Calculus- Hints and Tips

Calculus is the study of change. Differentiation looks at rates of change and slopes of tangents to curves.

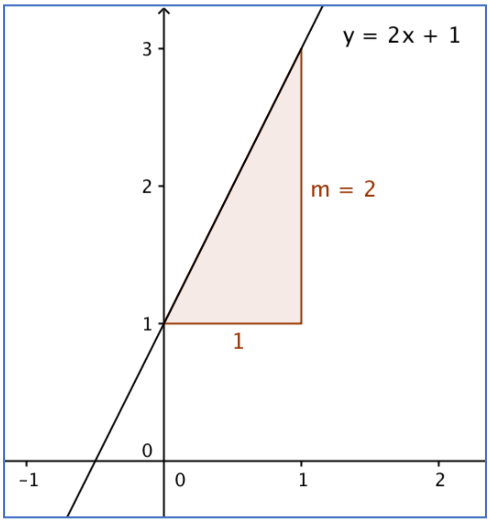
$\frac{dy}{dx}$ means how much does y change as x changes. If f(x) is a function of x, then f'(x) means how much f(x) changes as x changes.

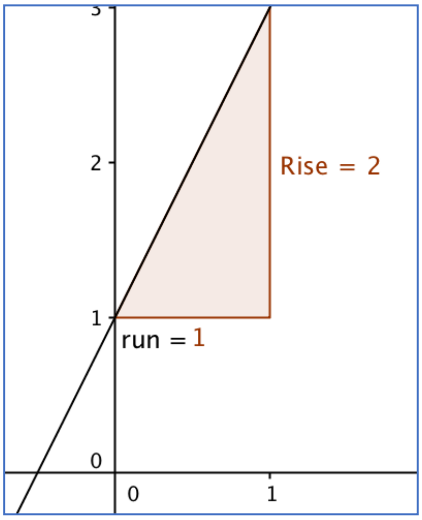
Differentiation of Linear Functions

The rate of change of a linear function is the slope of that line.

We know that the equation of a line can be written as y = mx + c, where m is the slope.

For example, if y = 2x + 1, then the slope is 2. I.e. every time x increases by 1, y increases by 2.





So the rate of change of y with respect to x is 2.

$$\text{I.e.} \quad \frac{dy}{dx} = 2$$

We see that the rate of change of a linear function is constant. No matter what part of the line you look at, y will always increase by 2 for every increase in x of 1. Now let's look at the rate of change of a non-linear function.

Differentiation of Non-Linear functions

Let's consider a quadratic function:

$$y = ax^2 + bx + c$$

The rate of change of y relative to x is always changing. In other words, the slope of y is constantly changing. See the illustration below:

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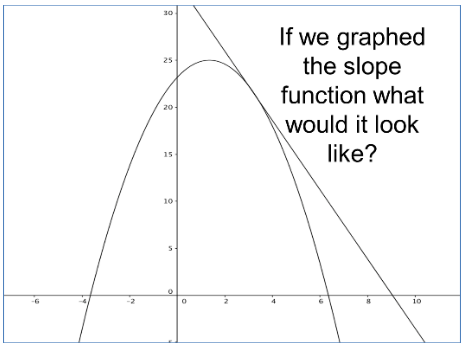
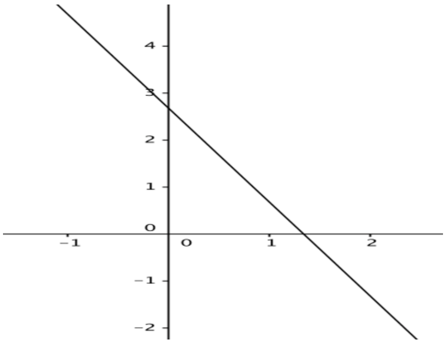


Answer:

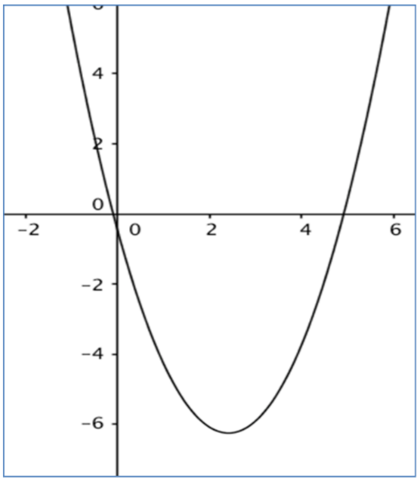


The second graph above shows $\frac{dy}{dx}$ and it tells you what the slope of the tangent to y is at any given x.

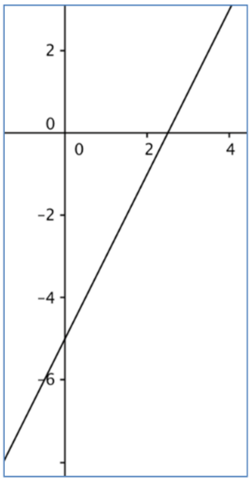
If you follow the curve above from left to right, you can see that y is increasing initially (i.e. the rate of change of y relative to x is positive) and then eventually y begins to decrease (i.e. the rate of change of y relative to x is negative). You can see this in the graph of $\frac{dy}{dx}$ because it is positive initially and then become negative somewhere between x = 1 and x = 2.

Below are some more illustrations to help you visualise the differentiation of different functions.

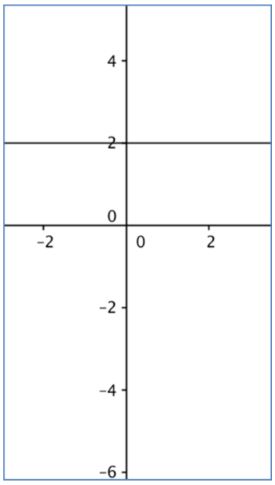
If f(x) is quadratic, i.e. f(x) = ax^2 + bx + c, then the first and second derivatives of f(x) look like this:



$f(x)$



$f'(x)$



$f''(x)$

If f(x) is cubic, i.e. f(x) = ax^3 + bx^2 + cx + d, then the first and second derivatives of f(x) look like this:

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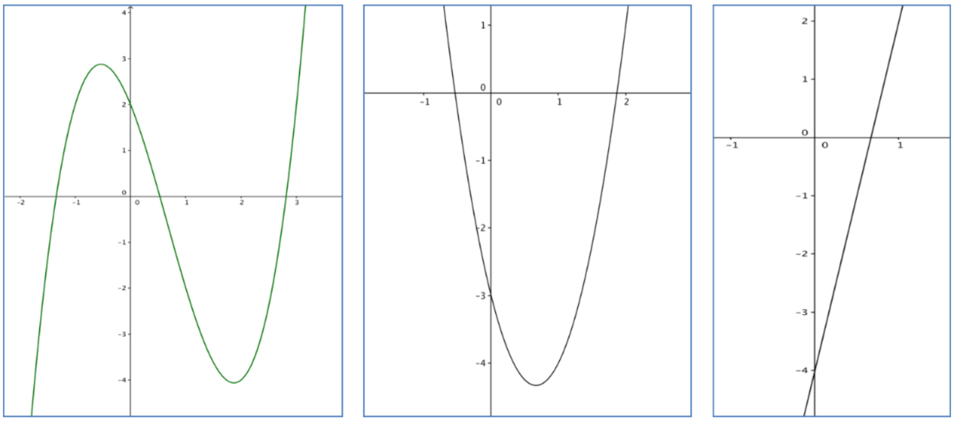

$g(x)$ $g'(x)$ $g''(x)$

Differentiation Rules – Page 25 of the tables

Note: Get used to looking for formulae in your tables. They contain lots of useful information that will help you get marks and knowing where to look will save you time in the exam!!!

Function	**Derivative**
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln(x)$	$\frac{1}{x}$
e^x	e^x
e^{ax}	$\mathrm{a}e^{ax}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$

Product Rule

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

When you have a function **multiplied by** a function

Quotient Rule (Fraction Rule)

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

When you have a function **divided by** a function

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Chain Rule

$$\boldsymbol{f(x) = u(v(x))}$$

$$\boldsymbol{f'(x) = \frac{du}{dv} * \frac{dv}{dx}}$$

When you have a function **within** a function

Differentiation by first Principles

Given $f(x)$, the derivative is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

You are looking for the slope of a function at a particular point, so you first look at the slope between two points (the point where x = x and the point where x = x + h). Then you find out what that slope is as h approaches 0.

Maximums and Minimums

The value of x for which f ’(x) = 0, identifies either a maximum or a minimum point on a curve.

(i.e. the slope of the tangent to the curve at that point is 0)

In order to determine if it’s a maximum or a minimum, we look at the rate of change of the slope of the curve.

$f''(x) > 0 \implies$ **Minimum turning point**

$f''(x) < 0 \implies$ **Maximum turning point**

Continuity

A function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Injective Functions

f:X→Yf:X→Y is injective if and only if:

- For x ≠ y ⇒ f(x) ≠ f(y) , or
- If f(x) = f(y) ⇒ x = y

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Another way of thinking about it:

A function $f: X \to Y$ is **not** injective if two **distinct** elements $a, b \in X$ exist with $f(a) = f(b)$