Society of Actuaries in Ireland

Calculus 1 Solutions – 27 Jan 2021

1)

Answers:

(i)$4x^3$ (ii)$21x^2$ (iii)$-x^{-2}$ (iv)1

2)

(i) $f'(x) = 10x + 7$

(ii) $g'(x) = 6x^2 + 8x + 1$

(iii) $\frac{dy}{dx} = 8$

3)

$\boldsymbol{y = 8x + 4}$

this is the equation of a line

$\boldsymbol{\frac{dy}{dx} = 8}$ this is the slope of that line

Prove it: $y = mx + c$

m = 8

4)

Product Rule - Example

$y = (4 + 3x^2)(6x + 4x^2)$

Find $\frac{dy}{dx}$

(hint let **u** = $(\mathbf{4 + 3x^2})$

and **v** = $(\mathbf{6x + 4x^2})$

$\frac{du}{dx} = 6x \quad \frac{dv}{dx} = 6 + 8x$

$\frac{dy}{dx} = (4 + 3x^2)(6 + 8x) + (6x + 4x^2)(6x)$

$\frac{dy}{dx} = 24 + 32x + 18x^2 + 24x^3 + 36x^2 + 24x^3$

$= 48x^3 + 54x^2 + 32x + 24$

5) Quotient rule: $y = \frac{(5+6x)}{2x^2}$ $Find\ \frac{dy}{dx}$

Quotient Rule - Example

$y = \frac{(5 + 6x)}{2x^2}$

Find $\frac{dy}{dx}$

(hint let **u** = $(\mathbf{5 + 6x})$

and **v** = $\mathbf{2x^2}$

$\frac{du}{dx} = 6 \quad \frac{dv}{dx} = 4x$

$$\frac{dy}{dx} = \frac{(2x^2)(6) - (5 + 6x)(4x)}{(2x^2)^2}$$

$$\frac{dy}{dx} = \frac{12x^2 - 20x - 24x^2}{4x^4}$$
$$\frac{dy}{dx} = \frac{-12x^2 - 20x}{4x^4}$$
(divide above and below by $4x$)
$$\frac{dy}{dx} = \frac{-3x - 5}{x^3}$$

6) Chain rule: $y = (2 + 3x)^{\ 4}\ Find\ \frac{dy}{dx}$

Chain Rule - Example

$y = (2 + 3x)^{\ 4}$

Find $\frac{dy}{dx}$

(hint let **v** = $(\mathbf{2} + \mathbf{3x})$

and **u** = $(v)^4$

$\frac{du}{dv} = 4v^3 \quad \frac{dv}{dx} = 3$

$$f'(x) = (4v^3)(3)$$
$$= (4(2 + 3x)^3)(3)$$
$$= (4(2 + 3x)^3)(3)$$
$$= 12(2 + 3x)^3$$

7) Find the following limits:

$$a)\ \lim_{x\to 1}\left(\frac{x^2+x-2}{x-1}\right) = \lim_{x\to 1}\left(\frac{(x-1)(x+2)}{x-1}\right) = \lim_{x\to 1}(x+2) = 1+2=3$$

$$b)\ \lim_{n\to\infty}\left(\frac{2n^2-3n+2}{6n^2+5n-6}\right) = \lim_{n\to\infty}\left(\frac{2-\frac{3}{n}+\frac{2}{n^2}}{6+\frac{5}{n}-\frac{6}{n^2}}\right) = \frac{2}{6} = \frac{1}{3}$$

8) Find the derivative of $f(x) = 5 - 2x$ by first principles.

First find $f(x + h)$

$$f(x + h) = 5 - 2(x + h)$$
$$= 5 - 2x - 2h$$

$$\frac{f(x+h) - f(x)}{h} = \frac{5 - 2x - 2h - 5 + 2x}{h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{5-2x-2h-5+2x}{h}$$
$$= -2$$

9) Fill in the table:

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$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$

10)

(a) Differentiate the function $2x^2-3x-6$ with respect to x from first principles.

$$f(x)=2x^2-3x-6$$
$$f(x+h)=2(x+h)^2-3(x+h)-6=2x^2+4xh+2h^2-3x-3h-6$$
$$f(x+h)-f(x)=4xh+2h^2-3h$$
$$\underset{h\to 0}{Limit}\left(\frac{f(x+h)-f(x)}{h}\right)=\underset{h\to 0}{Limit}\left(\frac{4xh+2h^2-3h}{h}\right)=4x-3$$

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(b) Let $f(x)=\dfrac{2x}{x+2}$, $x\neq -2$, $x\in\mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y=f(x)$ is $\dfrac{1}{4}$.

$$f(x)=\frac{2x}{x+2}$$

Let $u(x)=2x\Rightarrow u'(x)=2$ and $v(x)=x+2\Rightarrow v'(x)=1$

$$f'(x)=\frac{(x+2)(2)-2x(1)}{(x+2)^2}=\frac{4}{(x+2)^2}$$

$$f'(x)=\frac{1}{4}\Rightarrow\frac{4}{(x+2)^2}=\frac{1}{4}$$

$$\Rightarrow 16=(x+2)^2$$

$\Rightarrow x+2=4$ or $x+2=-4$

$\Rightarrow x=2$ or $x=-6$

or

$x^2+4x-12=0$

$(x-2)(x+6)=0$

$\Rightarrow x-2=0$ or $x+6=-0$

$\Rightarrow x=2$ or $x=-6$

$$f(-6)=\frac{-12}{-6+2}=3 \text{ and } f(2)=\frac{4}{2+2}=1$$

Points $(-6,3)$ and $(2,1)$

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11)

Q6	Model Solution – 25 Marks	Marking Notes
(a)	$f(x+h)-f(x)=(2x+2h+4)^2-(2x+4)^2$ $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=$ $\lim_{h\to 0}\frac{(2x+2h+4)^2-(2x+4)^2}{h}$ $=\lim_{h\to 0}\left(\frac{\left[\begin{matrix}(4x^2+8hx+4h^2+16x+16h+16)\\-(4x^2+16x+16)\end{matrix}\right]}{h}\right)$ $=\lim_{h\to 0}\frac{8hx+4h^2+16h}{h}$ $=8x+16$ **or** $f(x)=(2x+4)^2=4x^2+16x+16$ $f(x+h)=4(x+h)^2+16(x+h)+16$ $=4x^2+8hx+4h^2+16x+16h+16$ $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $\lim_{h\to 0}\frac{8hx+4h^2+16h}{h}$ $=8x+16$	Scale 10D (0, 2, 5, 8, 10) *Low Partial Credit* • any $f(x+h)$ *Mid Partial Credit* • limit of $\frac{f(x+h)-f(x)}{h}$ *High Partial Credit* • limit of $\frac{(2x+2h+4)^2-(2x+4)^2}{h}$ **Notes:** - *omission of limit sign penalised once only* - *answer not from 1st Principles merits 0 marks*

(b) (i)+ (ii)	$y=x.\sin\frac{1}{x}$ $\frac{dy}{dx}=\sin\frac{1}{x}+x\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$ $\frac{dy}{dx}=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ $\frac{dy}{dx}=\sin\frac{\pi}{4}-\frac{\pi}{4}\cos\frac{\pi}{4}$ $=0{\cdot}15$	Scale 15D (0, 4, 7, 11, 15) *Low Partial Credit* • any correct differentiation *Mid Partial Credit* • product rule applied *High Partial Credit* • correct differentiation **Note:** *one penalty for calculator in wrong mode*

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12)

Q3	Model Solution – 25 Marks	Marking Notes
(a)	$f(x+h) = \frac{1}{3}(x+h)^2 - (x+h) + 3$ $f(x) = \frac{1}{3}x^2 - x + 3$ $f(x+h) - f(x) = \frac{2xh}{3} + \frac{h^2}{3} - h$ $\frac{f(x+h) - f(x)}{h} = \frac{2x}{3} + \frac{h}{3} - 1$ $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{2x}{3} - 1$	**Scale 20D (0, 5, 14, 17, 20)** *Low Partial Credit* • any $f(x+h)$ *Mid Partial Credit* • $f(x+h) - f(x)$ *with some correct work* *High Partial Credit* • $\frac{\frac{1}{3}(x+h)^2 - (x+h) + 3 - \left(\frac{x^2}{3} - x + 3\right)}{h}$ simplified **Notes**: • omission of limit sign penalised once only • answer not from 1st Principles merits 0 marks
(b)	$\frac{d(fg(x))}{dx} =$ $\frac{1}{(3(x+5)^2 + 2)}(6(x+5))$ $\frac{d(fg\left(\frac{1}{4}\right))}{dx} = \frac{6(\frac{21}{4})}{3(\frac{21}{4})^2 + 2} = \frac{504}{1355}$ $= 0{\cdot}372$ **OR** $f(x) = \ln(3x^2 + 2)$ $g(x) = (x+5)$ $f[g(x)] = \ln[3(x+5)^2 + 2]$ $= \ln(3x^2 + 30x + 77)$ $f'(x) = \frac{6x + 30}{3x^2 + 30x + 77}$ $x = \frac{1}{4}$: $f'(x) = \frac{31{\cdot}5}{84{\cdot}6875} = 0{\cdot}3719$ $= 0{\cdot}372$	**Scale 5C (0, 3, 4, 5)** *Low Partial Credit:* • Any correct differentiation • $fg(x)$ formulated *High Partial Credit:* • $\frac{d(fg(x))}{dx}$ found **Note**: Work with $f(x) \times g(x)$ merits low partial credit at most

