Society of Actuaries in Ireland 1972 - 2022

Algebra 2 - Solutions

Warm Up Questions:

Question 1.

Let $f(x) = -x^2 + 12x - 27,\ x \in \mathbb{R}$.

(a) **(i)** Complete Table 1 below.

Table 1							
x	3	4	5	6	7	8	9
$f(x)$	0	5	8	9	8	5	0

Question 2. Solve for x: $\frac{x+7}{3} + \frac{2}{x} = 4$

Multiply across:

$\Rightarrow \frac{(x+7)(x)(3)}{3} + \frac{2(x)(3)}{x} = 4(x)(3)$

$\Rightarrow x(x + 7) + 2(3) = 12x$

$\Rightarrow x^2 + 7x + 6 = 12x$

$\Rightarrow x^2 + 7x - 12x + 6 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x - 3)(x - 2)$

$\Rightarrow x = 3, x = 2$

Question 3. Express $\sqrt{48} - \sqrt{12} + \sqrt{27}$ in the form $a\sqrt{b}$

=> $\sqrt{16 x 3} - \sqrt{4 x 3} + \sqrt{9 x 3}$

=> $\sqrt{16}\sqrt{3} - \sqrt{4}\sqrt{3} + \sqrt{9}\sqrt{3}$

=> $4\sqrt{3} - 2\sqrt{3} + 3\sqrt{3}$

=> $5\sqrt{3}$

Question 4. Simplify:

$(b + 1)^3 - (b - 1)^3$

$(b + 1)^3 = (b + 1)(b + 1)^2$

$= (b + 1)(b^2 + 2b + 1)$

$= b(b^2 + 2b + 1) + 1(b^2 + 2b + 1)$

$= b^3 + 2b^2 + b + b^2 + 2b + 1$

$= b^3 + 3b^2 + 3b + 1$ .....(1)

$(b-1)^3 = (b-1)(b-1)^2$

$= (b-1)(b^2 - 2b + 1)$

$= b^3 - 3b^2 + 3b - 1$ .....(2)

$(b+1)^3 - (b-1)^3 = (1) - (2)$

$= b^3 + 3b^2 + 3b + 1 - (b^3 - 3b^2 + 3b - 1)$

$= 6b^2 + 2$

Side note: It is useful to remember $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Which you can use to check:

$(b+1)^3 - (b-1)^3 = b^3 + 3b^2 + 3b + 1 - (b^3 - 3b^2 + 3b - 1)$

$= 6b^2 + 2$

-b Formula

Question 5.

$10x^2 + 6x - 52 = 0$

$a = 10 \qquad b = 6 \qquad c = -52$

$\Rightarrow \frac{-6+/-\sqrt{(6^2-4(10)(-52))}}{2(10)}$

$\Rightarrow \frac{-6+/-\sqrt{(36-(-2080))}}{20}$

$\Rightarrow \frac{-6+/-\sqrt{(2116)}}{20}$

$\Rightarrow \frac{-6+/-46}{20}$

$\Rightarrow \frac{-6+46}{20} = \frac{40}{20} = 2$

$\Rightarrow \frac{-6-46}{20} = \frac{-52}{20} = \frac{-13}{5}$

$X = 2 \text{ or } X = \frac{-13}{5}$

Question 6. 2011 Paper 1 Q1

(c) Solve the equation $x^2 - 2\sqrt{3}x - 9 = 0$, giving your answers in the form $a\sqrt{3}$, where $a \in \mathbb{Q}$.

$$x = \frac{2\sqrt{3} \pm \sqrt{\left(-2\sqrt{3}\right)^2 - 4(1)(-9)}}{2(1)} = \frac{2\sqrt{3} \pm \sqrt{48}}{2}$$

$$= \frac{2\sqrt{3} \pm 4\sqrt{3}}{2}$$

$$= \sqrt{3} \pm 2\sqrt{3}$$

$$x = -\sqrt{3} \text{ or } x = 3\sqrt{3}$$

Question 7. 2015 Paper 1 Q2 (25 marks)

Solve the equation $x^3 - 3x^2 - 9x + 11 = 0$.
Write any irrational solution in the form $a + b\sqrt{c}$, where $a, b, c \in \mathbb{Z}$.

Solution:

$f(x) = x^3 - 3x^2 - 9x + 11$
$f(1) = 1^3 - 3(1)^2 - 9 + 11 = 0$
$\Rightarrow x = 1$ is a solution.
$(x - 1)$ is a factor

$$\begin{array}{r|l} & x^2 - 2x \quad -11 \\ \hline x - 1 & x^3 - 3x^2 - 9x + 11 \\ & x^3 - x^2 \\ \hline & \quad -2x^2 - 9x + 11 \\ & \quad -2x^2 + 2x \\ \hline & \qquad -11x + 11 \\ & \qquad -11x + 11 \\ \hline \end{array}$$

Hence, other factor is $x^2 - 2x - 11$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} = \frac{2 \pm \sqrt{48}}{2} = \frac{2 \pm 4\sqrt{3}}{2} = 1 \pm 2\sqrt{3}$$

Solutions: $\{1,\ 1 + 2\sqrt{3},\ 1 - 2\sqrt{3}\}$

Inequalities

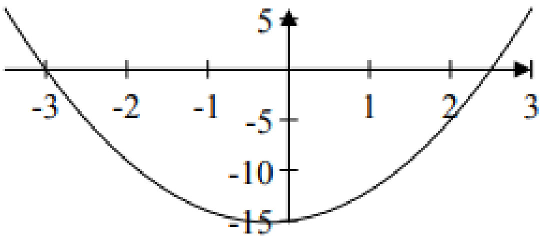
Question 8. 2013 Paper 1 Q2

(a) Find the set of all real values of x for which $2x^2 + x - 15 \geq 0$.

$2x^2 + x - 15 = 0$

$\Rightarrow (2x-5)(x+3) = 0 \Rightarrow x = 2\frac{1}{2}$ or $x = -3$

$2x^2 + x - 15 \geq 0$ for $\{x \mid x \leq -3\} \cup \{x \mid x \geq 2\frac{1}{2}\}$



OR

$f(x) = 2x^2 + x - 15 = (2x-5)(x+3)$

$(2x-5)(x+3) = 0$

$\Rightarrow x = \frac{5}{2}$ or $x = -3$

(i): $x \geq -3$ and $x \geq \frac{5}{2} \Rightarrow x \geq \frac{5}{2}$

(ii): $x \leq -3$ and $x \leq \frac{5}{2} \Rightarrow x \leq -3$

Solution Set: $\{x \mid x \leq -3\} \cup \{x \mid x \geq \frac{5}{2}\}$

Question 9. Solve the following inequality and graph the solution, $x \in R$:

$|3x+4| \leq |x+2|$

$(|3x + 4|)^2 \leq (|x+2|)^2$

⇨ $9x^2 + 24x + 16 \leq x^2 + 4x + 4$

⇨ $8x^2 + 20x + 12 \leq 0$

⇨ $2x^2 + 5x + 3 \leq 0$

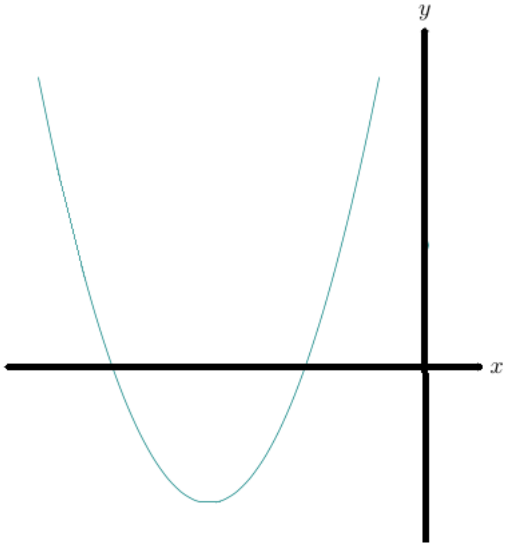
Solve $(2x + 3)(x + 1) = 0$

⇨ $(2x+3) = 0$ OR $(x+1) = 0$

⇨ $X = -3/2$ OR $x = -1$

Consider $2x^2 + 5x + 3$. This is a quadratic with a positive “a” (the x^2 coefficient).

So it is a quadratic curve with a U shape.



So $2x^2 + 5x + 3$ is less than 0 when:

$$\Rightarrow \frac{-3}{2} \le x \le -1$$

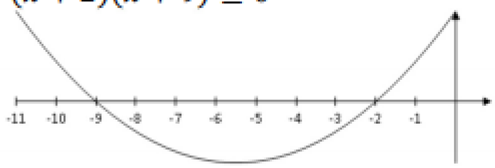
Question 10. <u>2018 Paper 1 Q1</u>

(b) Solve the inequality $\frac{2x-3}{x+2} \ge 3$, where $x \in \mathbb{R}$ and $x \neq -2$.

$$\frac{2x-3}{x+2} \geq 3 \qquad \times\ (x+2)^2$$

<- We multiply across by $(x + 2)^2$ as it is always non-negative

$$(2x-3)(x+2) \geq 3(x+2)^2$$
$$2x^2 + x - 6 \geq 3x^2 + 12x + 12$$
$$x^2 + 11x + 18 \leq 0$$
$$(x+2)(x+9) \leq 0$$



$$-9 \leq x < -2$$

Question 11. <u>2012 Paper 1 Q1</u>

(b) Find the set of all real values of x for which $\frac{2x-5}{x-3} \leq \frac{5}{2}$.

Multiply across by $2(x-3)^2$, which is non-negative:

$$2(x-3)(2x-5) \leq 5(x-3)^2$$
$$4x^2 - 22x + 30 \leq 5x^2 - 30x + 45$$
$$0 \leq x^2 - 8x + 15$$
$$0 \leq (x-5)(x-3)$$

$x \geq 5$ or $x < 3$.

or

$$\frac{2x-5}{x-3} - \frac{5}{2} \leq 0$$
$$\frac{2(2x-5) - 5(x-3)}{2(x-3)} \leq 0$$
$$\frac{-x+5}{2(x-3)} \leq 0$$

	$x<3$	$3<x<5$	$x>5$
$-x+5$	+	+	–
$x-3$	–	+	+
$\frac{-x+5}{2(x-3)}$	–	+	–

$x \geq 5$ or $x < 3$.

Simultaneous Equations

Question 12.

Solve the simultaneous equations:

$$a^2 - ab + b^2 = 3$$
$$a + 2b + 1 = 0$$

$$a = -2b - 1$$

$$(-2b-1)^2 + (2b+1)b + b^2 = 3$$
$$7b^2 + 5b - 2 = 0$$
$$(7b-2)(b+1) = 0$$
$$b = \tfrac{2}{7} \quad \text{or } b = -1$$
$$a = \tfrac{-11}{7} \quad \text{or } a = 1$$

Solution: $\{b = \tfrac{2}{7} \text{ and } a = \tfrac{-11}{7}\}$ or $\{b = -1 \text{ and } a = 1\}$.

Question 13.

(a) Solve the simultaneous equations.

$$2x + 3y - z = -4$$
$$3x + 2y + 2z = 14$$
$$x - 3z = -13$$

Solution:

$$(i) \quad 2x + 3y - z = -4 \qquad \times (2)$$
$$(ii) \quad 3x + 2y + 2z = 14 \qquad \times (-3)$$

$$4x + 6y - 2z = -8$$
$$-9x - 6y - 6z = -42$$

$$-5x - 8z = -50$$
$$(iii) \quad x - 3z = -13 \qquad \times (5)$$
$$-5x - 8z = -50$$
$$5x - 15z = -65$$
$$-23z = -115$$
$$z = 5$$
$$\Rightarrow x = 2$$
$$\Rightarrow y = -1 \qquad \{2, -1, 5\}$$

Logs

Question 14. Solve $log_x 8 = 3$

$$x^3 = 8$$

$x = 2$

Question 15. Solve $32^{x-1} = 28$ for x and give your answer to 2 decimal places

Solution is to take the natural log of both sides

$\ln(32^{x-1}) = \ln(28)$

$(x-1)\ln 32 = \ln 28$ ..... Using $\log(a^b) = b * \log(a)$

$x-1 = \ln 28/\ln 32$

$x = \ln 28/\ln 32 + 1$

$x = 1.96$

Question 16. 2016 P1 Q4 (10 marks):

Given $\log_a 2 = p$ and $\log_a 3 = q$, where $a > 0$, write each of the following in terms of p and q:

(i) $\log_a \frac{8}{3}$

$p = \log_a 2$, $q = \log_a 3$

$$\log_a \frac{8}{3} = \log_a 8 - \log_a 3$$
$$= \log_a (2)^3 - \log_a 3$$
$$= 3\log_a 2 - \log_a 3$$
$$= 3p - q$$

(ii) $\log_a \frac{9a^2}{16}$.

$$\log_a \frac{9a^2}{16} = \log_a (3a)^2 - \log_a (2)^4$$
$$= 2\log_a 3 + 2\log_a a - 4\log_a 2$$
$$= 2q + 2(1) - 4p$$
$$= 2q + 2 - 4p$$

Question 17. 2014 P1 Q2

Given that $p = \log_c x$, express $\log_c \sqrt{x} + \log_c (cx)$ in terms of p.

We know that

$$\log_c \sqrt{x} = \log_c x^{\frac{1}{2}} = \frac{1}{2}\log_c x = \frac{1}{2}p$$

using the power law for logarithms.
Also,

$$\log_c(cx) = \log_c c + \log_c x = \log_c c + p$$

using the product rule for logarithms.
But $\log_c c = 1$ since $c^1 = c$. Therefore

$$\log_c \sqrt{x} + \log_c(cx) = \frac{1}{2}p + 1 + p = \frac{3p}{2} + 1.$$

Additional Questions

Question 18.

(i) Let x = Stage number.

There are 4 times as many blue tiles than the Stage number

Blue tiles = 4x

There are 4 white tiles in every stage. This is a constant and remains 4 no matter what stage number we use.

The total number of green tiles is the square of the stage number.

Number of Green tiles = x^2

The total number of tiles (T) must be the green tiles + blue tiles + white tiles

$T = x^2 + 4x + 4$

(ii) $x^2 + 4x + 4 = 324$

Factorise $(x+2)(x+2)=324$

$(x+2)^2=324$

$x+2=18$

$x=16$

There are x^2 green tiles therefore $16^2 = 256$ green tiles.

(iii) Mary's kitchen is square. Therefore the length of each side = $\sqrt{6.76} = 2.6$m = 260 cm.
Each tile has sides of 20 cm each and 13 x 20 = 260. Therefore there are 13 tiles on each side or in each row.
In the first row there are two white tiles and the rest (13-2=11) are blue. Therefore this must be stage 11.

Green = x^2=121

Blue =4x =44

White = 4

Check: The total number of tiles = 121+44+4=169.

The area of each tile = 0.20 x 0.20 = 0.04 m^2. The total number of tiles needed = $6.76 \div 0.04 = 169$.

Question 19.

1)

200 m Race:

$$y = a(b - x)^c$$
$$y = 4{\cdot}99087(42{\cdot}5 - 23{\cdot}8)^{1{\cdot}81}$$
$$y = 1000$$

Javelin:

$$y = a(x - b)^c$$
$$y = 15{\cdot}9803(58{\cdot}2 - 3{\cdot}8)^{1{\cdot}04}$$
$$y = 1020$$

2)

$$y = a(x - b)^c$$
$$1295 = 15{\cdot}9803(x - 3{\cdot}8)^{1{\cdot}04}$$
$$81{\cdot}0373 = (x - 3{\cdot}8)^{1{\cdot}04} = z^{1{\cdot}04}$$
$$\log z = \frac{\log 81{\cdot}0373}{1{\cdot}04}$$
$$z = 68{\cdot}4343 = (x - 3{\cdot}8)$$
$$x = 72{\cdot}2343 = 72{\cdot}23 \text{ m}$$

3)

$$y = a(b - x)^c$$
$$1087 = 0{\cdot}11193(254 - 121{\cdot}84)^c$$
$$\frac{1087}{0{\cdot}11193} = (132{\cdot}16)^c$$
$$\log 9711{\cdot}426 = c \log 132{\cdot}16$$
$$c = \frac{\log 9711{\cdot}426}{\log 132{\cdot}16} = 1{\cdot}88$$