Society of Actuaries in Ireland

Algebra 1 – Solutions

Question 1

(1)(i) Solve for x : $2(4-3x) + 12 = 7x-5(2x-7)$

$8 - 6x + 12 = 7x - 10x + 35$

$-15 = 3x \Rightarrow x = -5$

(1)(ii) Verify your answer to (i) above

X = -5

$2 (4 - (-15)) + 12$

$38 + 12$

50

$7 (-5) - 5 (-10 - 7)$

$-35 + 85$

50

Question 2

(2) Solve the simultaneous equation $x = y = 7$

$x^2 + y^2 = 25$

$X = 7 - y$

$(7-y)^2 + y^2 = 25$

$y^2 - 7y + 12 = 0$

$(y-4) (y-3) = 0$

$y = 4$	$y = 3$
$x = 7-4$	$x = 7-3$
$x=3$	$x = 4$
(3,4)	(4,3)

Question 3

Simplify $\frac{x^2 - xy}{x^2 - y^2}$.

Factorise the numerator (the top line of the equation) and denominator (bottom line of the equation- difference of two squares)

$$\frac{x(x-y)}{(x+y)(x-y)}$$

$$\frac{x(x-y)}{(x+y)(x-y)}$$

$$= \frac{x}{(x+y)}$$

Question 4

Express the following as a single fraction in its simplest form:

$$\frac{6y}{x(x+4y)} - \frac{3}{2x}$$

Step 1:
Find the common denominator by multiplying the bottom lines:
$2x * x(x + 4y) = 2x^2(x+4y)$

So $2x^2(x+4y)$ is our **common denominator**

Step 2:
Find the numerator by cross multiplying (top lines by bottom lines):

$6y*2x - 3*x(x+4y) = 12xy - 3*(x^2+4xy)$
$= 12xy - 3x^2 - 12xy$
$= -3x^2$

So $-3x^2$ is our **numerator**

Step 3:
The answer is the numerator divided by the denominator:

$$\frac{6y}{x(x+4y)} - \frac{3}{2x} = \frac{Numerator}{Denominator}$$

$$= \frac{-3x^2}{2x^2(x+4y)}$$

$$= \frac{-3}{2(x+4y)}$$

$$= \frac{-3}{2x+8y}$$

Question 5

Solve the simultaneous equations:

$$x^2 + xy + 2y^2 = 4 \qquad \textit{Equation (1)}$$
$$2x + 3y = -1. \qquad \textit{Equation (2) Line}$$

Find x in terms of y using the linear equation; Equation (2) *2x + 3y = -1*

$$x = \frac{-3y - 1}{2}$$

Substitute $x = \frac{-3y-1}{2}$ *into Equation (1)*

$$\left(\frac{-3y - 1}{2}\right)^2 + \left(\frac{-3y - 1}{2}\right)y + 2y^2 = 4$$

Multiply across by 4

$$(-3y - 1)^2 + (-3y - 1)2y + 8y^2 = 16$$

Expand the bracket and take the 16 over to the left hand side

$$9y^2 + 6y + 1 - 6y^2 - 2y + 8y^2 - 16 = 0$$

Group terms together

$$11y^2 + 4y - 15 = 0$$

$$(11y + 15)(y - 1) = 0$$

$$y = \frac{-15}{11} \text{ or } y = 1$$

Substitute $y = \frac{-15}{11}$ *or* $y = 1$ *into Equation (2) to solve for x*

$$2x + 3\left(\frac{-15}{11}\right) = -1$$

$$2x + \left(\frac{-45}{11}\right) = -1$$

$$2x = -1 + \frac{45}{11}$$

$$x = \frac{17}{11}$$

OR

$$2x + 3(1) = -1$$

$$2x + 3 = -1$$

$$2x = -4$$

$$x = -2$$

Give the answer matching the appropriate x and y values:

Answer = $\left(\frac{17}{11}, \frac{-15}{11}\right)$ $and\ (-2,1)$

Question 6

Express the following as a single fraction in its simplest form:

$$\frac{x^2+4}{x^2-4} - \frac{x}{x+2}$$

Hint: $x^2 - 4$ **is the difference between two squares i.e. $(x)^2 - (2)^2 = (x + 2)(x - 2)$**

Step 1:

Find the common denominator by multiplying the bottom lines:

So $(x^2 - 4)(x+2)$ is our **common denominator**

Step 2:

Find the numerator by cross multiplying (top lines by bottom lines):

$(x^2 + 4)(x+2) - x(x^2 - 4) = (x+2) * [(x^2 + 4) - x(x-2)]$ .... because : $x^2 - 4 = (x + 2)(x - 2)$

So $(x+2) * [(x^2 + 4) - x(x-2)]$ is our **numerator**

Step 3:

The answer is the numerator divided by the denominator:

$$\frac{x^2+4}{x^2-4} - \frac{x}{x+2} = \frac{\boldsymbol{Numerator}}{\boldsymbol{Denominator}}$$

$$= \frac{(x+2) * [(x^2+4) - x(x-2)]}{(x^2-4)(x+2)}$$

$$= \frac{(x^2+4) - x(x-2)}{(x^2-4)}$$ ...(x+2) cancels

$$= \frac{(x^2+4) - x^2 + 2x)}{(x^2-4)}$$

$$= \frac{2x+4}{(x^2-4)}$$

$$= \frac{2(x+2)}{(x+2)(x-2)}$$ ...difference of two squares

$$= \frac{\mathbf{2}}{\mathbf{(x-2)}}$$

Question 7

Find the range of values of x for which $|x-4| \geq 2$, where $x \in \mathbb{R}$.

Method 1:

Expand the bracket	$x^2 - 8x + 16 \geq 4$
Take the 4 to the left hand side	$x^2 - 8x + 12 \geq 0$
Solve for the roots of the equation	$(x-2)(x-6) \geq 0$
	$x = 2$
	$x = 6$
Answer	$x \leq 2$ or $x \geq 6$

Method 2:

Split into 2 separate equations: $+(x-4) \geq 2 \text{ or } -(x-4) \geq 2$

Solve each equation separately:

$$x - 4 \geq 2$$
$$x - 4 + 4 \geq 2 + 4$$
$$x \geq 6$$

OR

$$-(x-4) \geq 2$$
$$+(x-4) \leq -2$$
$$x - 4 + 4 \leq -2 + 4$$
$$x \leq 2$$

Question 8

Find the set of all real values of x for which $2x^2 + x - 15 \geq 0$.

Step 1:

For inequalities, first set the equation = 0 and solve.

$2x^2 + x - 15 = 0$

(2x-5) (x+3) = 0

2x-5=0 x+3=0

x =2.5 x= -3

Step 2:

Then look at the sign in the inequality in the question.

If the sign is ≤ 0 we are “between the posts” which means the answer will be in the format of number ≤ x ≤ number e.g. -3 ≤ x ≤ 2.5

If the sign is ≥ 0 we are “outside the posts” which means the answer will be in the format of x ≤ number and x ≥ number e.g. x≤-3 and x ≥ 2.5

Step 3:
Therefore in this case the answer is
x≤-3 and x ≥ 2.5 *(You can sub in values to the original question to check if your answer is correct!)*

Question 9

$$x = \sqrt{x+6}$$
$$\Rightarrow x^2 = x+6$$
$$\Rightarrow x^2 - x - 6 = 0$$
$$\Rightarrow (x+2)(x-3) = 0$$
$$\Rightarrow x = -2, \quad x = 3$$
$$x = -2: \quad -2 \neq \sqrt{-2+6} = \sqrt{4} = 2 \ \times$$
$$x = 3: \quad 3 = \sqrt{3+6} = \sqrt{9} = 3 \ \checkmark$$

Question 10

Solve the following for x, y and z.

x + 2y – z = 1
2x + y + z = 4
x + 2y + z = 2

Solution:

Step 1:
Number each equation

x + 2y – z = 1 .... (1)
2x + y + z = 4 .... (2)
x + 2y + z = 2 .... (3)

Step 2:
Add two equations together to find equations (4) and (5):

x + 2y – z = 1 .... (1)
x + 2y + z = 2 .... (3)
2x+4y = 3 .... (4)

x + 2y – z = 1 .... (1)
2x + y + z = 4 .... (2)
3x + 3y = 5 .... (5)

Step 3:
Solve equations (4) and (5):

2x+4y = 3 .... (4)
3x + 3y = 5 .... (5)

6x + 12y = 9 .... (4) (x3)
6x + 6y = 10 .... (5) (x2)

6x + 12y = 9 .... (4)
-6x - 6y = -10 .... (5) (x-1)
6y = -1
y = -1/6

2x+4y = 3 (equation 4)
2x +4(-1/6) = 3

x = 11/6
x + 2y – z = 1 (Equation 1)
(11/6) + 2(-1/6) – z = 1
z = 1/2
(You can sub in values to the original question to check if your answer is correct!)

Question 11

Solve the equation

|4x – 3| > 5

Solution 2:
Step 1:
Set the equation = instead of less than/greater than and solve
|4x - 3| = 5
This could mean that

4x - 3= 5	or	4x - 3= -5
4x = 5+3		4x = -5 + 3
4x = 8		4x = -2
x = 2		x = -½

Step 2:
Is the answer "between the roots" or "outside the roots"?
In this question the sign is > so the answer is outside the roots. So the answer is:
x > 2 and x < -½

Question 12

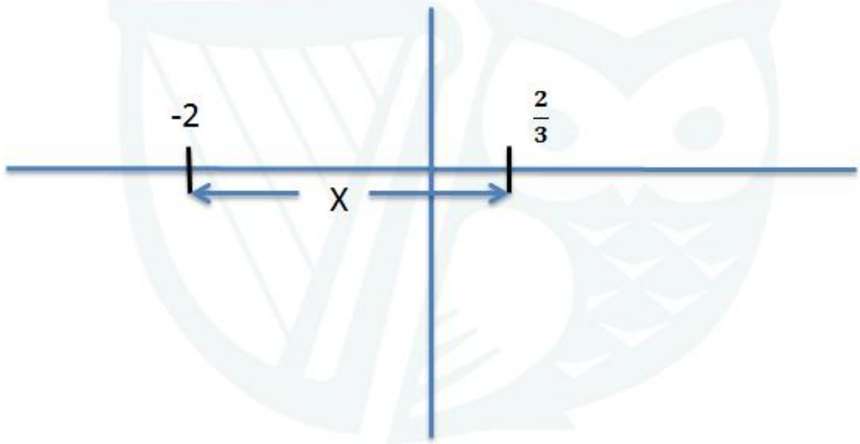
Step 1:
Set the equation = instead of less than/greater than and solve
|3x + 2| < 4

This could mean that

3x + 2= 4	or	3x + 2= -4
3x = 4-2		3x = -4 -2
3x = 2		3x = -6
x = ⅔		x = -2

Step 2:
Is the answer "between the roots" or "outside the roots"?
In this question the sign is < so the answer is between the roots. So the answer is:
-2 < x < ⅔
Step 3:
Is the answer between the roots or outside the roots?
(Note: that x = -2 and x = ⅔ should not be included on the number in the shaded region as inequality does not include equals, it's just less than)

This answer is between the roots so on a graph this looks like:



Question 13

<u>Step 1:</u>
Set the equation = 0

$$f(x) = 2x^3 - 4x^2 - 22x + 24 = 0$$

<u>Step 2:</u>
To find x, try a few values of x and see if they give you zero.
Try x=0 first:

$f(0) = 2*0 - 4*0 - 22*0 + 24$
$= 24 \neq 0$

This is not equal to zero so x=0 is not a solution (or "root").
Try x=1:

$f(1) = 2*1 - 4*1 - 22*1 + 24$
$= 0\,!$

This is equal to zero so **x=1** is a solution (or "**<u>root</u>**") which means that **(x-1)** is a **<u>factor</u>**.

<u>Step 3:</u>
Divide the equation by the factor you just found.

$$2x^2 - 2x - 24$$
$$(x-1)\sqrt{2x^3 - 4x^2 - 22x + 24}$$
$$2x^3 - 2x^2$$
$$-2x^2 - 22x + 24$$
$$-2x^2 + 2x$$
$$-24x + 24$$
$$-24x + 24$$

<u>Step 4:</u>

Factorise fully.
$2x^3 - 4x^2 - 22x + 24$ = $(x-1)(2x^2 - 2x - 24)$

= (x-1)(2x+6)(x-4)

Step 5:

Pull out the final answers, also called "roots".
x-1 = 0
x=1

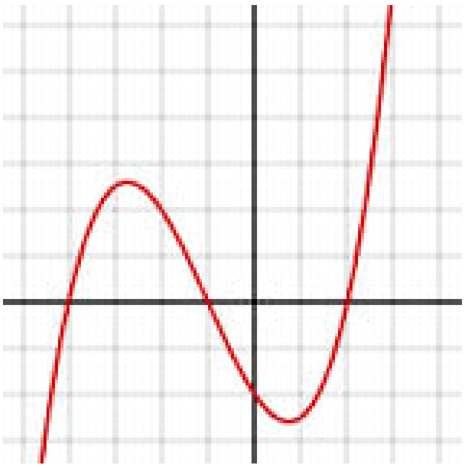
2x+6 = 0
2x = -6
x = -3

x-4 = 0
x = 4

x = -3 , 1 ,4 are the solutions

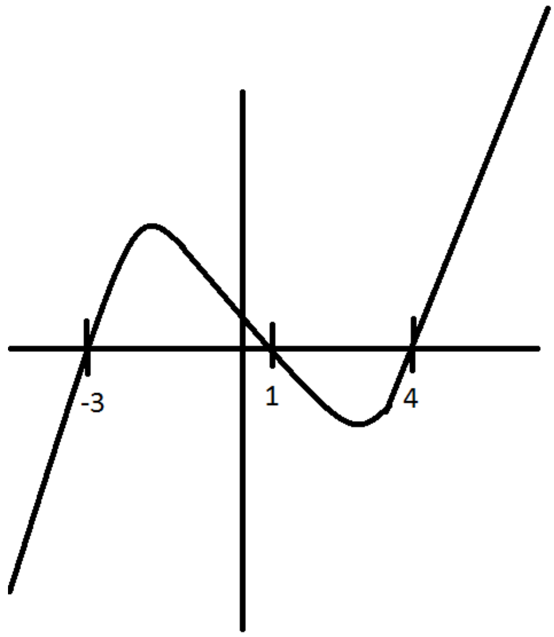
Step 6:

The graph always looks something like this:



So just make sure it crosses the x axis at your solutions x = -3 , 1 ,4:

Algebra 1 – Solutions



Question 14

(a)

$x=-3,\quad x=-1,\quad x=2$
$f(x)=(x+3)(x+1)(x-2)=x^3+2x^2-5x-6$

OR

$f(x)=x^3+2x^2-5x-6$

$f(-3)=-27+18+15-6=0 \Rightarrow (x+3)$ is a factor
$f(-1)=-1+2+5-6=0 \Rightarrow (x+1)$ is a factor
$f(2)=8+8-10-6=0 \Rightarrow (x-2)$ is a factor

$f(x)=(x+3)(x+1)(x-2)=x^3+2x^2-5x-6$

(b) (i)

$f(x)=g(x)$
$x^3+2x^2-5x-6=-2x-6$
$\Rightarrow x^3+2x^2-3x=0$
$\Rightarrow x(x^2+2x-3)=0$
$\Rightarrow x(x-1)(x+3)=0$
$\Rightarrow x=0,\quad x=1,\quad x=-3$
$\Rightarrow y=-6,\quad y=-8,\quad y=0$

Points: $(-3,0),\ (0,-6),\ (1,-8)$

(ii)

$g(x)=-2x-6$
$g(-3)=-2(-3)-6=6-6=0 \Rightarrow (-3,0)$
$g(0)=-2(0)-6=-6 \Rightarrow (0,-6)$

Question 15

a (i)

$f(x)=x^3+kx^2-4x-12$
$(x+3)$ is factor $\Rightarrow f(-3)=0$
$f(-3)=(-3)^3+k(-3)^2-4(-3)-12=0$
$-27+9k+12-12=0$
$9k=27 \Rightarrow k=3$

(ii)

$$\frac{3}{1+x^{p}}+\frac{3}{1+x^{-p}}=\frac{3\left(1+x^{-p}\right)+3\left(1+x^{p}\right)}{\left(1+x^{p}\right)\left(1+x^{-p}\right)}$$
$$=\frac{3\left(1+x^{-p}+1+x^{p}\right)}{1+x^{p}+x^{-p}+x^{0}}$$
$$=\frac{3\left(2+x^{-p}+x^{p}\right)}{\left(2+x^{-p}+x^{p}\right)}$$
$$=3$$

Question 16

(i) Let x = Stage number.

There are 4 times as many blue tiles than the Stage number

Blue tiles = 4x

There are 4 white tiles in every stage. This is a constant and remains 4 no matter what stage number we use.

The total number of green tiles is the square of the stage number.

Number of Green tiles = x^2

The total number of tiles (T) must be the green tiles + blue tiles + white tiles

$T = x^2 + 4x + 4$

(ii) $x^2 + 4x + 4 = 324$

Factorise $(x+2)(x+2)=324$

$(x+2)^2=324$

$x+2=18$

$x=16$

There are x^2 green tiles therefore $16^2 = 256$ green tiles.

(iii) Mary's kitchen is square. Therefore the length of each side = $\sqrt{6.76} = 2.6$m = 260 cm.
Each tile has sides of 20 cm each and 13 x 20 = 260. Therefore there are 13 tiles on each side or in each row.
In the first row there are two white tiles and the rest (13-2=11) are blue. Therefore this must be stage 11.

Green = x^2=121

Blue =4x =44

White = 4

Check: The total number of tiles = 121+44+4=169.

The area of each tile = 0.20 x 0.20 = 0.04 m^2. The total number of tiles needed = $6.76 \div 0.04 = 169$.