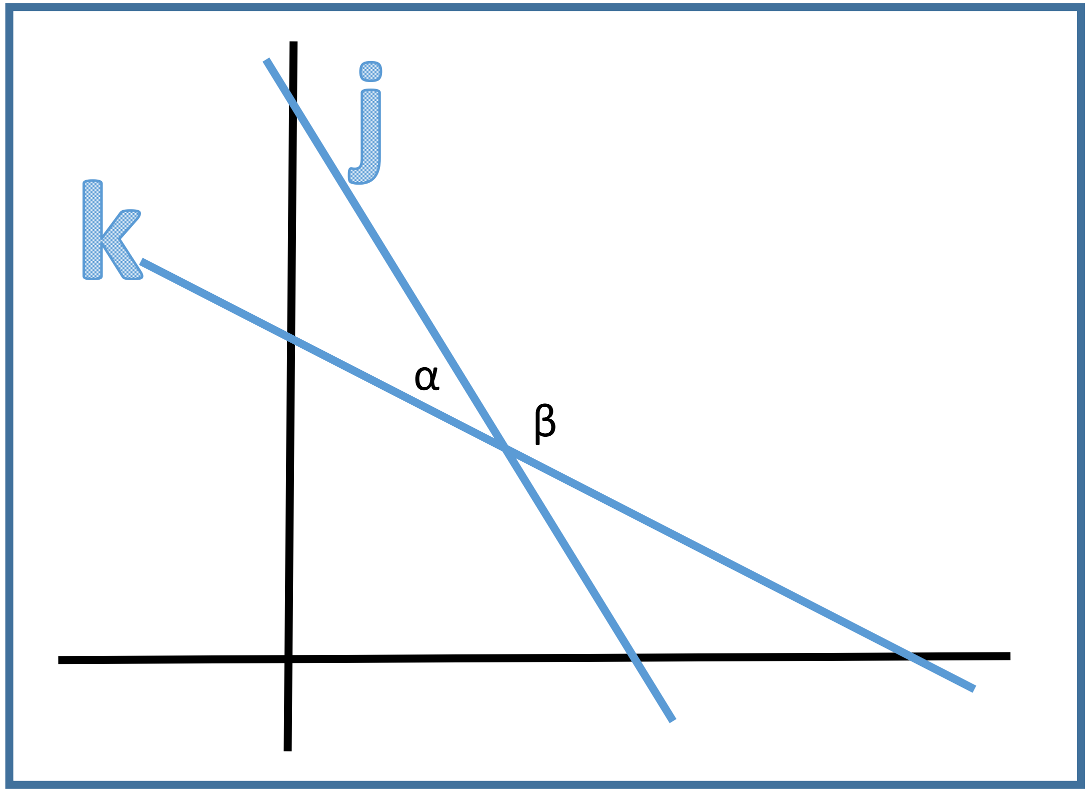
The Basics – Pages 18 & 19 of “Formulae and Tables”

The angle formed between two lines of slope m1 and m2 can be found using:

$$\tan \alpha = \pm \frac{m1 - m2}{1 + m1m2}$$

The positive answer gives the tan of the acute angle α

The negative answer gives the tan of the obtuse angle β



Question 1

The equations of six lines are given:

Hint! Calculate these

Line	Equation	y = mx + c	m
h	$x = 3 - y$	y=	
i	$2x - 4y = 3$	y=	
k	$y = -\frac{1}{4}(2x - 7)$	y=	
l	$4x - 2y - 5 = 0$	y=	
m	$x + \sqrt{3}y - 10 = 0$	y=	
n	$\sqrt{3}x + y - 10 = 0$	y=	

(a) Complete the table below by matching each description given to one or more of the lines.

Description	**Line(s)**
A line with a slope of 2.	
A line which intersects the y-axis at $(0, -2\frac{1}{2})$.	
A line which makes equal intercepts on the axes.	
A line which makes an angle of $150°$ with the positive sense of the x-axis.	
Two lines which are perpendicular to each other.	

(b) Find the acute angle between the lines m and n. **Hint! Check P19 of your log tables for a formula**

Question 1 Part (a)

Line	Equation	y = mx + c	m
h	x = 3 – y	y = -x + 3	-1
i	2x – 4y = 3	y = $\frac{1}{2}x$ - $\frac{3}{4}$	$\frac{1}{2}$
k	y = -$\frac{1}{4}$ $(2x\ - 7)$	y = -$\frac{1}{2}x + \frac{7}{4}$	-$\frac{1}{2}$
l	4x – 2y – 5 = 0	y = 2x - $\frac{5}{2}$	2
m	X + $\sqrt{3}y\ - 10 = 0$	y = -$\left(\frac{1}{\sqrt{3}}\right)x + \frac{10}{\sqrt{3}}$	- $\frac{1}{\sqrt{3}}$
n	$\sqrt{3}x + y\ - 10 = 0$	y = -$\sqrt{3}x + 10$	-$\sqrt{3}$

Description	Line(s)
A line with a slope of 2	l
A line which intersects the y-axis at (0, -2$\frac{1}{2}$)	l
A line which makes equal intercepts on the axes	h …. (0, 3) & (3, 0)
A line which makes an angle of 150° with the positive sense of the x-axis	m …. tan 150° = - $\frac{1}{\sqrt{3}}$
Two lines which are perpendicular to each other	k, l …. m1 * m2 = -1

Question 1 Part (b)
Find the acute angle between the lines m and n

Line	Equation	y = mx + c	m	
m	$X + \sqrt{3}y - 10 = 0$	y = $-\left(\frac{1}{\sqrt{3}}\right)x + \frac{10}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	m1
n	$\sqrt{3}x + y - 10 = 0$	y = $-\sqrt{3}x + 10$	$-\sqrt{3}$	m2

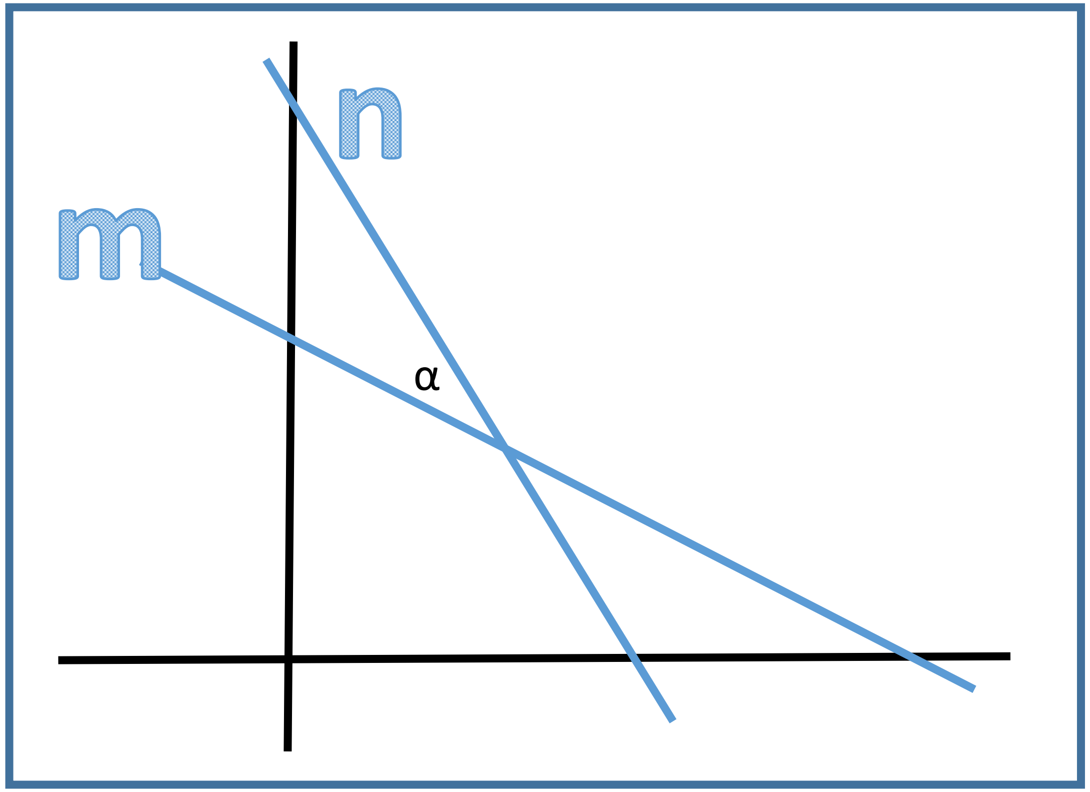
Tan α = $\frac{m1 - m2}{1 + m1m2}$ or $\frac{m2 - m1}{1 + m1m2}$ .... Page 19 of tables

Tan α = $\frac{-\frac{1}{\sqrt{3}} - (-\sqrt{3})}{1 + (-\frac{1}{\sqrt{3}})(-\sqrt{3})}$ or $\frac{-\sqrt{3} - (-\frac{1}{\sqrt{3}})}{1 + (-\frac{1}{\sqrt{3}})(-\sqrt{3})}$ ... discard -neg

Tan α = $\frac{1.154701}{2}$ or $\frac{2}{2\sqrt{3}}$

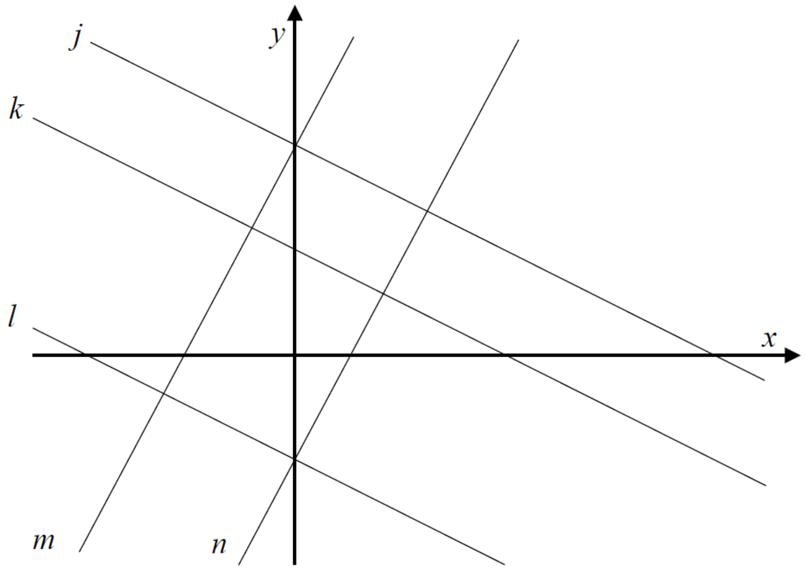
Tan α = 0.57735 or $\frac{1}{\sqrt{3}}$

Answer: α = 30°



Question 2

The equations of four of the five lines are given in the table below.



Equation	Line
$x+2y=-4$	
$2x-y=-4$	
$x+2y=8$	
$2x-y=2$	

(a) Complete the table, by matching four of the lines to their equations.

(b) Hence, insert scales on the x-axis and y-axis.

Hint! Calculate these

y = mx + c	Cuts y-axis	Cuts x-axis
y =		
y =		
y =		
y =		

(c) Hence, find the equation of the remaining line, given that its x-intercept and y-intercept are both integers.

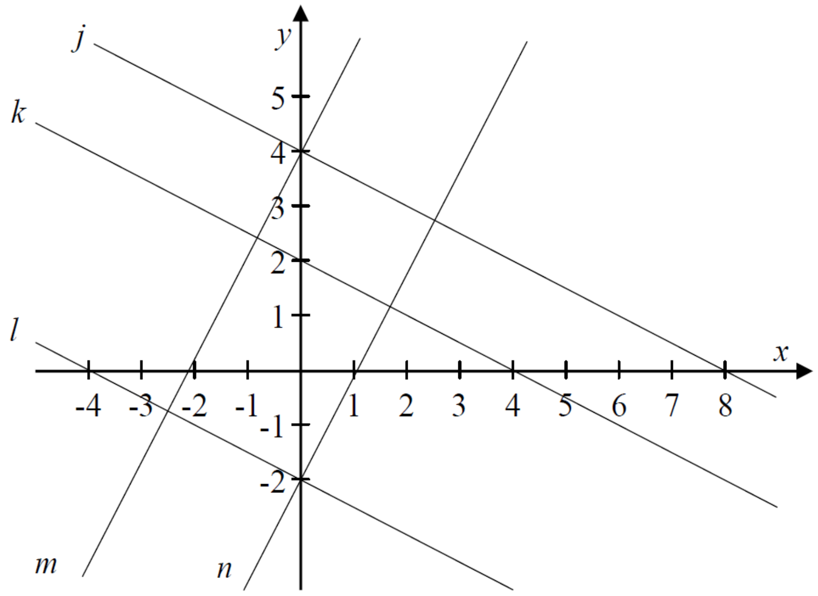
Question 2

Part (a)

Equation	y = mx + c	Cuts y-axis	Cuts x-axis	Line
x + 2y = -4	$y = -\frac{1}{2}x - 2$	(0, -2)	(-4, 0)	l
2x – y = -4	$y = 2x + 4$	(0, 4)	(-2, 0)	m
x + 2y = 8	$y = -\frac{1}{2}x + 4$	(0, 4)	(8, 0)	j
2x – y = 2	y = 2x - 2	(0, -2)	(1, 0)	n

Part (b)

Scale is 6mm per unit – add the numbers to the diagram



Part (c)

Intercepts for k are (0, 2) and (4,0) … from observation

Slope of k is $\frac{(y2 - y1)}{(x2 - x1)} = \frac{(0-2)}{(4-0)} = -\frac{1}{2}$

Equation of k: (y – y1) = m(x – x1)

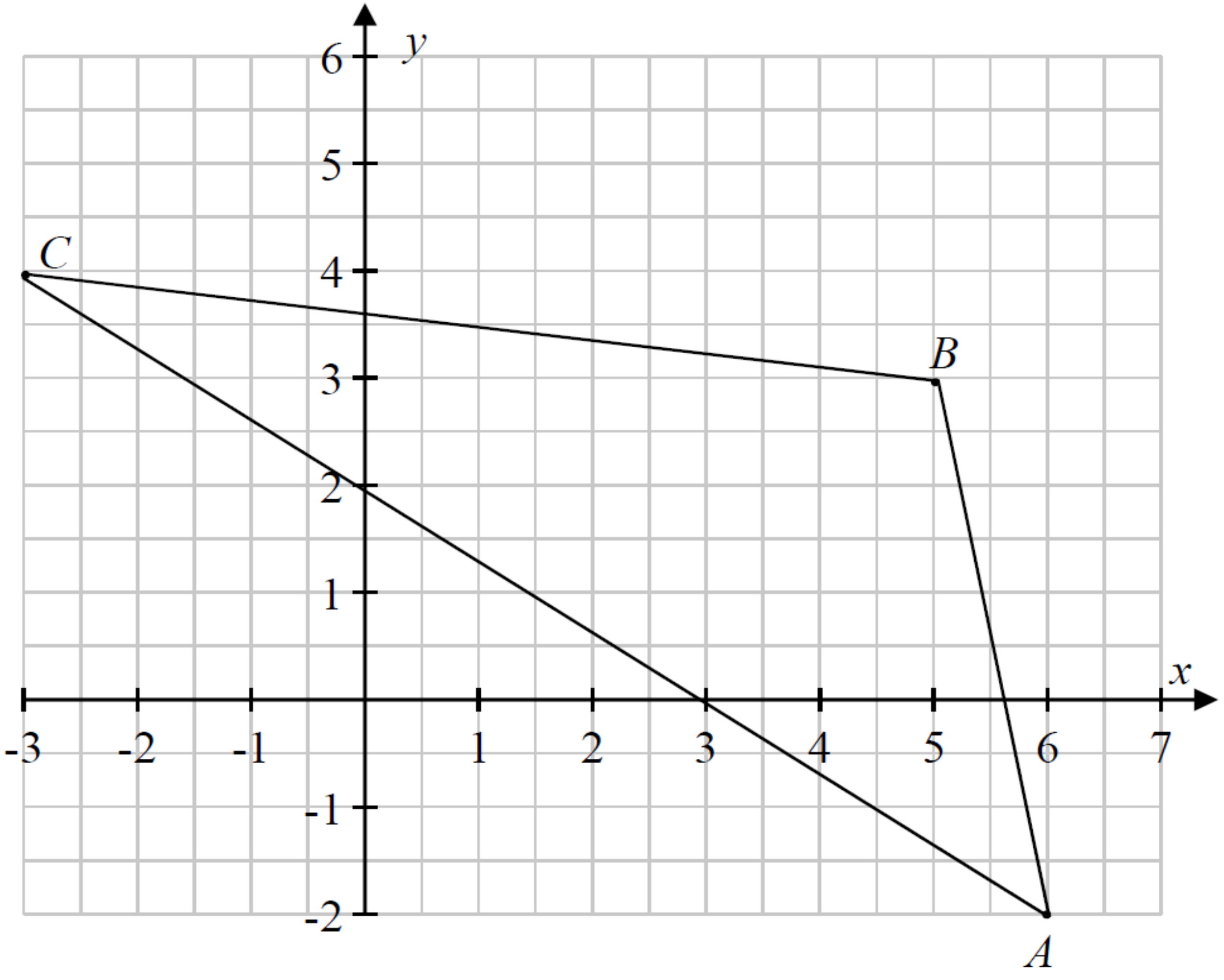
$(y - 2) = -\frac{1}{2}(x - 0)$

2y – 4 = -x

x + 2y – 4 = 0

Question 3

(a) Find the equation of the line through B which is perpendicular to AC.



Hint!

1) Find the slope of AC
2) Turn the fraction upside-down and multiply by -1
3) This gives you the slope of a perpendicular line to AC

(b) Use your answer to part **(a)** above to find the co-ordinates of the orthocentre of the triangle ABC.

Hint! 1) Find the equation of the line through C which is perpendicular to AB
2) Use your answer from (a) and simultaneous equations to get the answer

Question 3

(a) Find the equation of the line through B which is perpendicular to AC.

Slope of |AC| = $\frac{(y2 - y1)}{(x2 - x1)} = \frac{(-2 - 4)}{(6 - (-3))} = \frac{-6}{9} = -\frac{2}{3}$

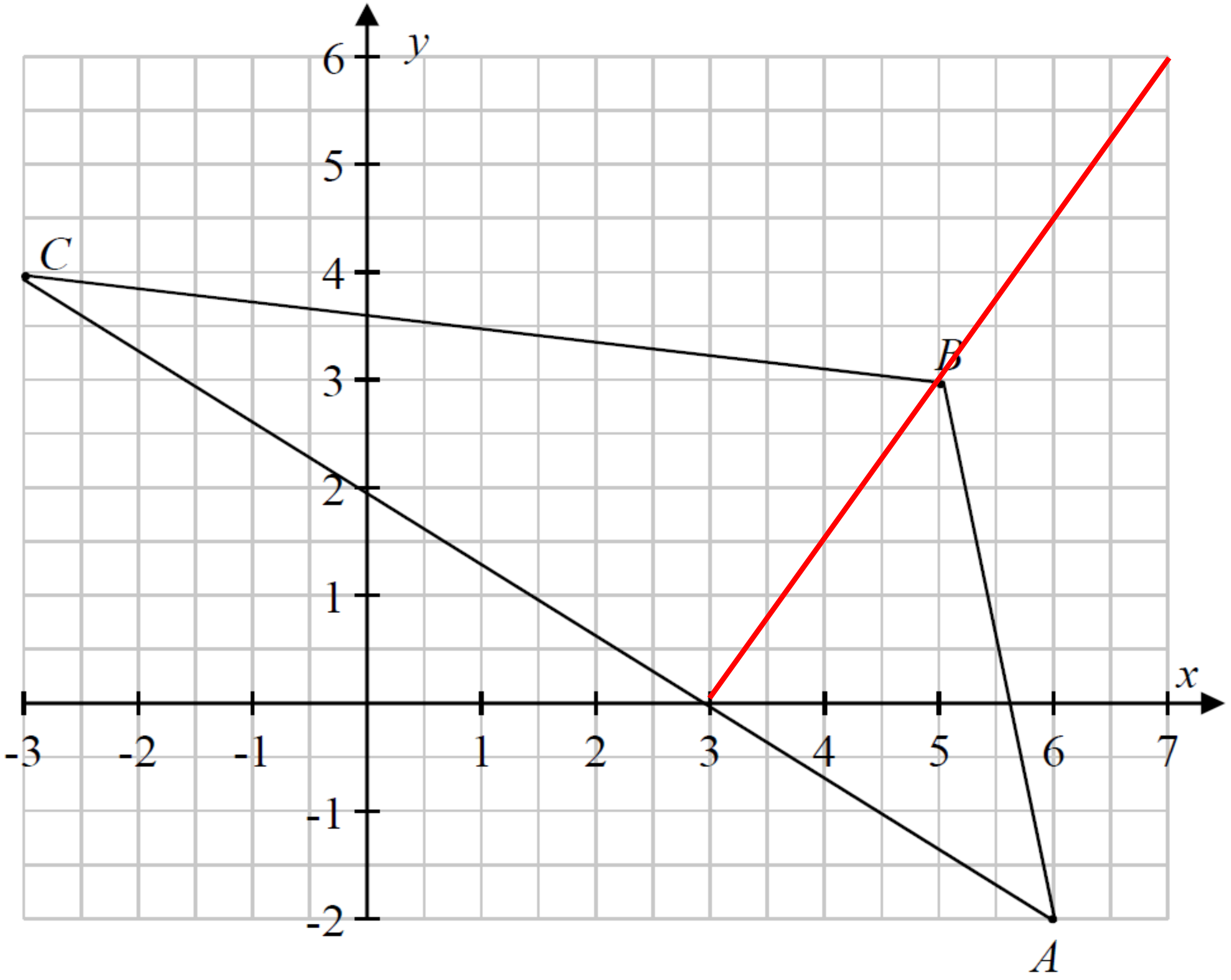
Slope of Perpendicular line through B is $\frac{3}{2}$

Use (y –y1) = m (x – x1) to find equation of line where (x1, y1) is (5, 3) and m is $\frac{3}{2}$

$(y - 3) = \frac{3}{2}(x - 5)$

$2y - 6 = 3x - 15$

$3x - 2y - 9 = 0$



Question 3

(b) Use your answer to part **(a)** above to find the co-ordinates of the orthocentre of the triangle *ABC*.

Slope of |AB| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(-2-3)}{(6-5)} = \frac{-5}{1} = -5$

Slope of Perpendicular line through C is $\frac{1}{5}$

Use (y –y1) = m (x – x1) to find equation of line where (x1, y1) is (-3, 4) and m is $\frac{1}{5}$

$(y-4) = \frac{1}{5}(x+3)$

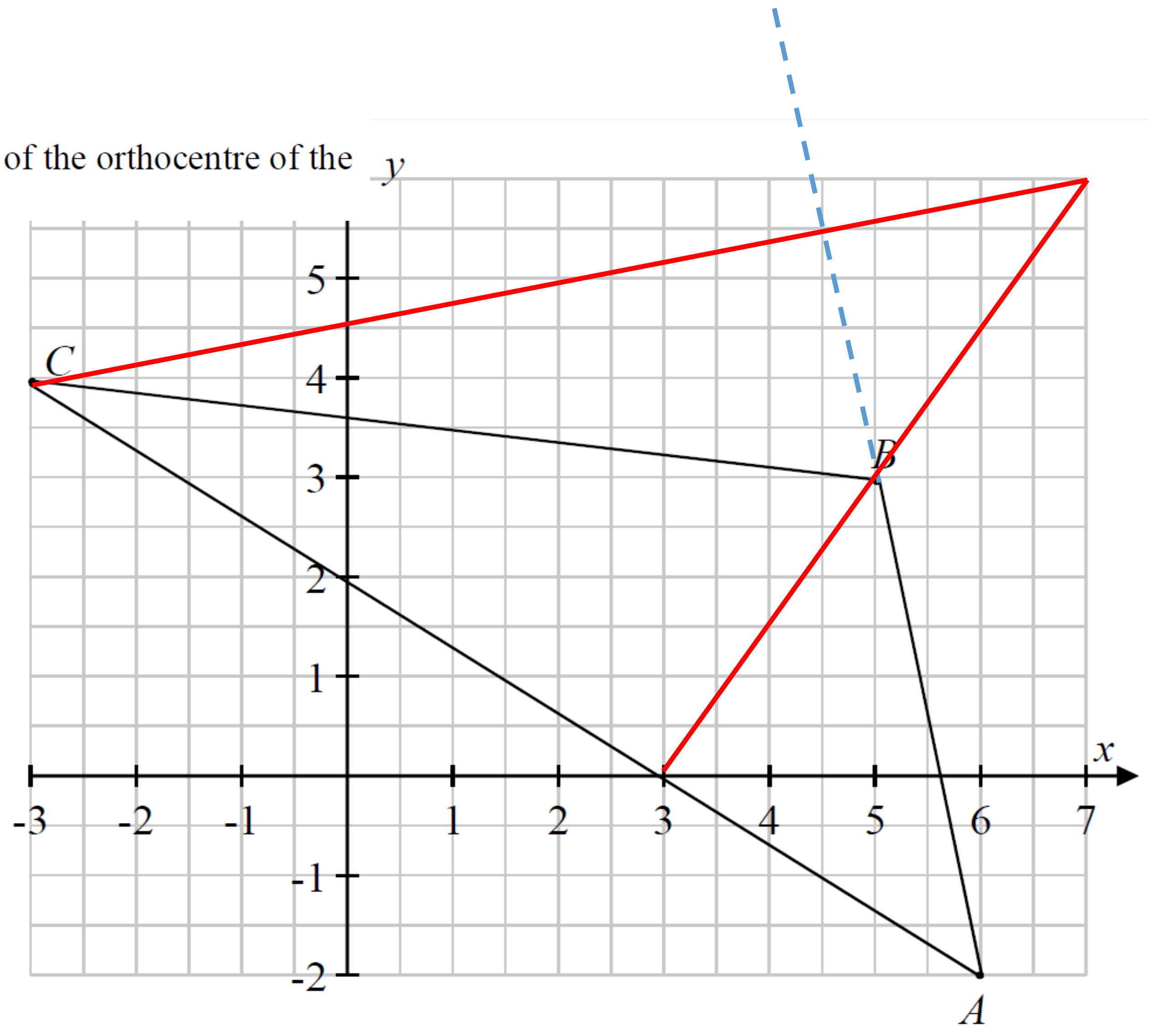
$5y - 20 = x + 3$

$x - 5y + 23 = 0$

Solve the simultaneous equations

$x - 5y + 23 = 0$ and $3x - 2y - 9 = 0$

Point of intersection is (7, 6) = orthocentre



Question 3

(b) Use your answer to part **(a)** above to find the co-ordinates of the orthocentre of the triangle *ABC*.

Slope of |BC| = $\frac{(y2-y1)}{(x2-x1)}$ =$\frac{(4-3)}{(-3-5)} = \frac{1}{-8} = -\frac{1}{8}$

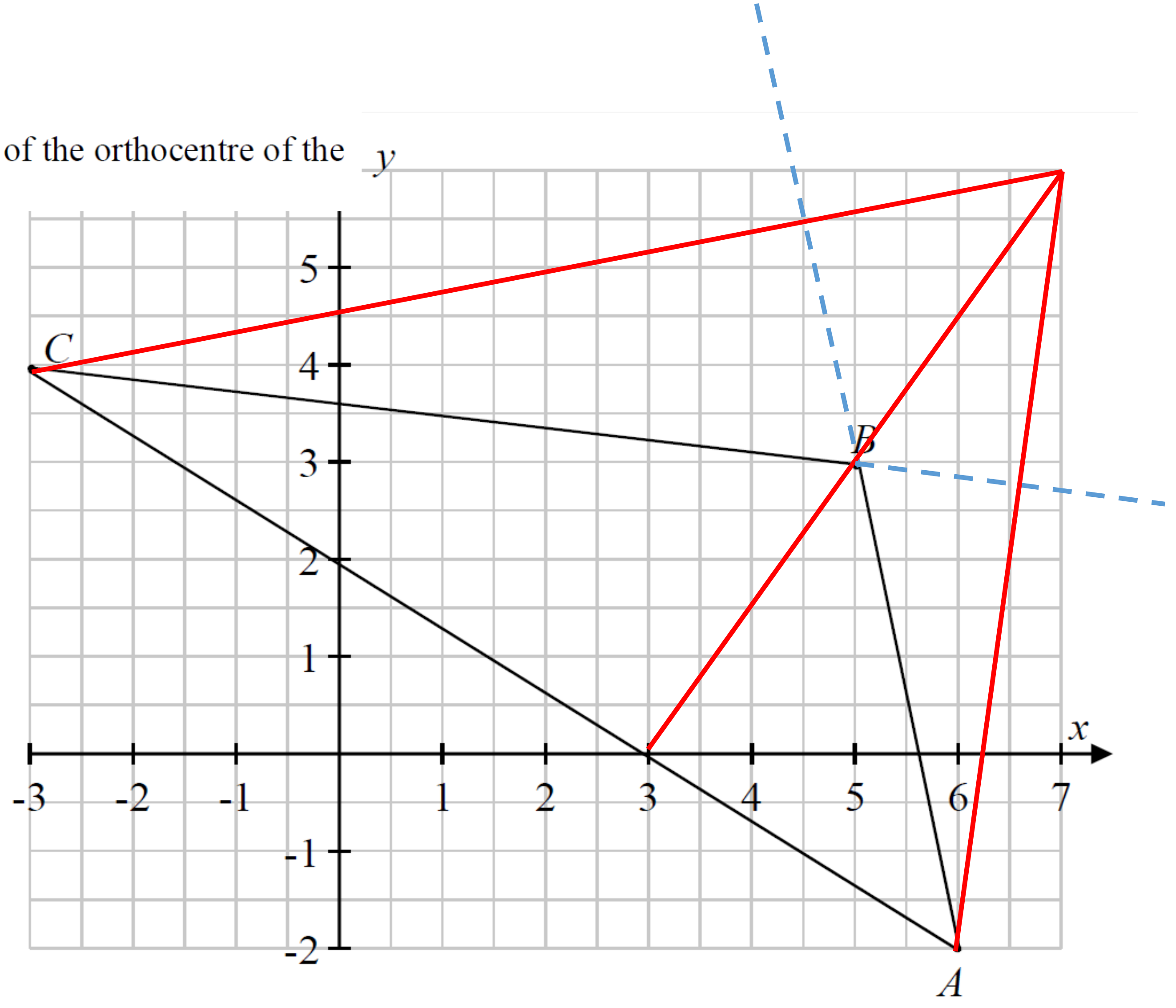
Slope of Perpendicular line through A is 8

Use (y –y1) = m (x – x1) to find equation of line where (x1, y1) is (6, -2) and m is 8

(y – (-2)) = 8 (x – 6)
y + 2 = 8x – 48
8x – y – 50 = 0

Solve the simultaneous equations
8x – y – 50 = 0 and 3x – 2y – 9 = 0

Point of intersection is (7, 6) = orthocentre



Question 4

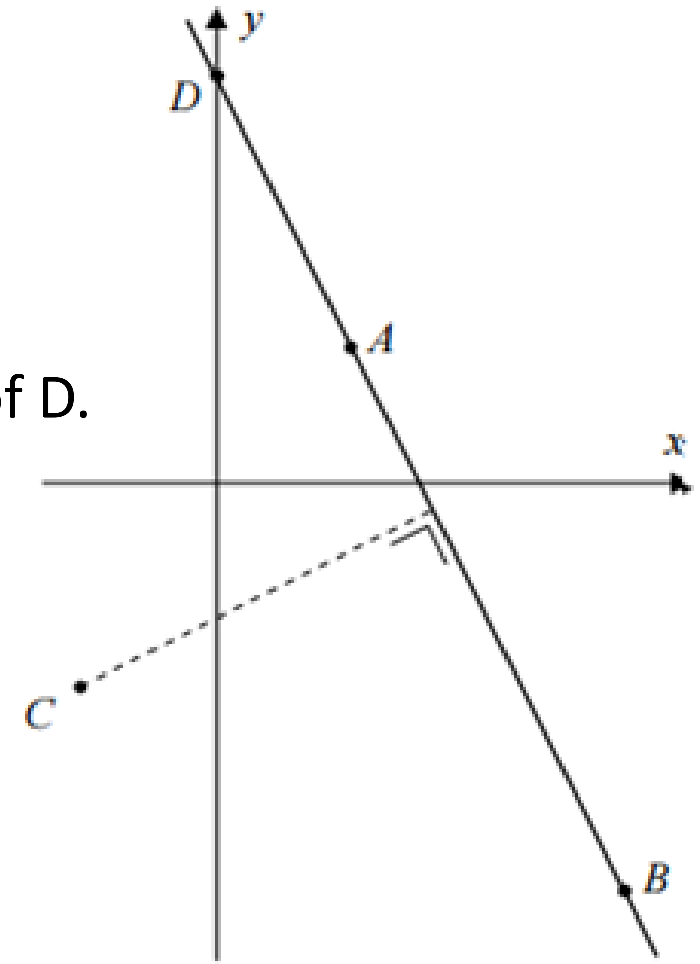
The co-ordinates of three points A, B and C are: A(2,2), B(6,-6), C(-2,-3)

(a) Find the equation of AB.

(b) The line AB intersects the y-axis at D. Find the coordinates of D.

(c) Find the perpendicular distance from C to AB.

(d) Hence, find the area of the triangle ADC.



Question 4

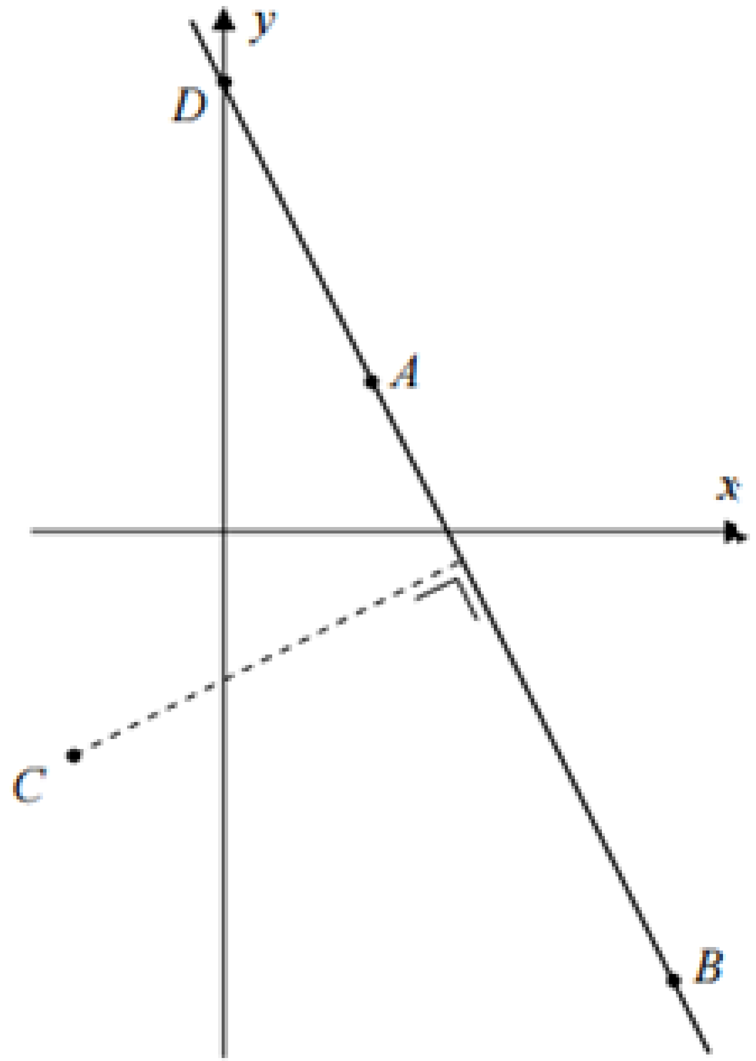
The co-ordinates of three points *A*, *B*, and *C* are: *A*(2, 2), *B*(6, –6), *C*(–2, –3).
(See diagram on facing page.)

(a) Find the equation of *AB*.

Slope of |AB| = $\frac{(y2 - y1)}{(x2 - x1)} = \frac{(-6 - 2)}{(6 - 2)} = \frac{-8}{4} = -2$

Use (y –y1) = m (x – x1) to find equation of line |AB| where (x1, y1) is (2, 2) and m is -2

(y – 2) = -2 (x – 2)
y – 2 = – 2x + 4
2x + y – 6 = 0



(b) The line *AB* intersects the *y*-axis at *D*.
Find the coordinates of *D*.

Rewrite |AB| in the format y = mx + c
2x + y – 6 = 0
y = – 2x + 6
c (the y-axis intercept) = 6 OR y = 6 when x = 0

So D is the point (0, 6)

Question 4

The co-ordinates of three points A, B, and C are: $A(2, 2)$, $B(6, -6)$, $C(-2, -3)$.
(See diagram on facing page.)

(c) Find the perpendicular distance from C to AB.

Use the formula $\frac{|ax1+by1+c|}{\sqrt{a^2+b^2}}$

(x1, y1) is C(-2, -3) and the line (ax +by + c =0) is 2x + y – 6= 0

$$\frac{|2(-2)+1(-3)-6|}{\sqrt{2^2+1^2}} = \frac{13}{\sqrt{5}}$$

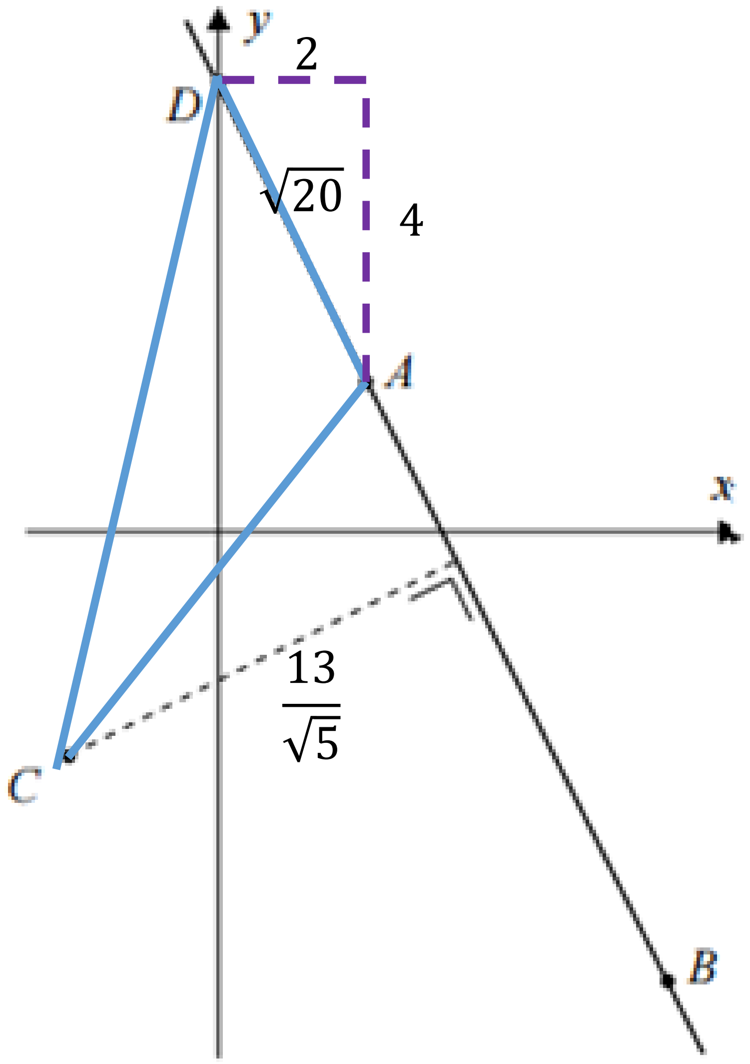
(d) Hence, find the area of the triangle ADC.

Area of Triangle = Half the base by the perpendicular height

Base = |AD| = $\sqrt{(x2 - x1)^2 + (y2 - y1)^2} = \sqrt{20}$

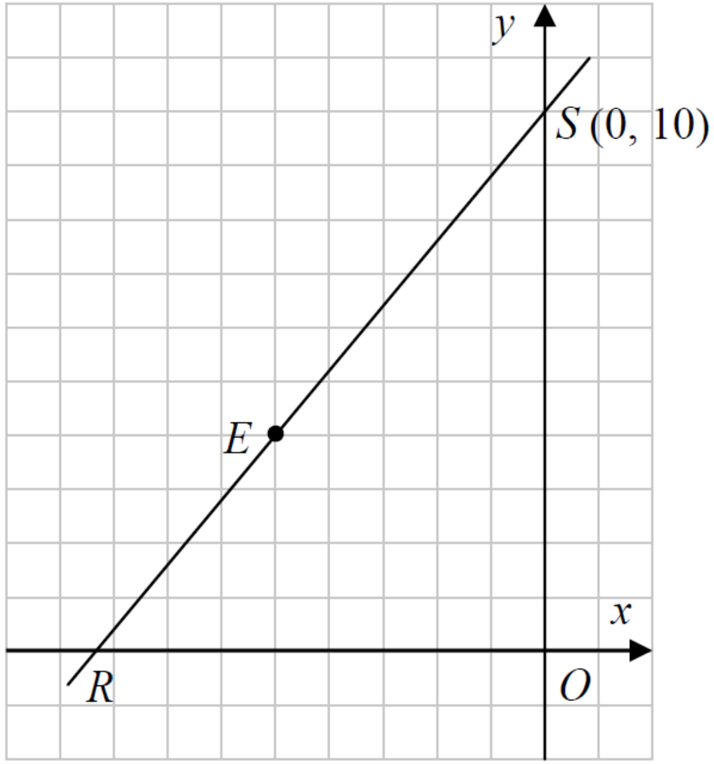
Perpendicular Height = $\frac{13}{\sqrt{5}}$

Area = $\frac{1}{2} * \sqrt{20} * \frac{13}{\sqrt{5}}$ = 13 square units



Question 5

The line *RS* cuts the x-axis at the point *R* and the y-axis at the point $S(0, 10)$, as shown. The area of the triangle *ROS*, where O is the origin, is $\frac{125}{3}$.



(a) Find the co-ordinates of *R*.

Hint!

Area = ½ base x perpendicular height

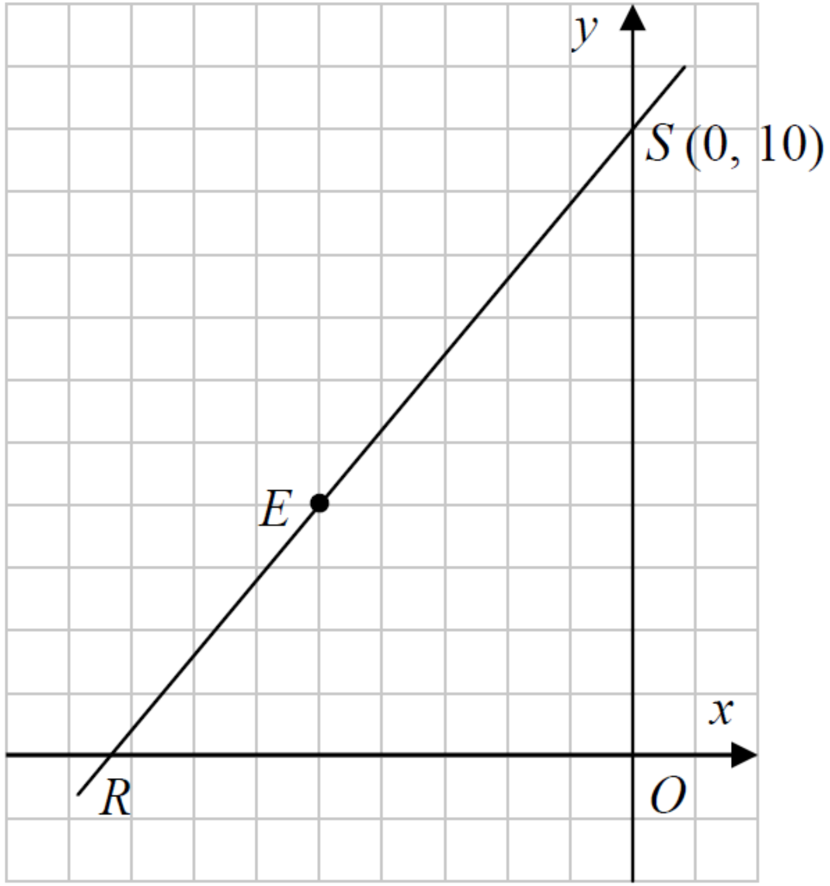
(b) Show that the point $E(-5, 4)$ is on the line *RS*.

(c) A second line $y = mx + c$, where m and c are positive constants, passes through the point *E* and again makes a triangle of area $\frac{125}{3}$ with the axes. Find the value of m and the value of c.

Question 5

The line RS cuts the x-axis at the point R and the y-axis at the point $S(0, 10)$, as shown. The area of the triangle ROS, where O is the origin, is $\frac{125}{3}$.

(a) Find the co-ordinates of R.



Use the formula for area on page 18 of the tables OR
Use the triangle area formula "half the base by the perpendicular height"

$$\frac{1}{2}\,|OR|.\,10 = \frac{125}{3}$$

$$|\mathrm{OR}| = \frac{25}{3}$$

$$\mathrm{R}\left(-\frac{25}{3}, 0\right)$$

Question 5 **(b)** Show that the point $E(-5, 4)$ is on the line RS.

METHOD 1: Get the equation for the line |RS| and show that E is on the line

Slope of |RS| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(10-0)}{(0-(-\frac{25}{3}))} = \frac{10}{(\frac{25}{3})} = \frac{30}{25} = \frac{6}{5}$

Equation of |RS| is (y – y1) = m (x – x1) …..use $\frac{6}{5}$ for m and (0, 10) for (x1, y1)

$(y - 10) = \frac{6}{5}(x - 0)$ ➡ 5y – 50 = 6x ➡ Equation of |RS| is 6x – 5y + 50 = 0

Put E (-5, 4) into the equation of the line

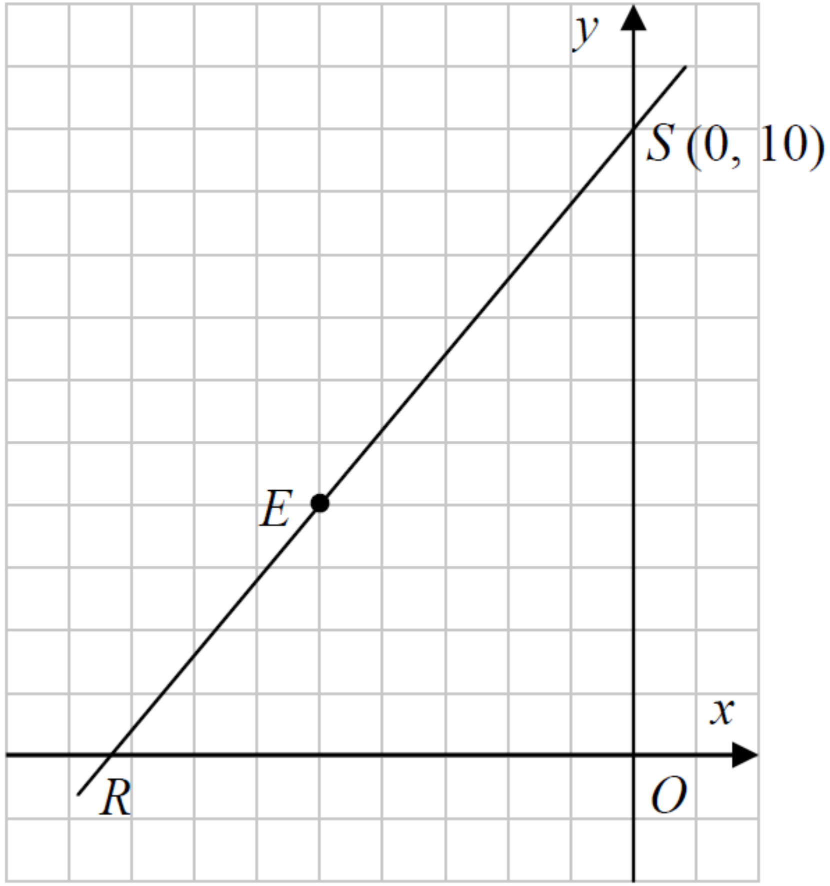
6(-5) -5(4) + 50 = - 30 – 20 + 50 = 0 …. So E is on the line |RS|

METHOD 2: Show that the slope of |RE| = slope of |ES|

Slope of |ES| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(10-4)}{(0-(-5))} = \frac{6}{5}$

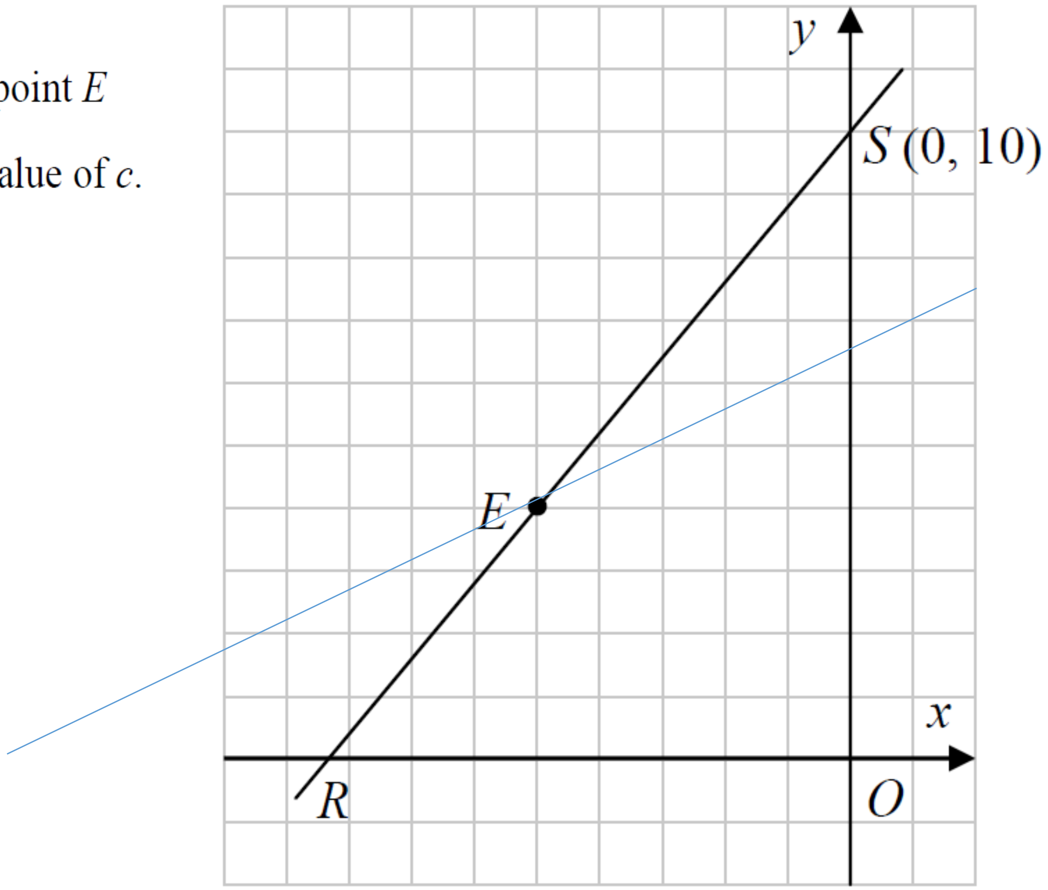
Slope of |RE| = $\frac{(y2-y1)}{(x2-x1)} = \frac{(4-0)}{(-5-(-\frac{25}{3}))} = \frac{4}{\frac{10}{3}} = \frac{12}{10} = \frac{6}{5}$

So E is on the line |RS|



Question 5

(c) A second line $y = mx + c$, where m and c are positive constants, passes through the point E and again makes a triangle of area $\frac{125}{3}$ with the axes. Find the value of m and the value of c.



y = mx + c

The point E(-5, 4) is on this line, so substituting for x and y

4 = -5m + c .... c = 4 + 5m .... A

This line cuts the y-axis at (0, c) and the x-axis at $(-\frac{c}{m}, 0)$

The area of the triangle is $\frac{125}{3}$

This equals $\frac{1}{2}\left|x1y2 - x2y1\right| = \frac{1}{2}\left|0 - c(-\frac{c}{m})\right| = \frac{1}{2}\left|\frac{c^2}{m}\right|$

Substituting from A above $\frac{125}{3} = \frac{1}{2}\left|\frac{(4+5m)^2}{m}\right|$

250 m = 75m^2 + 120 m + 48

75m^2 - 130m + 48 = 0

$(5m - 6)(15m - 8) = 0$ .... $(5m - 6)$ relates to the line |RS|, so

$m = \frac{8}{15}$ and c = $4 + 5\left(\frac{8}{15}\right) = \frac{20}{3}$

Question 6

(a) The co-ordinates of two points are $A(4, -1)$ and $B(7, t)$.

The line $l_1 : 3x - 4y - 12 = 0$ is perpendicular to AB. Find the value of t.

Hint! Find both slopes.

(b) Find, in terms of k, the distance between the point $P(10, k)$ and l_1.

Hint! Use the perpendicular distance formula

(c) $P(10, k)$ is on a bisector of the angles between the lines l_1 and $l_2 : 5x + 12y - 20 = 0$.

(i) Find the possible values of k.

(ii) If $k > 0$, find the distance from P to l_1.

Question 6

(a) The co-ordinates of two points are $A(4, -1)$ and $B(7, t)$.

The line $l_1 : 3x - 4y - 12 = 0$ is perpendicular to AB. Find the value of t.

Find the slope of L1: 3x – 4y -12 = 0

3x – 4y – 12 = 0

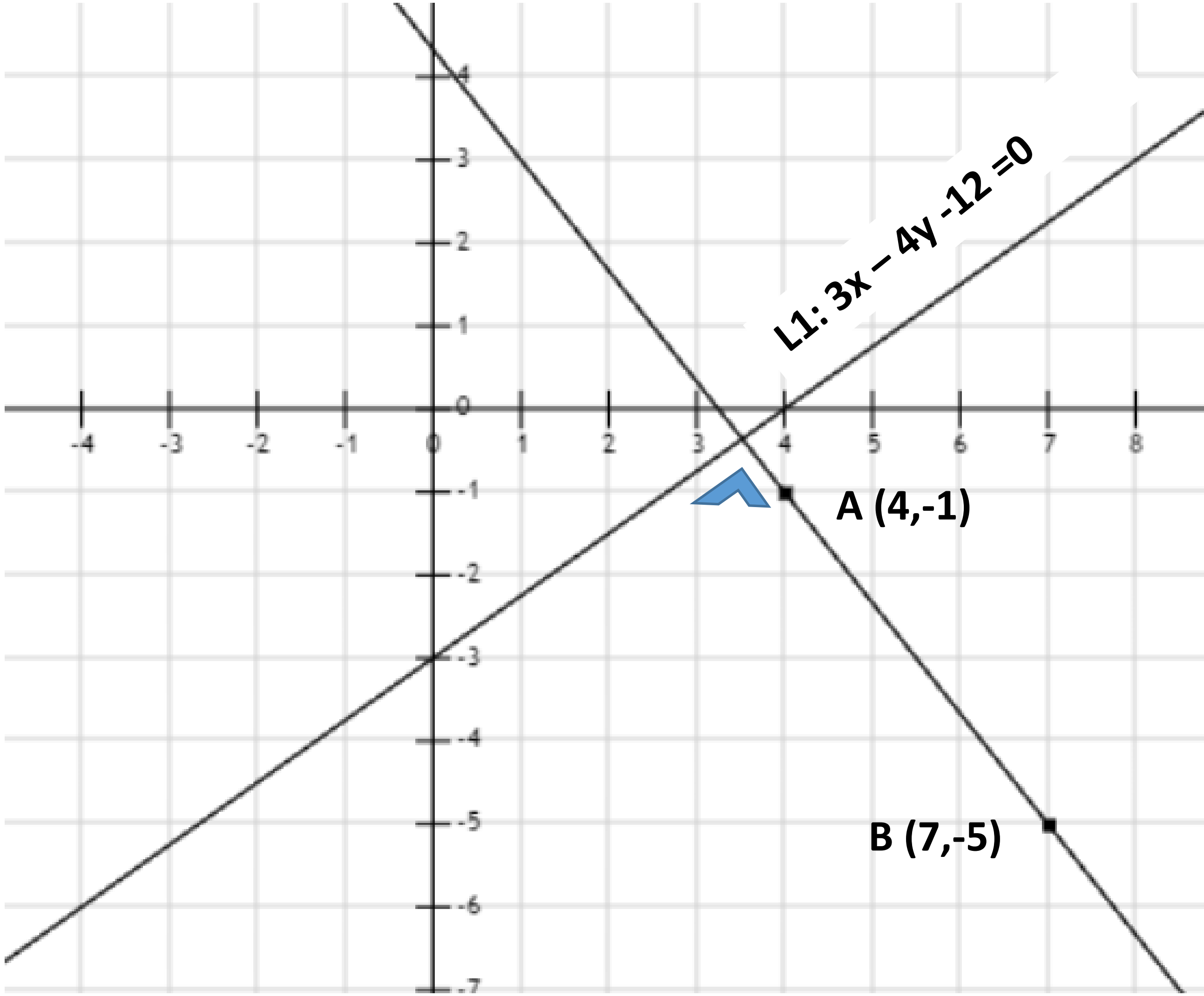
4y = 3x – 12

$y = \frac{3}{4}x - 3$ …. $slope\ of\ L1 = \frac{3}{4}$

$|AB|\ perpendicular\ to\ L1\ so\ slope\ of\ |AB| is\ -\frac{4}{3}$

Slope of |AB| = $\frac{(y2 - y1)}{(x2 - x1)} = \frac{(t - (-1))}{(7 - 4)} = \frac{(t + 1)}{(3)}$

$\frac{(t + 1)}{(3)} = -\frac{4}{3}$ ……. t + 1 = -4 …… t = -5



Question 6

(b) Find, in terms of k, the distance between the point $P(10,\ k)$ and l_1.

Use the formula for the perpendicular distance from point (x1, y1) to line ax +by + c =0

$$\frac{|ax1 + by1 + c|}{\sqrt{a^2 + b^2}}$$

(x1, y1) is (10, k) and the line (ax +by + c =0) is 3x – 4y – 12 = 0

$$\frac{|3(10) - 4(k) - 12|}{\sqrt{3^2 + 4^2}}$$

$$\frac{|18\ - 4k|}{5}$$

Question 6

(c) $P(10,\ k)$ is on a bisector of the angles between the lines l_1 and $l_2: 5x + 12y - 20 = 0$.

(i) Find the possible values of k.

Use the formula for the perpendicular distance to get the distance in terms of k from P to l2

(x1, y1) is (10, k) and the line (ax +by + c =0) is 5x + 12y – 20 = 0

$$\frac{|5(10) + 12(k) - 20|}{\sqrt{5^2 + 12^2}}$$

$$\frac{|30 + 12k|}{13}$$

P is equidistant from l1 and l2

So $\frac{(18 - 4k)}{5} = \frac{(30+12k)}{13}$ OR $\frac{(18 - 4k)}{5} = -\frac{(30+12k)}{13}$

$k = \frac{3}{4}$ OR $k = -48$

(ii) If $k > 0$, find the distance from P to l_1.

k > 0 so k = $\frac{3}{4}$

Use one of the previous results and insert the value for k

Perpendicular distance

$$= \frac{(18 - 4k)}{5} = \frac{(18 - 4(\frac{3}{4}))}{5} = \frac{(18 - 3)}{5} = 3$$

Next Week…

- Monday 27th October
- Same location
- 6-8pm
- Geometry 2
- **Note: Bring Formulae and Tables Booklet**